

Stable Quasicrystalline Ground States

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We give strong evidence that noncrystalline materials such as quasicrystals or incommensurate solids are not exceptions, but rather are generic in some regions of phase space. We show this by constructing classical lattice-gas models with translation-invariant finite-range interactions and with a unique quasi-periodic ground state which is stable against small perturbations of two-body potentials. More generally, we provide a criterion for stability of nonperiodic ground states.

KEY WORDS: Quasicrystals; nonperiodic tilings; classical lattice-gas models; ground states; stability.

1. INTRODUCTION

One of the important problems in physics, the so-called crystal problem, is to understand why matter is crystalline at low temperatures.⁽¹⁻⁷⁾ It is traditionally assumed (but has never been proved) that at zero temperature, minimization of the free energy of a system of many interacting particles can only be obtained by their periodic arrangements (a perfect crystal), which at nonzero temperature are disrupted by defects due to entropy. Recently, however, there has been a growing evidence that this basic phenomenon, the crystalline symmetry of low temperature matter, has exceptions; in particular incommensurate solids⁽⁸⁾ and, more recently, quasicrystals.⁽⁹⁾ It is very important to find out how generic these examples are. In other words, is nonperiodic order present in these systems stable against small perturbations of interactions between particles?

The problem of stability of quasiperiodic structures was studied recently in continuum models of particles interacting through a well,

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Lennard-Jones, and other potentials.⁽¹⁰⁻¹⁵⁾ However, no final conclusion was reached. After all, one has to compare chosen quasiperiodic structure with all possible arrangements of particles in the space, a really formidable task.

Here we will present two classical lattice gas (toy) models with a unique stable nonperiodic ground state. More precisely, every site of a square lattice can be occupied by one of several different particles. The particles interact through two-body finite-range translation-invariant potentials (we chose chemical potentials of all particles to be zero). Our models have only nonperiodic ground-state configurations (infinite-lattice configurations minimizing the energy density of the system). We will prove that if one perturbs our models by sufficiently small chemical potentials or two-body translation-invariant interactions, their ground-state configurations do not change. It means that there is an open set in a space of two-body interactions without periodic ground-state configurations. This constitutes a first generic counterexample to the crystal problem.

In Section 2, we discuss general classical lattice gas models with a unique nonperiodic ground state. We introduce a criterion, the so-called strict boundary condition, for nonperiodic ground-state configurations to be stable against small perturbations of finite-range potentials. In Section 3, we describe main features of Robinson's nonperiodic tilings of the plane, construct a classical lattice gas model based on them, and show why its unique nonperiodic ground state is not stable against small perturbations of chemical potentials. In Section 4, we present modifications of Robinson's tilings which allows us to construct models with a unique nonperiodic ground state satisfying our stability criterion. Section 5 contains a short discussion.

2. CLASSICAL LATTICE GAS MODELS AND NONPERIODIC GROUND STATES

A classical lattice gas model is a system in which every site of a lattice \mathbf{Z}^d can be occupied by one of n different particles. An infinite-lattice configuration is an assignment of particles to lattice sites, i.e., an element of $\Omega = \{1, \dots, n\}^{\mathbf{Z}^d}$. If $X \in \Omega$ and $A \subset \mathbf{Z}^d$, then we denote by $X(A)$ a restriction of X to A . Particles at lattice sites \mathbf{a} and \mathbf{b} interact through a two-body translation-invariant potential $\Phi(\mathbf{a} - \mathbf{b})$, which is a function on $\{1, \dots, n\}^{\{\mathbf{a}, \mathbf{b}\}}$ —the space of all possible assignments of particles to lattice sites \mathbf{a} and \mathbf{b} , and we assume that $\Phi(\mathbf{a} - \mathbf{b}) = 0$ if $\text{dist}(\mathbf{a}, \mathbf{b}) > r$. The Hamiltonian in a bounded region A can be then written as $H_A = \sum_{\mathbf{a}, \mathbf{b} \in A} \Phi(\mathbf{a} - \mathbf{b})$.

Y is a *local excitation* of X , $Y \sim X$, $Y, X \in \Omega$, if there exists a bounded $A \subset \mathbf{Z}^d$ such that $X = Y$ outside A .

For $Y \sim X$ the *relative Hamiltonian* is defined by

$$H(Y, X) = \sum_{\{\mathbf{a}, \mathbf{b}\} \cap A \neq \emptyset} (\phi(\mathbf{a} - \mathbf{b})(Y) - \Phi(\mathbf{a} - \mathbf{b})(X))$$

$X \in \Omega$ is a *ground-state configuration* of H if

$$H(Y, X) \geq 0 \quad \text{for any } Y \sim X$$

That is, we cannot lower the energy of a ground-state configuration by changing it locally.

The energy density $e(X)$ of a configuration X is

$$e(X) = \lim_{A \rightarrow \mathbf{Z}^d} \inf \frac{H_A(X)}{|A|}$$

where $|A|$ is the number of lattice sites in A . It can be shown that any ground-state configuration has the minimal energy density (for a proof see Appendix A). It means that local conditions present in the definition of a ground-state configuration force global minimization of the energy density.

Equilibrium behavior of a system of many interacting particles can be described by a grand-canonical ensemble which tells us what are probabilities of finding particles at given positions. An infinite-volume limit of this ensemble is called a translation-invariant Gibbs state or equilibrium state. Mathematically speaking, it is a measure on the space, Ω , of all configurations (for precise definitions see Appendix B). Then by a *ground state* we mean a limit of a nonzero-temperature equilibrium state as temperature approaches zero, with other variables, such as chemical potentials, kept fixed. A ground state is, therefore, a translation-invariant probability measure on Ω . It can be shown that the set of all ground-state configurations have probability one with respect to a ground-state measure.

If a system has a unique periodic ground-state configuration and its translations, then a unique ground-state measure assigns an equal probability to all these translations. For example, the Ising antiferromagnet has two alternating ground-state configurations but only one ground-state measure which assigns probability 1/2 to both of them.

Generally, a probability ground-state measure μ gives equal weights to all ground-state configurations and can be obtained as a limit of averaging over a given ground-state configuration X and its translations $\tau_{\mathbf{a}}X$ by lattice vectors $\mathbf{a} \in \mathbf{Z}^d$: $\mu = \lim_{A \rightarrow \mathbf{Z}^d} 1/|A| \sum_{\mathbf{a} \in A} \delta(\tau_{\mathbf{a}}X)$, where $\delta(\tau_{\mathbf{a}}X)$ is a probability measure assigning probability 1 to $\tau_{\mathbf{a}}X$.

For any potential, the set of ground-state configurations is nonempty but it may not contain any periodic configurations.⁽¹⁶⁻²³⁾ We restrict ourselves to systems in which, although all ground-state configurations are nonperiodic, there is a unique translation-invariant probability measure supported by them. Therefore, such measure is inevitably a zero-temperature limit of translation-invariant Gibbs states (equilibrium states) and hence it is a ground state of a given model. Uniqueness of a ground-state measure is equivalent to the statement that a uniformly defined frequency of any finite arrangement of particles is the same in all ground-state configurations (such measures are called uniquely ergodic measures and their supports are called uniquely ergodic sets).⁽²⁴⁾ As a support of a ground-state measure we will always choose a minimal set (if any arrangement appears in a ground-state configuration, it appears with a positive density), i.e., we exclude ground-state configurations with broken bonds (such as interface ground-state configurations of the ferromagnetic Ising model). More precisely, to find the frequency of a finite arrangement in a given configuration we first count the number of times it appears in a box of size l which is centered at the origin of the lattice, divide it by l^d , and then take the limit $l \rightarrow \infty$. If the convergence is uniform with respect to the position of the boxes, then we say that the configuration has a uniformly defined frequency of this arrangement.

Let us emphasize again that in our models the frequency of any finite particle arrangement is the same in all ground-state configurations. Yet, it is not true that all ground-state configurations are lattice translations of a single nonperiodic ground-state configuration. That is why, in order to deal with all ground-state configurations simultaneously, we have to introduce a ground-state measure. Such a situation is present in all tiling models of quasicrystals; for example in the Penrose tilings, where instead of ground-state configurations we have perfect tilings.

To get all ground-state configurations we have to close (in a topology described in Appendix B) a set of all translations of any ground-state configuration (the closure does not depend on the choice of a configuration). In this way we obtain uncountably many ground-state configurations. In fact, probability of any single configuration should be zero; otherwise, the measure of a countable union of its translations would be infinite. If there were countably many ground-state configurations (there are countably many translations), then the measure of the set of all ground-state configurations would be zero, and this is impossible.

We will introduce now a condition which is equivalent to the stability of nonperiodic ground states. It generalizes the so-called Peierls condition⁽²⁷⁾ for models without periodic ground-state configurations.

For clarity of presentation we assume that our models are two-dimensional. We also assume that our models have a unique ground state,

supported by ground-state configurations for which all interactions attain simultaneously their minima (we choose them to be equal to zero). As it was explained before, we do not have to consider ground-state configurations with broken bonds. Therefore, if Y is not a ground-state configuration, it contains pairs of particles with nonminimal energies (we choose them to be equal to 1), the so-called *broken bonds*. Denote by $B(Y)$ the number of broken bonds in Y . Thus $H(Y, X) = B(Y)$ if $Y \sim X$ and X is a ground-state configuration.

Condition. The strict boundary condition for local excitations.

Let X be a ground-state configuration and Y a local excitation of X ; $Y \sim X$. Let $n_{ar}(Y, X)$ denote the difference of the number of appearances of an arrangement ar (a particle or a pair of particles) in Y and the number of its appearances in X .

We say that a model satisfies the strict boundary condition for local excitations and an arrangement ar if there exists a $C_{ar} > 0$ such that for every ground-state configuration X and every $Y \sim X$

$$|n_{ar}(Y, X)| < C_{ar} B(Y) \quad (1)$$

Theorem 1. A unique ground state of a finite-range Hamiltonian, H , is stable against small perturbations of chemical potentials and two-body interactions of range smaller than r if and only if the strict boundary condition is satisfied for particles and pairs of particles at a distant smaller than r .

Proof. Assume first that the strict boundary condition does not hold for a ground-state configuration X and an arrangement ar (a particle or a pair of particles at a distance smaller than r). Hence, for any $C > 0$ there exists $Y_C \sim X$ such that $n_{ar}(Y_C, X) > CB(Y_C)$ or $n_{ar}(Y_C, X) < -CB(Y_C)$. We assume that the first case holds; the second one can be treated in an analogous way. We introduce a perturbation, Φ , which assigns a negative energy, $E = -1/C$, to the arrangement ar . This perturbation is small if C is large. Let $H'' = H + H'$, where $H' = \sum_A \Phi_A$. Then

$$H''(Y_C, X) = B(Y_C) - n_{ar}(Y_C, X)/C < 0 \quad (2)$$

Thus, X is not a ground-state configuration for the perturbed Hamiltonian H'' .

Assume now that $|n_{ar}(Y, X)| < C_{ar} B(Y)$ for every arrangement ar (a particle or a pair of particles at a distance smaller than r) and every local excitation $Y \sim X$ of a ground-state configuration X . Let Ψ be a potential of

a range smaller than r and $H'' = H + H'$ be a perturbed Hamiltonian, where $H' = \sum_A \Psi_A$. Then

$$H''(Y, X) = B(Y) + \sum_{ar} n_{ar}(Y, X) \Psi(ar) \quad (3)$$

$$> B(Y) - B(Y) \sum_{ar} C_{ar} |\Psi(ar)| > 0 \quad (4)$$

if

$$\sum_{ar} C_{ar} |\Psi(ar)| < 1 \quad (5)$$

Hence, X is a ground-state configuration for H'' for every sufficiently small perturbation.

So far we have proved that ground-state configurations of H (without broken bonds) are ground-state configurations of H'' . Now we will show that the set of ground-state configurations of H'' is a uniquely ergodic set. It will then follow that the ground state of H'' must be equal to that of H . Let μ'' be an ergodic measure different from the ground-state measure of H . It has therefore a nonzero density, ω , of broken bonds. Let Z be in the support of μ'' and have the density ω of broken bonds. We will now show that Z is not a ground-state configuration of H'' .

We construct two configurations: $W \sim Z$ such that $W = X$ on a square A of size l and $W = Z$ outside A , and $Y \sim X$ such that $Y = Z$ on A and $Y = X$ outside A .

Since Y is an excitation of X , then

$$|n_{ar}(Y, X)| < C_{ar} B(Y) < C_{ar} (\omega l^2 + o(l^2)) \quad (6)$$

When we change Z to X on A we decrease the number of broken bonds at least by $\omega l^2 - o(l^2)$ and change the number of appearances of ar at most by $C_{ar} (\omega l^2 + o(l^2))$ which follows from (6). Hence,

$$H''(W, Z) < -(\omega l^2 - o(l^2)) + \sum_{ar} |\Psi(ar)| (C_{ar} (\omega l^2 + o(l^2))) \quad (7)$$

Thus,

$$H''(W, Z) < 0 \quad (8)$$

if (5) holds and l is big enough. It follows that Z is not a ground-state configuration of H'' . ■

3. ROBINSON'S TILINGS AND UNSTABLE GROUND STATES

To construct our classical lattice gas models, we will use Robinson's nonperiodic tilings of the plane.^(28, 29) He designed 56 square-like tiles such that using an infinite number of their copies one can tile the plane only in a nonperiodic fashion. We describe now the Robinson tiles; we follow ref. 28 closely. There are five basic tiles represented symbolically in Fig. 1; all other tiles can be obtained from them by rotations and reflections. The first tile on the left is called a cross; the other tiles are called arms. Every tile is also furnished with one of the four parity markings shown in Fig. 2. Crosses can be combined with the parity marking at the lower left in Fig. 2. Vertical arms (the direction of long arrows) can be combined with the marking at the lower right and horizontal arms with the marking at the upper left. All tiles can be combined with the remaining marking. Two tiles "match" if arrow heads meet arrow tails separately for the parity markings and the markings of crosses and arms.

We will now describe main features of Robinson's nonperiodic tilings. Observe first that, in any tiling, the centers of the tiles form the square lattice \mathbf{Z}^2 . Therefore, any tiling can be described by an assignment of tiles to lattice sites, a lattice configuration. It is also easy to see that if the plane is tiled with tiles with the above parity markings, then these must alternate both horizontally and vertically in the manner shown in Fig. 2. Hence, every odd-odd position on the \mathbf{Z}^2 lattice (if columns and rows are suitably numbered) is occupied by crosses with the lower left parity marking. They form a periodic configuration with period 4 as shown in Fig. 3 (crosses are denoted there by \perp , where line segments denote double arrows). Then in the center of each "square" matching rules force a cross such that the previous pattern reproduces but this time with period 8.

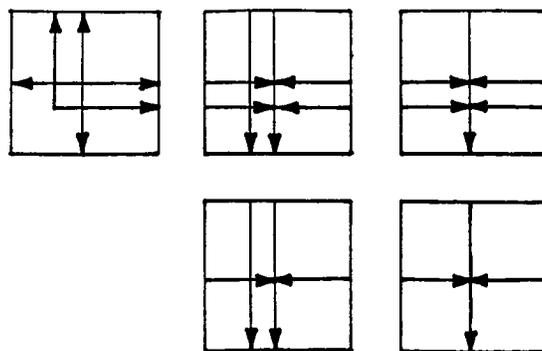


Fig. 1. Crosses and arms.

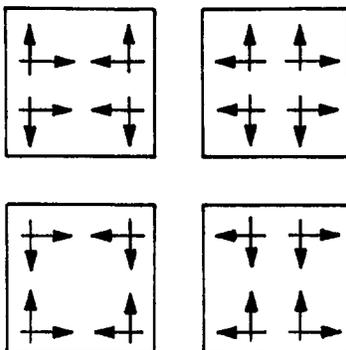


Fig. 2. Parity markings.

Continuing this procedure (the so-called principle of expanding squares) infinitely many times, we obtain a nonperiodic configuration. It has built in periodic configurations of period 2^n , $n \geq 2$, with tiles on sublattices of \mathbb{Z}^2 as is shown in Fig. 4. Each side of every square consists of two lines of arms with long double arrows at the left of Fig. 1 (called double arms) meeting half-way through double short arrows of arms at the top of Fig. 1. It is exactly the arms with asymmetrically located double short arrows which force crosses to be present at the center of each square. Crosses of different squares at the same sublattice are joined by lines of arms with single long arrows at the right of Fig. 1 (called single arms). For a complete construction of Robinson's tilings and proofs, we refer the reader to the original article of Robinson.⁽²⁸⁾

Now, using Robinson's tiles we will construct a lattice gas model.⁽¹⁶⁻²³⁾ Every site on a square lattice can be occupied by one of the particles

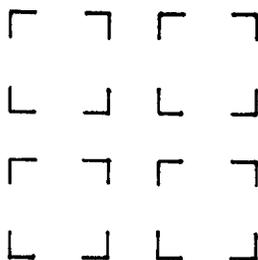


Fig. 3. Relative orientations of crosses.

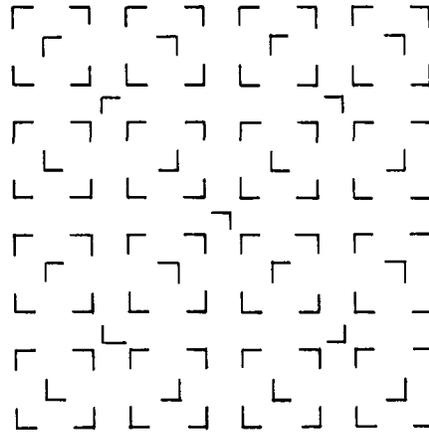


Fig. 4. Robinson's nonperiodic ground-state configuration.

corresponding to the tiles. Two nearest-neighbor particles which do not match as tiles have a positive interaction energy, say 1; otherwise, the energy is zero. We obtain a lattice gas model with nearest-neighbor translation-invariant interactions.

Any periodic configuration with period p corresponds to a periodic configuration of tiles. Therefore, it has, in any square of p^2 lattice sites, at least one pair of nearest neighbors with energy equal to 1. Hence, any periodic configuration has a nonzero energy density. On the other hand, nonperiodic configurations corresponding to tilings have energy density equal to zero. It follows that the model does not have periodic ground-state configurations.

It has been proven⁽¹⁹⁾ that the unique ground state of this model is not stable against small perturbations of chemical potentials. To see this let us introduce a small negative chemical potential h for single arms. Notice that every ground-state configuration contains arbitrarily long sequences of double arms. When we change such a sequence of length l into a sequence of single arms, we lower the energy by lh along the sequence, and only at two endpoints of the sequence the energy increases by 1. Hence, no matter how small h is, one can always change locally a ground-state configuration of an unperturbed model to lower its energy. It is easy to see that the strict boundary condition is not satisfied.

In fact, we showed that frequencies of single and double arms change continuously with respect to their chemical potentials. This behavior of continuous change in stoichiometries at zero temperature is not present in previously known models and real alloys.

4. STABLE QUASIPERIODIC GROUND STATES

We will present in a moment models with a unique nonperiodic ground state satisfying our stability criterion. This will be a modification of the model based on Robinson's tiles such that a sufficiently small chemical potential for one kind of arm and a small two-body perturbation do not change its unique ground state.

We cannot simply get rid of the double arms because it is exactly the double arms which create squares whose central crosses are forced. The absence of double arms will destroy the principle of expanding squares and we will no longer have a unique nonperiodic ground state.

However, let us recall that the forcing is done by four arms located exactly at the middle point between crosses forming a square; these are the arms at the top of Fig. 1. The main idea of our construction is to force similar arms in the middle of every side of every square, as in the original Robinson tiling, by using periodic sequences of new markings instead of translation-invariant sequences of single and double arms. We will use sequences of alternating A , B , and C markings propagating from crosses. Whenever two B or C markings meet at the middle of a side of a square, then our special arms are forced there.

Observe that we do not have long sequences of different arms anymore, just three translations of lines $\dots ABCABC\dots$. It is impossible to distinguish between these translations by chemical potentials and also by two-body interactions because of the symmetry of the tiles and the fact that 2^n "periods" of Robinson's tilings and period 3 of the sequences are relatively prime.

MODEL I

We will describe first our tiles. These are modified crosses and arms with the parity markings as before. We have also two additional levels of letter markings. Each side of each tile is marked by A , B , or C and by T , O , M , E , or K (not all choices are present as explained below). All possible markings of crosses are shown in Fig. 5. Next we allow all rotations of crosses but we do not allow any reflections. This feature makes our model somewhat different from the original Robinson one. We have 15 types of arrows present in arms: all combinations of $A \rightarrow B$, $B \rightarrow C$, $C \rightarrow A$ and $T \rightarrow O$, $O \rightarrow M$, $M \rightarrow E$, $E \rightarrow K$, $K \rightarrow T$. Line segments present in arms are marked $A-A$, $B-B$, or $C-C$ while $B-B$ and $C-C$ segments are also marked $T-O$ or $O-T$ as is shown in Fig. 6. Again, we allow rotations but not reflections.

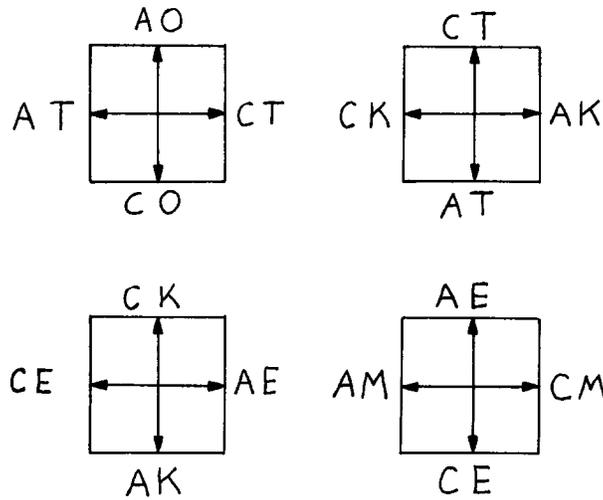


Fig. 5. Additional markings of crosses in Model I.

Two tiles match, if in addition to previous requirements, letter markings match separately on both levels. We also assume that line segments $A-A$ can be put next to any letter marking of the second level (this is not a tiling condition).

Our matching rules can be translated as before into a lattice gas model with nearest-neighbor interactions. In addition, we have to take care of the fact that letter markings of the second level which are opposite nearest neighbors of $A-A$ line segments are not forced to be correlated. We introduce a two-body interaction of range 2 between two arrows facing one another such that the energy is zero if one of them has T marking and the other one is marked by O ; otherwise, the energy is 1.

We will show that the unique ground-state measure of the above model is of the Robinson type. More precisely, looking at crosses and

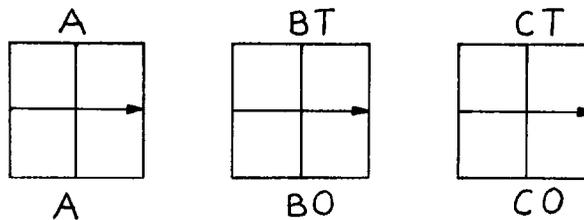


Fig. 6. Additional markings of arms in Model I.

ignoring A arrows on every sublattice $2^n\mathbf{Z}^2$ with n odd and C arrows for n even, we see the self-similar structure of expanding squares shown in Fig. 4.

Proposition 1. The unique ground-state measure of Model I is of the Robinson type as described above.

Proof. First, let us recall that the parity markings force crosses to occupy every site of a $2\mathbf{Z}^2$ sublattice. Letter markings of the second level allow only crosses which are rotations of the cross at the upper left in Fig. 5.

Arms with the upper left and lower right parity marking force relative orientations of these crosses to be such that squares of length 2 are created as is shown in Fig. 3 (arrows of crosses with A markings and arms are not drawn and remaining arrows are represented by line segments). The above arms with $C-C$ line segments force crosses to appear in the center of every square just created. New crosses (four rotations of the cross at the upper right in Fig. 5) form a $4\mathbf{Z}^2$ sublattice.

Once we have crosses on a sublattice, lines of arrows propagate from every cross and meet half-way through line segments of arms. For the crosses on a $2^n\mathbf{Z}^2$ sublattice with n odd, these are alternating $C-C$ and $A-A$ line segments and alternating $B-B$ and $A-A$ segments for n even. These segments force crosses to have relative orientations on every sublattice as is shown in Fig. 3. $C-C(B-B)$ segments force crosses in centers of squares just created. Crosses on successive sublattices have to be chosen in clockwise order from Fig. 5. The upper left corner of every square has to be occupied by an appropriate cross from Fig. 5 and other corners by clockwise rotations. An infinite nonperiodic configuration is inductively created. ■

Proposition 2. Model I satisfies the strict boundary condition for particles and pairs of particles, if we do not distinguish between their different rotations and reflections.

Proof. Let a broken bond be a unit segment on the dual lattice separating two nearest-neighbor particles with a positive interaction energy (a common side of two nearest-neighbor tiles which do not match) or separating two arrows at a distance 2 with the positive energy. Let us then divide an excitation Y of a ground-state configuration X into connected components without broken bonds (two lattice sites are connected if they are nearest neighbors) such that on every component crosses having relative orientations shown in Fig. 3 form a $2\mathbf{Z}^2$ sublattice. This is achieved

by paths on the lattice dual to \mathbf{Z}^2 with lengths not bigger than 16 and joining broken bonds.

When one considers only $2\mathbf{Z}^2$ sublattices and lines of arms joining their sites, the absolute value of the difference between the number of appearances of a given particle in Y and the number of its appearances in X is not bigger than 16 times the number of the above paths.

Now we decompose every connected component into components with crosses that form $4\mathbf{Z}^2$ sublattices. This is achieved by paths on the dual lattice with lengths not bigger than 32 and again joining broken bonds. When one considers $4\mathbf{Z}^2$ sublattices and lines of arms joining their sites, the additional difference of the number of appearances of our particle is not bigger than 16 times the number of new paths plus $16/2$ times the number of old paths.

At the n th step we divide every connected component constructed so far into components with crosses forming $2^n\mathbf{Z}^2$ sublattices and we use paths with lengths not bigger than 2^{n+3} . Again, the additional difference of the number of appearances of our particle is bounded by 16 times the number of new paths plus $1/2$ of the difference at the $(n-1)$ th step.

We repeat this process m times, where m is the smallest number such that $\text{diam}\{\mathbf{a}, Y(\mathbf{a}) \neq X(\mathbf{a})\} < 2^m$. Now, broken bonds and paths can be regarded as vertices and edges of a planar graph. It follows from Euler's formula that the total number of all paths is bounded by 3 times the number of broken bonds. This shows that $|n(Y, X)| \leq 96B(Y)$ for a given particle.

We will now discuss two-particle arrangements. For a pair of particles at a distance D , at least one particle is located on a sublattice $2^k\mathbf{Z}^2$ with $2^k \leq 2D$ or on a line joining sites of this sublattice. To get a bound for a pair of particles we must take into account the effect of shifting one line of arms joining two sites of a sublattice with respect to the other parallel line. This can be done by multiplying the previous bound by $2D$. ■

Let us now perturb our model. We will consider only covariant interactions. That is, a chemical potential of a particle-tile or a two-body interaction between a pair of particles should be the same for any of their rotations and reflections. Combining Theorem 1 and Proposition 2 we get

Theorem 2. The unique nonperiodic ground state of Model I is the unique ground state for any sufficiently small chemical potential and two-body covariant perturbation.

MODEL II

Let us now describe tiles of our second model. We have modified crosses with A and C arrows shown in Fig. 7, arms with $A \rightarrow B, B \rightarrow C,$

and $C \rightarrow A$ arrows and $A-A$, $B-B$, and $C-C$ line segments shown in Fig. 8 and 9. Arms in Fig. 8 are combined with the upper right parity marking. Arms in Fig. 9 can be combined with the lower right or upper left parity marking. Crosses are combined with the parity markings as before. We allow all rotations and reflections of our new tiles as in the original Robinson model.

We do not have anymore the previous second level of letter markings. However, $B-B$ and $C-C$ line segments are not located in the middle of arms anymore so they force crosses to create squares. Arms with central $A-A$ segments from Fig. 8 cannot prevent these squares from going out of phase; a fault shown in Fig. 10 can be created. Such faults are also present in some of the original Robinson tilings. One may prove that any tiling has at most one fault (follow the proof of the analogous theorem in ref. 28, look also at the proof of Proposition 3). Therefore, as in the previous model, the unique ground state of Model II is of the Robinson type.

Proposition 3. Model II satisfies the strict boundary condition for particles and pairs of nearest-neighbor particles, if we do not distinguish between their different rotations and reflections.

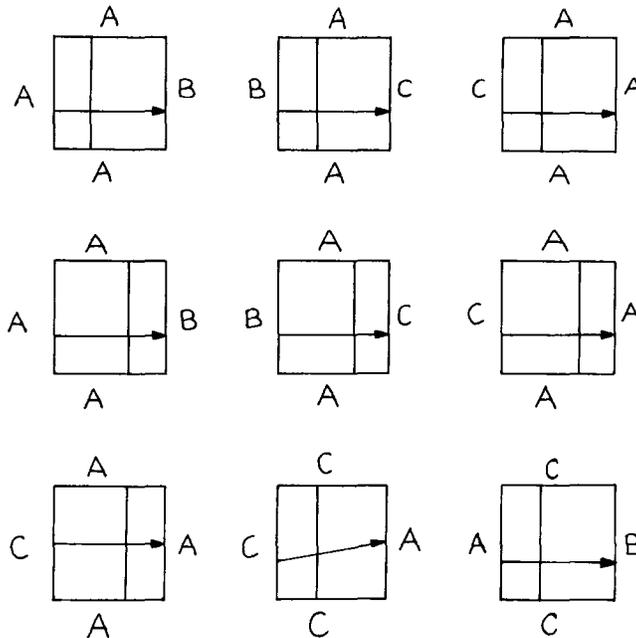


Fig. 9. Additional markings of arms in Model II.

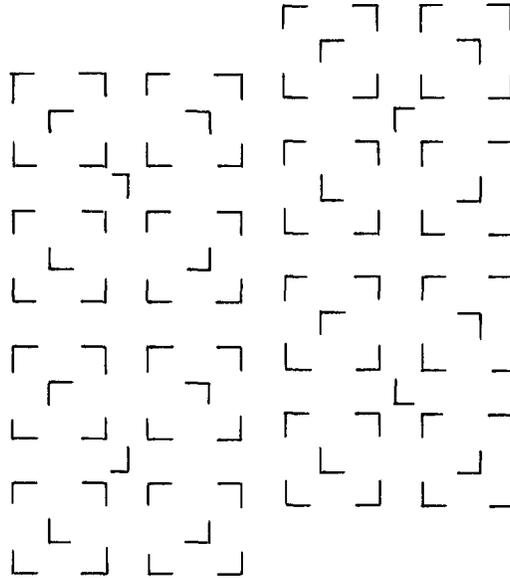


Fig. 10. A ground-state configuration with a fault.

Proof. Observe first that arms with the lower right and the upper left parity marking have asymmetric $A-A$ line segments. Therefore, there are no faults on $2\mathbb{Z}^2$ sublattices. The first step is the same as in the proof of Proposition 2.

In the second step we decompose every connected component without broken bonds into components with crosses that form $4\mathbb{Z}^2$ sublattices but not necessarily having relative orientations as in Fig. 3; faults may be present. This is achieved by paths on the dual lattice with lengths not bigger than 32 and joining broken bonds.

We may have an arbitrarily long faults perpendicular to $A-A$ line segments without any broken bonds except at the endpoints of the fault. The presence of two arms on the fault with line segments parallel to the fault forces a cross (or a broken bond) between them and then a broken bond is forced at a distance not bigger than 16 from the cross. Arms with $B-B$ or $C-C$ line segments perpendicular to the fault also force a cross on the fault or a broken bond at a distance not bigger than 16. Now, when one considers only $4\mathbb{Z}^2$ sublattices and lines of arms joining their sites, then the absolute value of the difference between the number of appearances of a given particle in an excitation Y and the number of its appearances in a ground-state configuration X is not bigger than 16 times the number of paths in the present decomposition plus $16/2$ times the number of paths in

the first step (there is no change of the number of particles and pairs of nearest neighbors along a fault with $A-A$ line segments perpendicular to it).

Now we decompose every connected component into components with crosses that form $8Z^2$ sublattices. These crosses go out of phase along previous faults and they may also create their own faults. We repeat this process m times, as in the proof of Proposition 2, to get $|n(Y, X)| < 96B(Y)$ for particles and pairs of nearest-neighbor particles. ■

Theorem 3. The unique nonperiodic ground state of Model II is the unique ground state for any sufficiently small chemical potential and nearest-neighbor covariant perturbation.

Let us notice that particles of our models have shapes and interactions between them are covariant but not symmetric: an interaction between nearest neighbors RP is the same as between $\mathcal{P}\mathcal{R}$ but not necessarily the same as between PR . There is, however, a construction due to Radin⁽¹⁷⁾ who enlarges a family of possible particles and makes them shapeless, and interactions between them fully symmetric (dependent only on the distance between given particles). He constructs in this way a classical lattice gas model with a finite-range fully symmetric interactions and with a unique nonperiodic ground state. His method requires the original family of particles-tiles to be both rotation and reflection-invariant. Therefore, it can be applied only to our second model. We use it to construct a model with shapeless particles and with a unique nonperiodic ground state which is stable in a space of fully symmetric interactions.

Theorem 4. The unique ground state of the symmetric version of Model II is the unique ground state for any sufficiently small chemical potential and symmetric perturbation of range smaller than three lattice spacings.

5. CONCLUSIONS

We have constructed two finite-range classical lattice gas models with a unique quasiperiodic ground state which is stable against small perturbations of interactions.

The family of particles of our first model is rotation-invariant but not reflection-invariant. Our quasiperiodic ground state is stable in the space of covariant two-body interactions.

In the second model, particles are shapeless and interaction between them depends only on the distance between them. The unique quasiperiodic

ground state is stable in the space of fully symmetric interactions of range smaller than three lattice spacings.

In the construction of our models we used the presence of order in nonperiodic ground-state configurations. Although nonperiodic, they are not, however, completely chaotic. They exhibit long-range positional order in the sense that arrangements of particles at distant regions are correlated. In fact, our ground-state configurations possess highly ordered structures: if a certain fraction of particles is ignored, the rest of a ground state configuration is periodic; the smaller the fraction, the larger the period. It would be interesting to see if other, more disordered, ground states are stable. Modifying the principle of expanding squares responsible for nonperiodicity of Robinson's tilings, many models without periodic ground states were recently constructed.^(30, 23, 31)

Let us point out that the Robinson tilings share with the Penrose tilings many essential features like forced nonperiodicity, self-similarity, uniqueness of a structure for given matching rules. In fact, using the so-called Ammann bars, the Penrose tilings can be also represented by configurations on a square lattice.⁽²⁹⁾ We hope that investigating Robinson's tilings one may provide some information about behavior of quasicrystals associated with the Penrose tilings. Let us also note that models based on the Penrose tilings and with particles on a square lattice were recently investigated in Ref. 32.

Finally, let us say few words about positive temperatures, where to construct stable equilibrium phases of a system of many interacting particles one has to minimize not the energy but the free energy of the system taking into account disruptions due to entropy. For lattice gas models with a finite number of periodic ground-state configurations satisfying the so-called Peierls condition, there exists a complete theory due to Pirogov and Y. Sinai.⁽²⁵⁻²⁷⁾ It tells us that for every Hamiltonian with the above properties, a low-temperature phase diagram is a small deformation of a zero-temperature phase diagram. We would like to generalize the Pirogov-Sinai theory to be able to apply it to lattice models without any periodic ground-state configurations. One of the important problems is to construct a quasiperiodic equilibrium phase for a finite-range interaction—a microscopic model of a quasicrystal. We conjecture the existence of quasiperiodic equilibrium phases in a modified three-dimensional version of our second model.

APPENDIX A

Theorem 5. Let X be a ground-state configuration of a finite-range Hamiltonian H . Then X has the minimal energy density: $e(X) \leq e(Y)$ for any $Y \in \Omega$.

Proof by contraposition. For clarity of presentation we consider two-dimensional models. Assume that there exists a configuration Y with smaller energy density than that for a ground-state configuration X : $e(Y) < e(X)$. Then there is a square, $Sq(N)$, of N^2 lattice sites and $\delta > 0$ such that

$$H_{Sq(N)}^\Phi(Y) < H_{Sq(N)}^\Phi(X) - N^2\delta \quad (9)$$

and

$$8N \sum_{0 \in A} \max_{W \in \Omega_A} |\Phi_A(W)| < N^2 \delta \quad (10)$$

Now we construct an excitation $Z \sim X$ as follows: $Z(Sq(N)) = Y(Sq(N))$ and $Z = X$ outside $Sq(N)$. It follows that $H(Z, X) < 0$ so X is not a ground-state configuration.

APPENDIX B

We equip $\{1, \dots, n\}$ with the discrete topology (every subset of $\{1, \dots, n\}$ is both open and closed). Ω is then compact in the product topology. Let ar be a finite arrangement of particles, $ar \in \{1, \dots, n\}^A$ for some bounded A , then

$$C_A^{ar} = \{X \in \Omega : X(A) = ar\}$$

is a *cylinder set* based in A . Cylinder sets are open and they generate all open sets in Ω (every open set is a union of some cylinders). It follows that a sequence of configurations X_m converges to X ,

$$X_m \rightarrow X \quad \text{as } m \rightarrow \infty$$

if for every bounded $A \subset \mathbf{Z}^d$ there is $m(A)$ such that for every $m > m(A)$, $X_m(A) = X(A)$.

Borel sets are generated from open sets by taking complements and countable unions.

A *probability measure*, μ , on a configuration space, Ω , is a function which assigns a number between 0 and 1 to every Borel set such that $\mu(\Omega) = 1$, $\mu(\emptyset) = 0$, and

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

for Borel sets A_i and A_j if $A_i \cap A_j = \emptyset$ for all $i \neq j$.

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