



Three-Player Games with Strategy-Dependent Time Delays

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Abstract

We analyze replicator dynamics with strategy-dependent time delays in a certain three-player game with one pure and two mixed Nash equilibria. In such a model, new players are born from parents who interacted in the past. We show that stationary states depend on time delays. Moreover, at certain time delays, interior equilibria cease to exist.

Keywords Evolutionary game theory · Multi-player games · Replicator dynamics · Time delays

1 Introduction

Evolutionary game theory describes behavior of populations consisting of many individuals. Usually their strategic interactions are decomposed into a sum of two-player games [13,14,21,22,33,34]. Such a decomposition can be treated as an approximation of real social processes involving multi-player games. One of the first papers investigating multi-player games is that of Haigh and Cannings [12] where multi-player War of Attrition was discussed. Systematic studies of multi-player games are presented in [5–9,18]. In particular, Kim [18] investigated an asymptotic and stochastic stability of Nash equilibria in multi-player games, Broom, Cannings, and Vickers [5] defined evolutionarily stable strategies for multi-player games and analyzed their properties, Bukowski and Miękiś [8] provided a classification of symmetric three-player games with two strategies, Broom and Cannings modeled dominance hierarchy formation as a multi-player game [6], Broom and Rychtář [7] provided a general framework for multi-player games in networks, and fixation probabilities in multi-player games were discussed by Gokhale and Traulsen [9], see also [10,11]. Particular multi-player games were also investigated; Pacheco et al. [29] analyzed a multi-player Stag Hunt game, and Souza et al. [31] and Santos et al. [30] discussed a multi-player Snowdrift game. Stochastic stability

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of three-player games on lattice and in well-mixed populations were discussed by Miękisz in [17,23],

All social and biological processes take a certain amount of time. It is natural, therefore, to introduce time delays into evolutionary games. Effects of time delay in evolutionary games were discussed in [1–3,15,16,24,25,27,28,32,35,36]. It was shown that for certain models and time delays (above a critical value where the Hopf bifurcation appears), interior Nash equilibria cease to be asymptotically stable and evolutionary dynamics exhibits oscillations and cycles. However, it was pointed out in [1] that in the so-called biological model, where it is assumed that the number of players born in a given time is proportional to payoffs received by their parents in a certain moment in the past, the interior Nash equilibrium is asymptotically stable for any time delay. Such a microscopic model gives rise to a system of differential equations of replicator dynamics, one for a frequency of a first strategy and the other one for a size of the population.

Recently, there were studied models with strategy-dependent time delays. In particular, Moreira et al. [27] discussed multi-player Stag Hunt game with time delays, Ben Khalifa et al. [3] investigated asymmetric games in interacting communities, and Wesson and Rand [35] studied Hopf bifurcations in two-strategy delayed replicator dynamics.

Recently [26], strategy-dependent time delays were introduced in a biological-type model of a two-player game presented in [1]. A novel behavior was reported, namely it was shown that, unlike in all previous models, stationary states of the resulting replicator-type dynamics depend on time delays.

Here, we discuss the effect of strategy-dependent time delays on the long-run behavior of a biological-type model of a certain three-player game with two strategies and three Nash equilibria, one pure and two interior ones. We show that stationary states depend on time delays and at certain time delays, interior states may disappear and a pure strategy becomes globally asymptotically stable.

In Sect. 2, we introduce our model and derive an equation for stationary states. In Sect. 3, we present numerical solutions of this equation for a particular game. Some analytical results concerning a number of stationary states are also presented. Discussion follows in Sect. 4. Appendix contains proofs of analytical results.

2 Replicator Dynamics with Strategy-Dependent Time Delays

We consider a certain symmetric three-player game with two strategies, A and B, given by the following payoff matrices:

$$\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & b \\ b & c+b \end{pmatrix} \right)$$

where the left matrix gives payoffs for the row player, when the third player uses A, whereas the right matrix provides payoffs in the case of the third individual playing B.

A complete classification of such games with respect to Nash equilibria and evolutionarily stable strategies is provided in [8]. Here, we will present a case study of possible effects of strategy-dependent time delays on the long-run behavior of three-player games in the replicator dynamics. We choose payoff parameters such that our game has three Nash equilibria, one pure (A, A, A) and two interior ones. In three-player games, there are at most two interior equilibria so this is the most complex situation in such games [8]. We know that a, b, c should satisfy the following conditions: $a > 0$, $c < 0$ and $b > \sqrt{|ac|}$. Then the mixed Nash

equilibria, represented by probabilities x_1 and x_2 of playing the strategy A , are given by the following formulas [8]:

$$x_1 = \frac{(b - c) - \sqrt{b^2 + ac}}{a + 2b - c} \quad \text{and} \quad x_2 = \frac{(b - c) + \sqrt{b^2 + ac}}{a + 2b - c}. \tag{1}$$

To see possible effects of time delays on stationary states, we consider here only a special case study, namely we set $a = 1, b = 2$ and $c = -1$; hence, $x_1 = \frac{3-\sqrt{3}}{6} \approx 0.21$ and $x_2 = \frac{3+\sqrt{3}}{6} \approx 0.79$. Let us note that $x_0 = 1$ and x_1 are asymptotically stable Nash equilibria, whereas x_2 is an unstable one.

To construct replicator dynamics with strategy-dependent time delays, we repeat the procedure from [1]. Let $p_i(t), i = A, B$, be the number of individuals playing at the time t the strategy A and B , respectively, $p(t) = p_A(t) + p_B(t)$ the total number of players, and $x(t) = \frac{p_A(t)}{p(t)}$ the fraction of the population playing A . Individuals are randomly matched to play a three-player game and receive average payoffs with respect to the state of the population $x(t)$:

$$U_A(t) = ax^2(t) + b(1 - x(t))^2 \quad \text{and} \quad U_B(t) = 2b(1 - x(t))x(t) + (c + b)(1 - x(t))^2. \tag{2}$$

We assume that individuals replicate at time t due to payoffs obtained at some earlier time; time delays depend on strategies and are equal to τ_A and τ_B respectively.

Let us assume that during a time interval of the length ε , only an ε -fraction of the population takes part in competitions, that is plays games.

We propose the following equations:

$$p_i(t + \varepsilon) = p_i(t) + \varepsilon p_i(t - \tau_i)U_i(t - \tau_i); \quad i = A, B, \tag{3}$$

$$p(t + \varepsilon) = p(t) + \varepsilon \left(p_A(t - \tau_A)U_A(t - \tau_A) + p_B(t - \tau_B)U_B(t - \tau_B) \right). \tag{4}$$

We divide (3) by (4) for $i = A$, obtain an equation for $x(t + \varepsilon) \equiv x_A(t + \varepsilon)$, subtract $x(t)$, divide the difference by ε , take the limit $\varepsilon \rightarrow 0$, and get an equation for the frequency of the first strategy,

$$\frac{dx}{dt} = \frac{p_A(t - \tau_A)U_A(t - \tau_A)(1 - x(t)) - p_B(t - \tau_B)U_B(t - \tau_B)x(t)}{p(t)} \tag{5}$$

which can be also written as

$$\frac{dx}{dt} = \frac{x(t - \tau_A)p(t - \tau_A)U_A(t - \tau_A)(1 - x(t)) - (1 - x(t - \tau_B))p(t - \tau_B)U_B(t - \tau_B)x(t)}{p(t)}. \tag{6}$$

Let us notice that unlike in the standard replicator dynamics, the above equations are not closed, there appear in them variables describing the size of the population at various times. One needs corresponding equations for the population size. From (3) and (4), we have

$$\frac{dp_i(t)}{dt} = p_i(t - \tau_i)U_i(t - \tau_i); \quad i = A, B, \tag{7}$$

$$\frac{dp(t)}{dt} = \left(p_A(t - \tau_A)U_A(t - \tau_A) + p_B(t - \tau_B)U_B(t - \tau_B) \right). \tag{8}$$

To trace the evolution of the population, we have to solve the system of Eqs. (6) and (8) together with initial conditions on the interval $[-\tau_M, 0]$, where $\tau_M = \max\{\tau_A, \tau_B\}$. We assume that

$$x(t) = \varphi_x(t), \quad p(t) = \varphi_p(t), \quad \text{for } t \in [-\tau_M, 0]. \tag{9}$$

Proposition 1 *If the initial functions φ_x, φ_p are continuous on $[-\tau_M, 0)$ and non-negative, then there exists a unique, non-negative solution of system ((6), (8)) with initial condition (9) well defined on the interval $[0, +\infty)$.*

Proof The local existence of the solutions follows immediately from a standard theory of delay differential equations [20]. The non-negativity follows from [4]. To prove the global existence, it is enough to use the step method and to observe that on the interval $[0, \min\{\tau_A, \tau_B\}]$ system ((6), (8)) becomes a system of non-autonomous ordinary differential equations. The equation for p becomes linear and it can be solved and then the equation for x is also linear with respect to $x(t)$. \square

2.1 Stationary States

We derive here an equation for a stationary state of the frequency of the first strategy. Let us assume that there exists a stationary frequency \bar{x} such that $x(t) = \bar{x}$ for all $t \geq 0$ and for some suitably chosen function $p(t)$. Then average payoffs of each strategy are constant and are equal to

$$\bar{U}_A = a \bar{x}^2 + b(1 - \bar{x})^2 \quad \text{and} \quad \bar{U}_B = 2b(1 - \bar{x})\bar{x} + (c + b)(1 - \bar{x})^2. \tag{10}$$

Thus, equation (8) becomes a linear delay differential equation

$$\frac{dp}{dt} = \bar{x}\bar{U}_A p(t - \tau_A) + (1 - \bar{x})\bar{U}_B p(t - \tau_B). \tag{11}$$

Note, that solutions of (11) with non-negative initial conditions are non-negative. This implies that the leading eigenvalue of this equation is real. The eigenvalues λ of (11) satisfy

$$\lambda = \bar{x}\bar{U}_A e^{-\lambda\tau_A} + (1 - \bar{x})\bar{U}_B e^{-\lambda\tau_B}. \tag{12}$$

Assume now that λ is a solution of (12) and $p(t) = p_0 \exp(\lambda t)$.

We plug such p into (6) and we get two stationary solutions of (6), $\bar{x} = 0, 1$ and possibly interior ones – solutions of

$$\bar{U}_A e^{-\lambda\tau_A} = \bar{U}_B e^{-\lambda\tau_B}. \tag{13}$$

If $\tau_A = \tau_B$, then (13) gives us mixed Nash equilibria of the non-delayed replicator dynamics.

If $\tau_A \neq \tau_B$, we have then $\lambda = \ln(\bar{U}_A/\bar{U}_B)/(\tau_A - \tau_B)$. Plugging it into (12), we conclude that \bar{x} satisfies an equation $F(\bar{x}) = 0$, where

$$F(x) = \frac{1}{\tau_A - \tau_B} \ln \left(\frac{ax^2 + b(1-x)^2}{2b(1-x)x + (c+b)(1-x)^2} \right) - \frac{\left(2b(1-x)x + (c+b)(1-x)^2 \right)^{\tau_A/(\tau_A-\tau_B)}}{\left(ax^2 + b(1-x)^2 \right)^{\tau_B/(\tau_A-\tau_B)}}. \tag{14}$$

First, we state a proposition that explains in which sense \bar{x} is a stationary state of the replicator dynamics.

Proposition 2 *Assume that $\tau_A \neq \tau_B$, let \bar{x} be a solution of $F(x) = 0$, where F is defined by (14) and let*

$$\bar{\lambda} = \frac{1}{\tau_A - \tau_B} \ln \left(\frac{a\bar{x}^2 + b(1 - \bar{x})^2}{2b(1 - \bar{x})\bar{x} + c(1 - \bar{x})^2} \right).$$

Then the functions

$$x(t) = \bar{x}, \quad p(t) = p_0 e^{\bar{\lambda}t}, \quad t \geq 0$$

are solutions of system (6)–(8) with initial conditions

$$\varphi_x(t) = \bar{x}, \quad \varphi_p(t) = p_0 e^{\bar{\lambda}t}, \quad \text{for } t \in [-\tau_M, 0].$$

3 Results

Here, we state some general results (proofs are given in the ‘‘Appendix’’) concerning the number of stationary states and provide numerical solutions for the equation for the stationary state as a function of time delays, τ_A and τ_B .

To show a variety of possible behaviors, we limit ourselves to the special case when

$$a = 1, \quad b = 2, \quad c = -1. \tag{15}$$

For such payoffs, the function F reads

$$F(x) = \frac{1}{\tau_A - \tau_B} \ln \left(\frac{x^2 + 2(1 - x)^2}{4(1 - x)x + (1 - x)^2} \right) - \frac{(4(1 - x)x + (1 - x)^2)^{\tau_A/(\tau_A - \tau_B)}}{(x^2 + 2(1 - x)^2)^{\tau_B/(\tau_A - \tau_B)}}. \tag{16}$$

In order to formulate a theorem concerning the number of stationary states inside the interval $(0, 1)$ we need the Lambert W function W_L that is a positive solution of the equation $y = x e^x$. Thus, $y = W_L(y) \exp(W_L(y))$.

Theorem 1 For $a = 1, b = 2$ and $c = -1$, the equation $F(x) = 0$ has at most two solutions in the interval $(0, 1)$. Moreover, the following statements are true:

- (i) For $\tau_A > \tau_B$, there always exists exactly one solution of the equation $F(x) = 0$ in the interval $\left(\frac{8}{9 + \sqrt{33}}, 1\right)$. On the other hand, if

$$\left(1 - \frac{\ln 2}{W_L(2\tau_A)}\right) \tau_A < \tau_B,$$

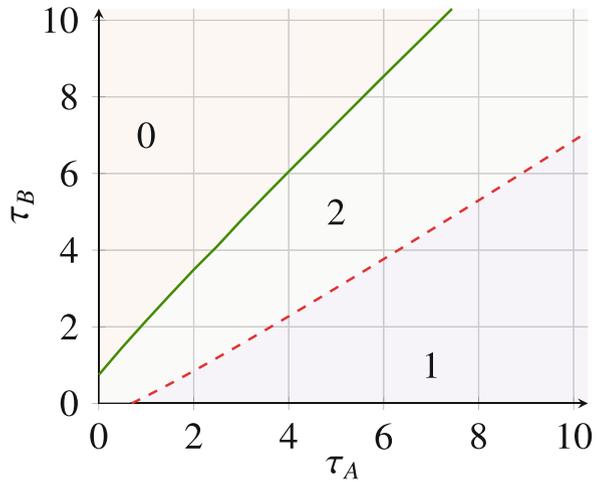
then there exists a solution of $F(x) = 0$ in the interval $x \in \left(0, \frac{1}{3 + \sqrt{3}}\right)$. It implies that if

$$\tau_B \leq \left(1 - \frac{\ln 2}{W_L(2\tau_A)}\right) \tau_A,$$

then there exists only one solution to $F(x) = 0$ in the interval $(0, 1)$.

- (ii) For $\tau_A < \tau_B$ and τ_B sufficiently close to τ_A (i.e., it is enough to take $\tau_A < \tau_B < \tau_A + \frac{4}{3} \ln \frac{5}{3}$), there exist two solutions of $F(x) = 0$ in the interval $(0, 1)$. On the other hand, for $\tau_B - \tau_A$ sufficiently large (e.g., larger than $2 \ln \frac{1 + \sqrt{33}}{4}$) there is no solution of $F(x) = 0$ in the interval $(0, 1)$.

Fig. 1 Number of stationary states inside the interval $(0, 1)$ for various values of τ_A and τ_B , $a = 1, b = 2, c = -1$



The proof is given in the “Appendix”.

In Fig. 1, we visually present the number of stationary states inside the interval $(0, 1)$ for various values of τ_A and τ_B .

To see the dependence of interior stationary states on time delays, we have solved numerically (16) using formulas (17) and (18) derived in the Appendix, results are presented in Fig. 2. We have also performed computer simulations of replicator dynamics to check stabilities of stationary states. We solved numerically system ((6), (8)) for various values of time delays. As initial functions, we took constant ones or constants ones for x and exponential for p . We have not noticed any significant difference in asymptotic behavior of solution if different initial functions were taken. We also checked the behavior of solutions for different initial values of the population obtaining again no significant differences. The asymptotic behavior of the strategy frequency depends on its initial value. In Fig. 2, we indicated stability of the stationary states as it was suggested by numerical simulations. Results of exemplary simulations are presented in Fig. 3.

First, we set $\tau_A = \tau, \tau_B = 1.5\tau$ (see Fig. 2b). When τ increases, the stable stationary state increases and the unstable one decreases until they disappear at a certain τ . Beyond that point, $x = 1$ becomes globally asymptotically stable.

Now, we set $\tau_B = 4$ (see Fig. 2c). For small τ_A , there are no interior states and $x = 1$ is globally asymptotically stable. At a certain τ_A , there appear two interior states, a stable and an unstable one. At another value of τ_A , the stable one disappears.

Finally, we set $\tau_A = 1$ (see Fig. 2d). For small τ_B , there is one interior unstable state. At a certain τ_B , there appears a stable interior state. At another value of τ_B , both interior states disappear and $x = 1$ becomes globally asymptotically stable.

4 Discussion

Social and biological processes are usually described by ordinary or partial differential equations, or by Markov processes if we take into account stochastic perturbations. However, interactions between individuals, players or molecules, naturally take time. It is therefore important to study the effect of time structure of interactions on evolution of populations. Recently, the effect of the duration of interactions between two players on their payoffs and

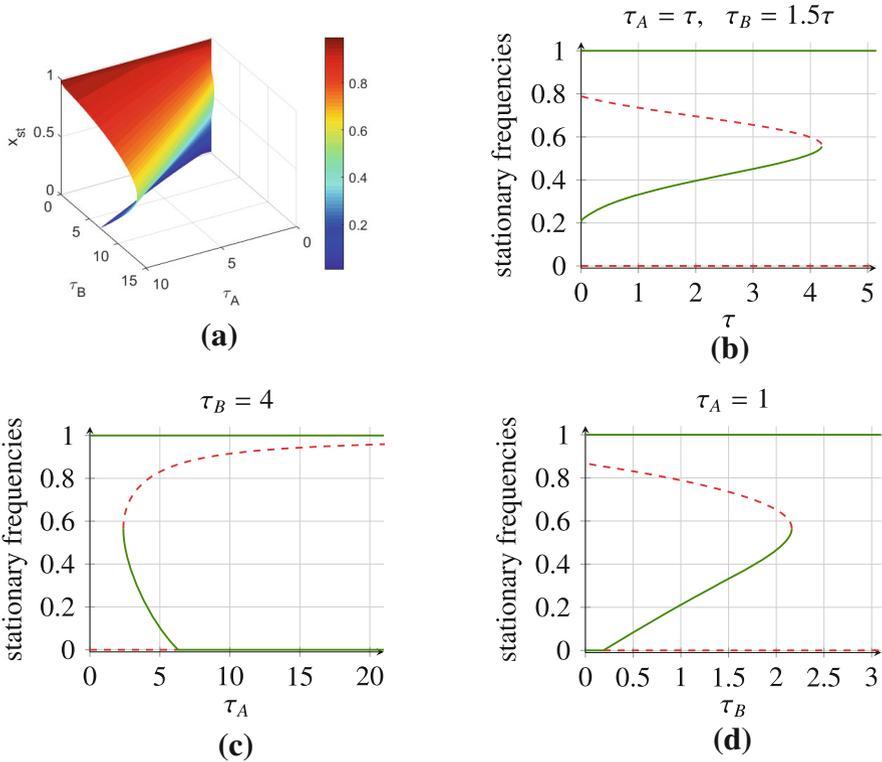


Fig. 2 Dependence of stationary states on τ_A and τ_B , $a = 1, b = 2, c = -1$. Solid lines denote stable states, dotted lines denote unstable states

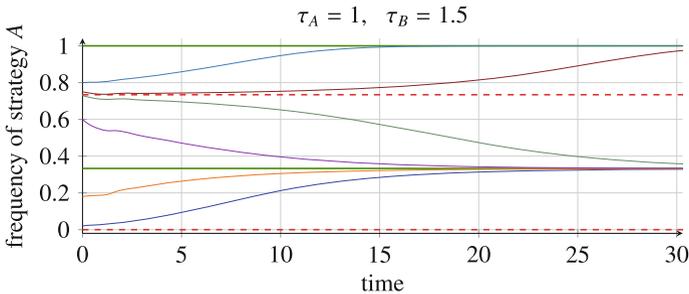


Fig. 3 Time evolution of frequency of strategy A for $\tau_A = 1, \tau_B = 1.5, a = 1, b = 2, c = -1$

therefore on evolutionary outcomes were discussed by Křivan and Cressman [19]. In their models, the duration of interactions depend of strategies involved. This naturally can be interpreted as strategy-dependent time delays. They showed that interaction times change stationary states of the system.

Another approach is to consider ordinary differential equations with time delays [32]. It was shown there that for small time delays, the stationary state is asymptotically stable and at a certain critical time delay the system undergoes the Hopf bifurcation—the interior state loses stability, oscillations arise. However, it was pointed out in [1] that in the so-called biological

model, where it is assumed that the number of players born in a given time is proportional to payoffs received by their parents at a certain moment in the past, the interior state is asymptotically stable for any time delay. Strategy-dependent time delays were introduced to this model in [26]. A novel behavior was observed. It was shown that strategy-dependent time delays change stationary states in two-player games. Here, we analyzed replicator dynamics with strategy-dependent time delays in a certain three-player game with one pure and two mixed Nash equilibria (the most complex behavior in such games). We showed that interior stationary states depend on time delays. Moreover, at certain time delays, interior equilibria cease to exist. Our results are qualitatively similar to those obtained in [19].

Here, we considered only a special case study, more systematic investigations are needed, especially concerning classic multi-player games describing social dilemmas.

It would be also interesting to analyze strategy-dependent time delays in stochastic dynamics of finite populations.

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Appendix

Proof of Theorem 1. Let us denote $\alpha = \frac{\tau_A}{\tau_A - \tau_B}$. The equation $F(x) = 0$ is equivalent to $F_L(x) = F_R(x)$, where

$$F_L(x) = \frac{\alpha}{\tau_A} \ln \left(\frac{x^2 + 2(1-x)^2}{4(1-x)x + (1-x)^2} \right),$$

$$F_R(x) = \left(x^2 + 2(1-x)^2 \right) \left(\frac{4(1-x)x + (1-x)^2}{x^2 + 2(1-x)^2} \right)^\alpha.$$

First, we consider the case $\tau_B > \tau_A$. We calculate the second derivative of F_L with respect to x and we get

$$F'_L(x) = -\frac{2\alpha}{\tau_A} \frac{3x^2 - 9x + 4}{(1-x)(3x+1) \left(\left(x\sqrt{3} - \frac{2}{\sqrt{3}} \right)^2 + \frac{2}{3} \right)},$$

$$F''_L(x) = \frac{\alpha}{\tau_A} \left(\frac{1}{(x-1)^2} + \frac{9}{(3x+1)^2} + \frac{8}{(3x^2 - 4x + 2)^2} - \frac{6}{3x^2 - 4x + 2} \right).$$

After some computations (reducing the expression in brackets to a common denominator and plotting—or estimating—a polynomial of the fifth degree of the numerator) it is possible to show that $F''_L(x) < 0$ for all $x \in (0, 1)$ and $\alpha < 0$. Thus, F_L is concave. On the other hand, the derivatives of F_R read

$$F'_R(x) = \frac{(4(1-x)x + (1-x)^2)^{\alpha-1}}{(x^2 + 2(1-x)^2)^\alpha} \left(2\alpha(x^2 - 5x^2 + 2) + 2(x(x-1)(3x+1)) \right),$$

$$F''_R(x) = 2 \frac{(4(1-x)x + (1-x)^2)^{\alpha-2}}{(x^2 + 2(1-x)^2)^{\alpha+1}} h(x),$$

where

$$h(x) = 2\alpha^2 (3x^2 - 9x + 4)^2 - \alpha \left(54x^5 - 171x^4 + 120x^3 + 111x^2 - 156x + 50 \right) + 3(x-1)^2 (3x^2 - 4x + 2) (3x+1)^2.$$

Plotting the polynomial of the fifth degree shows that the coefficient in α is positive for $x \in (0, 1)$, and therefore, $h(x) < 0$ for $x \in (0, 1)$. Thus, F_R is convex. This proves that equation $F_L(x) = F_R(x)$ has at most two solutions in $(0, 1)$.

Note that in this case ($\tau_A < \tau_B$), we have

$$\lim_{x \rightarrow 0^+} F_L(x) = -\frac{|\alpha|}{\tau_A} \ln 2 < 0, \quad \lim_{x \rightarrow 1^-} F_L(x) = -\infty,$$

and

$$\lim_{x \rightarrow 0^+} F_R(x) = 2^{1-\alpha} > 0, \quad \lim_{x \rightarrow 1^-} F_R(x) = 0.$$

If we compare the value of F_L and F_R at $x = \frac{1}{2}$, we deduce that if

$$\tau_A < \tau_B < \tau_A + \frac{4}{3} \ln \frac{5}{3},$$

then $F_L(\frac{1}{2}) > F_R(\frac{1}{2})$ so there are exactly two solutions of $F_L(x) = F_R(x)$ in the interval $(0, 1)$.

On the other hand, one can easily estimate that

$$F_L(x) \leq \frac{\alpha}{\tau_A} \ln \frac{\sqrt{33} - 1}{8}$$

for all $x \in (0, 1)$. Because $(1 - x)^2 + 4x(1 - x) \leq 4/3$ and $x^2 + 2(1 - x)^2 \geq \frac{2}{3}$, one deduce that $F_R(x) \geq \frac{1}{2}$. This completes the proof of part (ii) of the theorem.

Now, assume that $\tau_A > \tau_B$ which implies that $\alpha > 1$. In this case, it is convenient to change variables. Let

$$y = \frac{1 - x}{x} \iff x = \frac{1}{1 + y}.$$

The problem of finding solution of $F_L(x) = F_R(x)$ for $x \in (0, 1)$ transforms to the problem of finding solution of $G_L(y) = G_R(y)$ for $y > 0$, where

$$G_L(y) = -\frac{\alpha}{\tau_A} \ln \left(\frac{4y + y^2}{1 + 2y^2} \right), \quad G_R(y) = \frac{1 + 2y^2}{(1 + y)^2} \left(\frac{4y + y^2}{1 + 2y^2} \right)^\alpha.$$

It is very easy to calculate limits of G_L and G_R at 0 and at $+\infty$ and get

$$\begin{aligned} \lim_{y \rightarrow 0^+} G_L(y) &= +\infty, & \lim_{y \rightarrow +\infty} G_L(y) &= \frac{\alpha}{2} \ln 2, \\ \lim_{y \rightarrow 0^+} G_R(y) &= 0, & \lim_{y \rightarrow +\infty} G_R(y) &= 2^{1-\alpha}. \end{aligned}$$

We calculate the derivative of G_L and we get

$$G'_L(y) = \frac{2\alpha}{\tau_A} \cdot \frac{4y^2 - y - 2}{y(4 + y)(1 + 2y^2)}.$$

It is easy to see that for $\alpha > 1$, the function G_L is decreasing for $y \in (0, \tilde{y})$ with $\tilde{y} = \frac{1 + \sqrt{33}}{8}$, and it is increasing for $y > \tilde{y}$. Note also that

$$G_L(\tilde{y}) = -\frac{\alpha}{\tau_A} \ln \left(\frac{2\tilde{y}(\tilde{y} + 2\tilde{y}^2) + \tilde{y}(2 + \tilde{y} - 4\tilde{y}^2)}{1 + 2\tilde{y}^2} \right) = -\frac{\alpha}{\tau_A} \ln(2\tilde{y}) < 0,$$

while $G_R(Y) > 0$ for $y > 0$. Now, we prove that G_R is increasing on $(0, \tilde{y})$. We calculate the derivative of G_R with respect to y and we obtain

$$G'_R(y) = \frac{2(4y + y^2)^{\alpha-1}}{(1 + 2y^2)^\alpha(1 + y)^3} g(y)$$

with

$$g(y) = (1 + 2y^2)(2 - y) + (1 + y)(2 + y - 4y^2)(\alpha - 1).$$

Because $\tilde{y} < 2$ and $\alpha > 1$, we easily see that $g(y) > 0$ for $y \in (0, \tilde{y})$. Thus, G_R is increasing in $(0, \tilde{y})$, and therefore, there exists exactly one solution of $G_L(y) = G_R(y)$ in $(0, \tilde{y})$.

One can see that $G_L(x) = 0$ for $\hat{y}_1 = 2 - \sqrt{3}$ and $\hat{y}_2 = 2 + \sqrt{3}$. As $\hat{y}_2 > 2 > \tilde{y}$, we see that $g(y) < 0$ for $y > \hat{y}_2$, thus G_R is decreasing while G_L is increasing for $y > \hat{y}_2$. Thus, the second solution to $G_L(y) = G_R(y)$ exists in the interval $(\tilde{y}, +\infty)$ if and only if $G_L(+\infty) > G_R(+\infty)$, that is $\alpha \ln 2 > 2^{1-\alpha}$. Some simple algebra finishes the proof of the theorem. \square

Formula Used for Drawing Stationary Solutions

Here, we derive a formula that gives us a better relation between τ_A , τ_B , and \bar{x} than the equation $F(x) = 0$. As before, let us denote

$$\alpha = \frac{\tau_A}{\tau_A - \tau_B}, \quad U_A = x^2 + 2(1 - x)^2, \quad U_B = 4(1 - x)x + (1 - x)^2.$$

The equation $F(x) = 0$ is then equivalent to

$$\frac{1}{\tau_A} \alpha \ln \frac{U_A}{U_B} = U_A \left(\frac{U_A}{U_B} \right)^\alpha \iff \tau_A U_A = \ln \left(\frac{U_A}{U_B} \right)^\alpha \exp \left(\ln \left(\frac{U_A}{U_B} \right)^\alpha \right).$$

Thus,

$$\ln \left(\frac{U_A}{U_B} \right)^\alpha = W_L(\tau_A U_A) \iff \alpha = \frac{W_L(\tau_A U_A)}{\ln \frac{U_A}{U_B}},$$

where W_L is a Lambert W function, that is $x = W_L(x) \exp(W_L(x))$. Now, using the definition of α we get

$$\frac{\tau_A}{\tau_A - \tau_B} = \frac{W_L(\tau_A U_A)}{\ln \frac{U_A}{U_B}} \iff \tau_B = \tau_A \left(1 - \frac{\ln \frac{U_A}{U_B}}{W_L(\tau_A U_A)} \right).$$

In this manner, we get a relation between τ_A and τ_B at the stationary state x , namely

$$\tau_B = \tau_A \left(1 - \frac{\ln \frac{x^2 + 2(1-x)^2}{4(1-x)x + (1-x)^2}}{W_L((x^2 + 2(1-x)^2)\tau_A)} \right). \tag{17}$$

In an analogous manner—denoting $\beta = \frac{\tau_B}{\tau_A + \tau_B}$ —we get another relation between τ_A and τ_B at the stationary state x , namely

$$\tau_A = \tau_B \left(1 + \frac{\ln \frac{x^2 + 2(1-x)^2}{4(1-x)x + (1-x)^2}}{W_L((4(1-x)x + (1-x)^2)\tau_B)} \right). \tag{18}$$

We used formulas (17) and (18) to draw Fig. 2a.

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