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L. Sadun

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Tilings, tiling spaces and topology

L. SADUN*

Department of Mathematics, The University of Texas at Austin,
Austin, TX 78712-1082, USA

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To understand on aperiodic tiling (or a quasicrystal modeled on an aperiodic tiling), we construct a space of similar tilings, on which the group of translations acts naturally. This space is then an (abstract) dynamical system. Dynamical properties of the space (such as mixing, or the spectrum of the translation operator) are closely related to bulk properties of individual tilings (such as the diffraction pattern). The topology of the space of tilings, particularly the Cech cohomology, gives information on how original tiling may be deformed. Tiling spaces can be constructed as inverse limits of branched manifolds.

1. Physical and mathematical questions

Tilings are studied by mathematicians, physicists and material scientists. To a physicist, an aperiodic tiling, such as the Penrose tiling, is a model for a form of matter that is neither crystalline nor disordered. The most interesting questions are then about the physical properties of the material being modeled:

- P1. What is the X-ray diffraction pattern of the material? This is equivalent to the Fourier transform of the autocorrelation function of the positions of the atoms. Sharp peaks are the hallmark of ordered materials, such as crystals and quasicrystals.
- P2. What are the possible energy levels of electrons in the material? The locations of the atoms determine a quasiperiodic potential, and the spectrum of the corresponding Schrödinger Hamiltonian has infinitely many gaps. What are the energies of these gaps, and what is the density of states corresponding to each gap?
- P3. Can you really tell the internal structure of the material from diffraction data? What deformations (either local or non-local) of the molecular structure are consistent with the combinatorics of the molecular bonds? Which of these are detectable from diffraction data?

As an example of P3, consider the tilings in figure 1. The fist is a piece of the standard Penrose (rhomb) tiling, with each rhomb cut into two triangles. The second

*Email: sadun@math.utexas.edu

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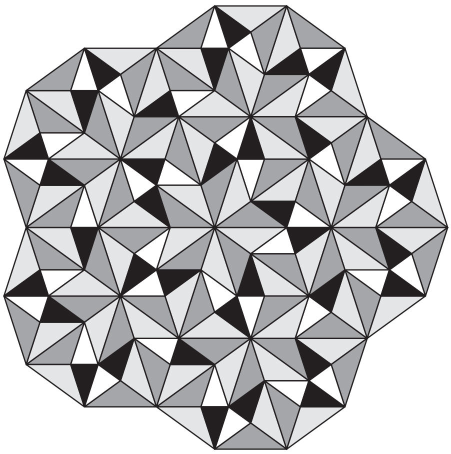

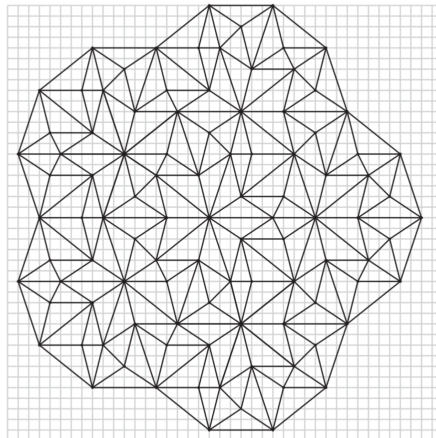

Figure 1. Patches of a Pensore tiling and a deformed Penrose tiling.

tiling is combinatorially identical to the first, but the shapes and sizes of the (40) different species of tiles have been changed. In the first tiling, the diffraction pattern has rotational symmetry, since the underlying tiling has symmetry. In the second tiling, the vertices all lie on lattice points, so the diffraction pattern is periodic with period $(2\pi\mathbb{Z})^2$. However, we will see that the diffraction peaks of the two tilings are related by a simple linear transformation. In other words, the first tiling and a linear transformation of the second have very different geometry but qualitatively similar diffraction patterns.

A mathematician takes a different approach to tilings. To a mathematician, a tiling T is a point in a tiling space X_T. The other points of X_T are tilings with the same properties as T (see the construction below). If T has nice properties (finite local complexity, repetitivity, well-defined patch frequencies), then X_T will have nice properties (compactness, minimality as a dynamical system, unique ergodicity), and we can ask the following mathematical questions:

- M1. What is the topology of X_T? What does the neighbourhood of a point of X_T look like? What are the (Čech) cohomology groups of X_T?
- M2. There is a natural action of the group $\mathbb{R}^d$ of translations on X_T. This makes X_T into a dynamical system, with d commuting flows. What are the ergodic measures on X_T? For each such measure, what is the spectrum of the generator of translations (think: momentum operator) on $L^2(X_T)$? This is called the *dynamical spectrum* of X_T.
- M3. From the action of the translation group on X_T, one can construct a C^* algebra. What is the K-theory of this C^*-algebra?

Remarkably, each math question about X_T answers a physics question about a material modeled on T. M1 answers P3, M2 answers P1, and M3 (in large part) answers P2. In this paper we will review the construction of X_T and explain the connection between M1 and P3, relying on [1–3] for details and proofs. The connection between dynamical and diffraction spectrum is an old story, and we refer the reader to [4–8]. For the relation between K-theory and gap-labeling (i.e., the density of states associated to gaps) see [9–11].

2. The space X_T

A point in our tiling space will be a tiling of $\mathbb{R}^d$ (for simplicity, we'll stick with $d=2$, but the construction is the same in all dimensions). Note that $\mathbb{R}^2$ isn't just a plane — it is a plane with a distinguished point, namely the origin. Translating the tiling to the right is equivalent to moving the origin to the left, *and results in a different tiling*. The orbit of T is the set of all translates of T.

We put a metric on tilings as follows. We say two tilings T_1 and T_2 are ϵ-close if they agree on a ball of radius $1/\epsilon$ around the origin, up to a further translation by ϵ or less. If $T_1=T-x$ and $T_2=T-y$ are translates of T, this implies that a ball of radius $1/\epsilon$ around x in T looks just like a ball of radius $1/\epsilon$ around y. The space X_T is the completion of the orbit of T in this metric. Put another way, it is the set of tilings S with the property that every patch of S can be found somewhere in T. Other names for X_T are the *continuous hull of T* and the *local isomorphism class of T*.

The first topological question is whether X_T is compact. If T has an infinite number of tile types, or if the tiles can fit together in an infinite variety of ways (e.g., a continuous shear along a common edge), then it is easy to construct a sequence of tilings whose behaviour at the origin never settles down, and which therefore does not have a convergent subsequence. However, if T has only a *finite* number of tile types, and these tiles only fit together in only a finite number of ways, then there are only a finite number of patterns (up to translation) that can appear in a fixed region. This is called *finite local complexity*, or FLC. If T is an FLC tiling, then any sequence of translates of T has a subsequence that converges on a ball of radius 1 around the origin. That sequence has a subsequence that converges on a ball of radius 2, etc. Using a Cantor diagonalization argument, we can easily produce a subsequence that converges everywhere. In other words, we have proved that

Theorem 2.1: *The tiling space X_T is compact if and only if T has finite local complexity.*

Every FLC tiling space is MLD-equivalent (see below) to a tiling space where the tiles are polygons that meet full-edge to full-edge. (See [12] for a discussion of the derived Voronoi tilings that accomplish this equivalence.) Without loss of generality, then, we can restrict our attention to tilings of this sort.

A tiling T is called *repetitive* if, for every patch P in T, there is a radius R_P such that every ball of radius R_P contains at least one copy of P. This guarantees that every tiling $S \in X_T$ contains the patch P, hence that the set of patches of S is exactly the same as the set of patches of T, and hence that $X_S=X_T$. In other words, there is nothing special about T itself, and the entire tiling space can be recovered from any tiling in the space. Equivalently, the orbit of every tiling $S \in X_T$ is dense in X_T. In dynamical systems language, the tiling dynamical system X_T is *minimal*.

If T is a periodic tiling (e.g., a checkerboard), invariant under translation by a lattice L, then it is easy to see that X_T is the torus $\mathbb{R}^2/L$. When X_T is totally non-periodic, meaning there are *no* vectors $x \in \mathbb{R}^2$, for which $T=T+x$, then X_T is much more complicated. Also much more interesting.

Locally, X_T looks like a disk crossed with a totally disconnected set. By definition, if $S \in X_T$, then an ϵ-neighbourhood of S is all tilings that agree with a small translate of S on a ball of radius $1/\epsilon$. This gives 2 continuous degrees of freedom (the small translations), and a number of discrete degrees of freedom (choices on how to extend the tiling beyond the $1/\epsilon$ ball). If the original tiling T is repetitive and non-periodic, then there are infinitely many discrete degrees of freedom, and the discrete choices yield a Cantor set.

To somebody who is used to smooth manifolds, a set that is locally a disk crossed with a Cantor set is bizarre. For instance X_T is connected (since it has a path-connected dense set, namely the orbit of T) but not path-connected (since the path component of T is merely the oribit of T). However, such structures are common in dynamical systems.

There are several important notions of equivalence for tiling spaces. Let S and T be different tilings. We say that X_S and X_T are *topologically conjugate* if there is a continuous map $\rho : X_S \to X_T$, with a continuous inverse, that commutes with translations. As dynamical systems, X_T and X_S then have identical properties. In particular, they have the same mixing properties and the same dynamical spectrum.

An even stronger equivalence is *mutual local derivability*, or MLD. We say that X_S and X_T are MLD if they are topologically conjugate and the map ρ is local. That is, if there exists a radius R such that, when $T_1, T_2 \in X_T$ agree exactly on a ball of radius R around a point x, then $\rho(T_1)$ and $\rho(T_2)$ agree exactly on a ball of radius 1 around x. We shall see that the two tiling spaces of figure 1 are topologically conjugate (up to a fixed linear transformation), but not MLD.

3. Inverse limits

One way to get a handle on tiling spaces is via inverse limits. If $K_1, K_2, \ldots,$ are topological spaces and $\sigma_n : K_n \to K_{n-1}$ are continuous maps, then the *inverse limit space* $\lim_{\leftarrow} K_n$ is defined as

$$\lim_{\leftarrow} K_n = \left\{ (x_1, x_2, \ldots) \in \prod_n K_n | \sigma_n(x_n) = x_{n-1} \right\} \tag{1}$$

In other words, a point in $\lim_{\leftarrow} K_n$ is a point $x_1 \in K_1$, together with a point $x_2 \in K_2$ that maps to x_1, together with a point $x_3 \in K_3$ that maps to x_2 and so on. Two sequences are considered close if their first N terms are close, for N large. The space K_n is called the n-th *approximant* to K_n; x_n determines $x_{n-1}, x_{n-2}, \ldots, x_1$.

A simple example of an inverse limit is when each K_n is the circle $\mathbb{R}/\mathbb{Z}$ and each σ_n a multiplication by 2. This is called the *dyadic solenoid*. For each $x_1 \in \mathbb{R}/\mathbb{Z}$, there are two possibilities for x_2, for each of these there are two possibilities for x_3, for each of these there are two possibilities for x_4, and so on. The point within ϵ of $(x_1, x_2, \ldots)$ can differ by a continuous motion (adding δ to x_1, $\delta/2$ to x_2, $\delta/4$ to x_3, and so on), or by keeping the first N terms the same and then making arbitrary choices for x_{N+1}, x_{N+2}, etc. This is exactly the same local structure as a 1-dimensional tiling space, namely one continuous degree of freedom and infinitely many discrete possibilities.

A nice property of inverse limit spaces is that their (Čech) cohomologies are easy to compute. The cohomology of the *inverse* limit is the *direct* limit of the cohomologies of the approximants:

$$H^k(\lim_{\leftarrow} K_n) = \lim_{\rightarrow} H^k(K_n) \tag{2}$$

For the dyadic solenoid, each K_n has $H^0 = H^1 = \mathbb{Z}$. The pullback map σ_n^* is multiplication by 1 on H^0 and multiplication by 2 on H^1, so the dyadic solenoid has $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}$ [1/2]. That is, H^1 is the set of rational numbers whose denominators are powers of 2.

Most computations of the cohomology of tiling spaces use inverse limits and equation (2) [11, 13–15]. However, for certain "cut-and-project" tilings, other techniques have been developed [16].

4. Tiling spaces are inverse limits

Spaces of tilings can always be viewed as inverse limits. Anderson and Putnam [13] first observed this for substitution tilings. Others extended the idea to larger classes of tiling space [11, 14, 17]. The following general construction is due to Gähler [15], and was further generalized in [3].

Theorem 4.1: *If T is a tiling with finite local complexity, then X_T is the inverse limit of a sequence of compact branched surfaces $K_1, K_2, \ldots$ and continuous maps $\sigma_n : K_n \to K_{n-1}$.*

A point in K_n will be a set of instructions for placing a tile containing the origin, a ring of tiles around it (the *first corona*), a second ring around that, and on out to the n-th corona. The map $\sigma_n : K_n \to K_{n-1}$ simply forgets the outermost corona. A sequence $(x_1, x_2, \ldots)$ is then a consistent set of instructions for tiling larger and larger regions of the plane, which is tantamount to the tiling itself.

Consider the tiles t_1, t_2 in (possibly different) tilings in X_T to be equivalent if a patch of the first tiling, containing t_1 and its first n coronas, is identical, up to translation, to a patch of the second tiling containing t_2 and its first n coronas. By finite local complexity, there are only finitely many equivalence classes, which we call n-collared tiles.

For each n-collared tile t_i, we consider how such a tile can be placed at the origin. This merely means picking a point in t_i to sit at the origin. Hence the set of all possible instructions for placing t_i over the origin is just t_i itself! A patch of a tiling in which the origin is on the boundary of two or more tiles is described by points on the boundary of two or more cells, and these points must be identified. The branched manifold K_n is then the union of all the n-collared tiles t_i, modulo this identification. Since we are using n-collared tiles, each of the points being identified carries complete information about the placement of the tiles touching the origin, plus their first $n-1$ coronas. This completes the construction.

Note that Theorem 4.1 does not require the tiling to be generated from a substitution. A substitution tiling space (such as the space of Penrose tilings) may be constructed as an inverse limit as above, or by Anderson and Putnam's original construction [13]. Although the construction of Theorem 4.1 is conceptually simple, it does not lend itself to explicit computations, and the cohomology of substitution tiling spaces is almost always done using the Anderson–Putnam construction.

5. Deformations of tilings

Since was assumed our tiles to be polygons, describing the shape and size of a tile means specifying the displacement vector corresponding to each edge of the polygon.

For instance, the shape of a triangle is determined by three vectors that add up to zero. If, somewhere in a tiling, tiles t_1 and t_2 meet along a common edge, then the edge of t_1 and the matching edge of t_2 must have exactly the same displacement vector. Furthermore, the sum of all displacement vectors for the edges of t_i must add up to zero.

That is, the shapes of the tiles are determined by a function f from the set of tile edges, modulo certain identifications, to $\mathbb{R}^2$. But the set of tiles, modulo these identifications, is precisely the approximant K_0! In other words, a shape is a vector-valued 1-cochain on K_0. The fact that the sum of the vectors around a tile add to zero means that this is a *closed* cochain.

We can consider slightly more general deformations, where the new shape of a tile depends not only on its type but on the types of its nearest neighbours. In that case the shape is a vector-valued closed 1-cochain on K_1. For maximum generality, we may allow closed 1-cochains on any approximant K_n.

Not all deformations are interesting. If the 1-cochain is d of a 0-cochain on K_n, then the resulting tiling space is MLD to the original one [2]. Our deformations, up to MLD equivalence, are then described by closed 1-cochains modulo exact 1-cochains, i.e., by the first cohomology of K_n with values in $\mathbb{R}^2$. Since we can take n arbitrarily large, this means we want an element of the direct limit. But that's precisely the cohomology of X_T itself. In other words,

Theorem 5.1 [CS]: *Deformations to a tiling T are parametrized by closed vector-valued 1-cochains on approximants to X_T. If two shape functions define the same class in $H^1(X_T, R^2)$, then the two tilings are MLD.*

Topological conjugacy is subtler. Some cohomology classes are *asymptotically negligible*, meaning they represent deformations that do not change the topological conjugacy class of a tiling space. For general tiling spaces these classes can be difficult to compute, but for substitution tilings (such as Penrose) they are easy. The substitution defines a map $\psi : X_T \to X_T$. This induces a pullback map $\psi^* H^1(X_T, \mathbb{R}^2) \to H^1(X_T, \mathbb{R}^2)$. The eigenspaces of ψ^* whose eigenvalues have magnitude strictly less than 1 are asymptotically negligible. [CS]

For the Penrose tiling, $H^1(X_T, \mathbb{R}^2)$ is 10-dimensional. The eigenvalues of ψ^* are the golden mean $\tau = (1 + \sqrt{5})/2$, with multiplicity 4, $1 - \tau$ with multiplicity 4, and -1 with multiplicity 2. The 4-dimensional τ eigenspace describe the 4-dimensional space of linear transformations (rotations, stretches and shears) that could be applied uniformly to the tiling. The -1 eigenspace breaks the (statistical) 180-degree rotational symmetry of the Penrose system, and the $1 - \tau$ eigenspace is asymptotically negligible. Since the second tiling in figure 1 preserves the 180-degree rotational symmetry, its shape must be described by a combination of τ and $1 - \tau$ eigenvectors, while the shape of the original Penrose tiling is entirely a τ eigenvector. Thus the second tiling space is topologically conjugate (but not MLD!) to a linear transformation applied to the first.

Topological conjugacies preserve the dynamical spectrum, and therefore preserve locations (but not necessarily intensities) of the peaks of the diffraction pattern. As a result, the locations of the Bragg peaks of the second tiling are related to the Bragg peaks of the first tiling by a linear transformation. (For an investigation on how shape changes affect the intensities of the peaks of the 1-dimensional Fibonacci tiling, see [18]).

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