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Classical Lattice-Gas Models of Quasicrystals

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One of the fundamental problems of quasicrystals is to understand their occurrence in microscopic models of interacting particles. We review here recent attempts to construct stable quasicrystalline phases. In particular, we compare two recently constructed classical lattice-gas models with translation-invariant interactions and without periodic ground-state configurations. The models are based on nonperiodic tilings of the plane by square-like tiles. In the first model, all interactions can be minimized simultaneously. The second model is frustrated; its nonperiodic ground state can arise only by the minimization of the energy of competing interactions. We put forward some hypotheses concerning stabilities of nonperiodic ground states. In particular, we introduce two criteria, the so-called strict boundary conditions, and prove their equivalence to the zero-temperature stability of ground states against small perturbations of potentials of interacting particles. We discuss the relevance of these conditions for the low-temperature stability, i.e., for the existence of thermodynamically stable nonperiodic equilibrium states.

KEY WORDS: Quasicrystals; nonperiodic tilings; classical lattice-gas models; nonperiodic ground states; nonperiodic Gibbs states; stability; frustration.

1. INTRODUCTION

Since the discovery of quasicrystals in 1984 by D. Schechtmann, I. Blech, D. Gratias, and J. W. Cahn,[(1)] one of the fundamental problems in condensed matter physics is to understand their occurrence in microscopic models of interacting particles. The equilibrium behavior of a system of many interacting particles results from the competition between its energy E and entropy S, i.e., the minimization of its free energy $F = E - TS$, where T is the temperature. At zero temperature this reduces to the minimization of

This paper is dedicated to John Cahn on the occasion of his 70th birthday.

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the energy. Configurations of particles which minimize the energy density of the system are called ground-state configurations. Here we will discuss models of interacting particles without periodic ground-state configurations. Then we would like to pass to low but nonzero temperature and controlling entropy (due to thermal fluctuations of particles) to show the existence of thermodynamically stable nonperiodic equilibrium states. This is an outline of a general program. In this paper,we review some partial results, prove the equivalence of two stability conditions, and put forward some open problems.

Let us mention that besides this "local rule" scenario there is another one. In random tiling models, in which there exist both periodic and nonperiodic ground states, quasicrystals are stabilized by the entropy.(2)

In the following we will discuss classical-lattice gas (toy) models of quasicrystals. More precisely, every site of the square lattice can be occupied by one of several different particles. The particles interact through finite-range, translation-invariant potentials. Our models have only nonperiodic ground-state configurations which belong to one local isomorphism class. It means that locally they cannot be distinguished one from another. Every local pattern of particles present in one ground-state configuration appears in any other within a bounded distance. We then say that our models have unique nonperiodic ground states.

In Section 2, we compare two recently constructed lattice-gas models with unique nonperiodic ground states. They are based on nonperiodic tilings of the plane by square-like tiles such that interactions between particles correspond to nearest-neighbor and next-nearest-neighbor matching rules. The first model is nonfrustrated. All interactions can be minimized at the same time. The second one is a model of a frustrated quasicrystal. It is also based on tilings but in addition it has nonzero chemical potentials for certain types of particles. The unique nonperiodic ground state of this model cannont be stabilized by matching rules alone, competing interactions are necessary. In Section 3, we discuss the stability of nonperiodic ground states against small perturbations of interactions between particles. In particular, we present two criteria, the so-called strict boundary conditions (their equivalence is proven in the Appendix), which are equivalent to the zero-temperature stability. In Section 4, we discuss the relevance of these conditions for the low-temperature stability of nonperiodic ground states, review some partial results and put forward some open problems.

2. LATTICE-GAS MODELS OF QUASICRYSTALS

Let us begin with tilings. We have at our disposal a finite number of polygons called prototiles. Using an infinite number of their copies we would like to tile the whole plane without overlaps and empty spaces. In Fig. 1 we present Robinsons tiles.[3,4] These are squares with some notches and dents on their sides representing matching rules which tell us which tiles can be put next one to another. Robinson showed that his tiles allow tilings of the plane and that every such tiling is nonperiodic—there is no nonzero shift of the plane which takes a tiling to itself. Later on (but still before the discovery of quasicrystals) Penrose constructed his famous tiles[5] which now form the canonical example of quasicrystalline structures.

Let us observe that if we tile the plane with the Robinson tiles, then the centers of tiles form the square lattice $\mathbf{Z}^2$. Therefore, every Robinson tiling can be represented by a function assigning particles to lattice sites, i.e., an element of $\{1,..., 56\}^{Z^2}$ (we consider all rotations and reflexions of the Robinson tiles as different tiles).

Now we present two recently constructed classical lattice-gas models of interacting particles which do not have periodic ground-state configurations (see (1) for a formal definition of a ground-state configuration).

The first model is based on the Robinson tilings.[6–16] Every site of the $\mathbf{Z}^2$ lattice can be occupied by one of 56 different particles which correspond to tiles. Two nearest-neighbor or next-nearest-neighbor particles which do not match as tiles contribute a positive energy, say 1, otherwise the energy is zero. We chose chemical potentials of all particles to be zero. Such a model does not have any periodic ground-state configurations. It follows from the fact that every periodic configuration would have a positive density of pairs of particles with the positive energy (corresponding to tiles that do not match). There is a one-to-one correspondence between ground-state configurations without interfaces and perfect tilings of the plane.

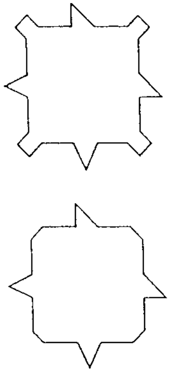

Fig. 1. Robinson's tiles.

Obviously, all interactions can be minimized simultaneously—in a perfect tiling all matching rules are satisfied. Such models are called nonfrustrated.

Every ground-state configuration of the above model has a very precise structure. For every $n \geqslant 1$, a restriction of a ground-state configuration to a certain sublattice $2^n\mathbf{Z}^2$ is a periodic configuration with the period 2^{n+1}. The relative orientations of particles on these sublattices are shown in Fig. 2 (hooks represent different orientations of the particle corresponding to the tile at the lower left of Fig. 1).

Our second model[17] is based on tiles shown in Fig. 3. These are squares with markings represented by vertical, horizontal, and diagonal lines. The first family of matching rules says that these lines cannot be broken on common sides of nearest-neighbor tiles and at common corners of next-nearest-neighbor tiles. These rules correspond to nearest-neighbor and next-nearest-neighbor lattice-gas interactions as it was described before. The second family of matching rules allows only certain patterns of five vertically or horizontally successive tiles. Namely, among five vertically successive tiles there must be at least one arm with the horizontal marking or a cross and there cannot be two such tiles at a distance smaller than four. Analogously, among five horizontally successive tiles there must be at least one arm with the vertical marking or a cross and there cannot be two such tiles at a distance smaller than four. This is translated into a five-body interaction by simply assigning a positive energy to all forbidden patterns; allowed five-particle patterns have zero energy. Finally, we have a rule which forces every arm with diagonal markings to have a cross as one of its nearest neighbors. It gives rise to a three-body interaction.

Ground-state configurations of this model are two-dimensional analogs of one-dimensional, "most homogeneous," nonperiodic ground-state

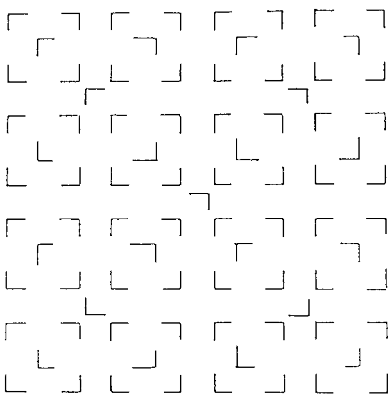

Fig. 2. Robinson's nonperiodic ground-state configuration.

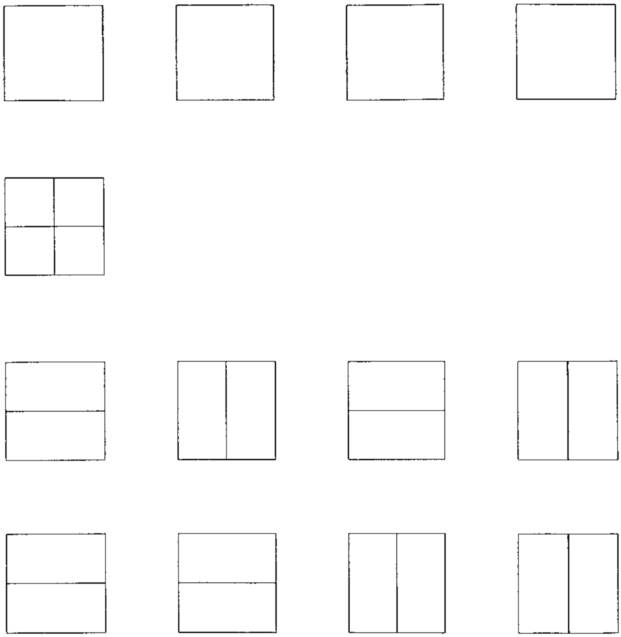

Fig. 3. Tiles of the frustrated model.

configurations of infinite-range, convex, repulsive interactions in models with devil's staircases.[17–20] More precisely, in our ground-state configurations, horizontal and vertical lines cannot be broken, they extend to infinity creating a two-dimensional grid shown in Fig. 4. Distances between lines follow the rule of the most-homogeneous configurations. It means that for every ground-state configuration there exists a sequence of natural numbers d_i such that the distance between two ith neighboring horizontal or vertical lines can be either d_i or $d_i + 1$. Now, to fix ground-state configurations with a given density of particles and therefore a unique sequence d_i, we introduce a negative chemical potential for crosses (particles located at intersection of a horizontal and a vertical line, corresponding to a tile shown in the second row of Fig. 3) and a positive chemical potential for arms (particles located along vertical or horizontal lines, corresponding to tiles shown in the last two rows of Fig. 3).

Our model is frustrated. One cannot minimize simultaneously all interactions—in every ground-state configuration there are crosses with a negative chemical potential, arms with a positive chemical potential and other particles with the zero chemical potential. Frustration is much more severe for any choice of chemical potentials which fix the density of particles

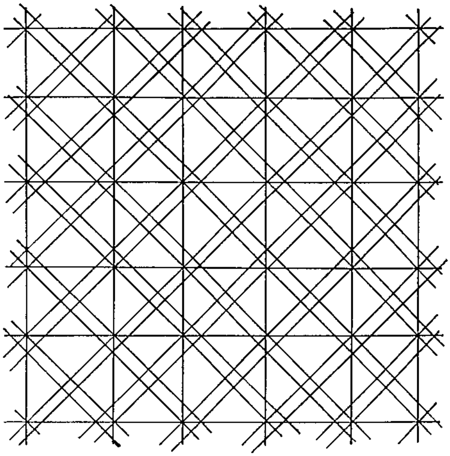

Fig. 4. A nonperiodic ground-state configuration of the frustrated model.

to be an irrational number. In this case one cannot minimize the energy of interacting particles by minimizing their energy in a finite volume and all its translates, no matter how big is the volume (the first model with such a property was constructed in ref. 14). Moreover, there does not exist any nonfrustrated finite-range interaction having our ground states as its unique ground state. Let us note that the canonical example of frustration, the antiferromagnetic Ising model on the triangular lattice, does not have such problems—frustrated two-spin interactions are grouped into triangles and the equivalent Hamiltonian expressed in terms of three-body triangle interactions is not frustrated.

Our model is a microscopic model of a quasicrystal which cannot be stabilized by matching rules alone; competing interactions are necessary.

Tiling models with interactions were already investigated by several authors. Jeong and Steinhardt[21–23] constructed Penrose tilings without imposing matching rules. Two Penrose rhombi allow many tilings (some even periodic). Jeong and Steinhardt proved that perfect Penrose tilings are chosen from this random tiling ensemble by maximizing the density of some cluster of tiles. Such a cluster represents atoms with low-energy interactions. Minimization of the energy corresponds to maximization of the cluster density and therefore forces quasiperiodicity. This is an outstanding result. Let us emphasize, however, that Penrose tilings can be obtained by (maybe nonphysical) matching rules. This is definitely not the case in our model. Situation analogous to ours was investigated by Gähler and Jeong.[24] They studied octagonal Amman-Beenker tilings which do not allow for perfect matching rules. They assigned negative energies to certain clusters of tiles and provided some numerical evidence that ground-state configurations of such interactions form a single isomorhism class consisted of octagonal Amman-Beenker tilings.

In all our models, although all ground-state configurations are nonperiodic, they all belong to one isomorphism class. It means that a uniformly defined frequency of any finite arrangement of particles is the same in all ground-state configurations. We then say that the model has a unique nonperiodic ground state (there is a unique translation-invariant probability measure supported by ground-state configurations which is therefore inevitably the zero-temperature limit of translation-invariant Gibbs states—equilibrium states). More precisely, to find the frequency of a finite arrangement of particles in a given configuration, we first count the number of times it appears in a box of size l which is centered at the origin of the lattice, divide it by l^d, and then take the limit $l \to \infty$. If the convergence is uniform with respect to the position of the boxes, then we say that the configuration has a uniformly defined frequency of this arrangement. We will show that stability of nonperiodic ground-state configurations is intimately connected with the rate of this convergence.

Let us emphasize again that in our models the frequency of any finite particle arrangement is the same in all ground-state configurations. Yet, it is not true that all ground-state configurations are lattice translations of a single nonperiodic ground-state configuration. Such a situation is present in all tiling models of quasicrystals; for example in the Penrose tilings, where instead of ground-state configurations we have perfect tilings.

3. ZERO-TEMPERATURE STABILITY CONDITIONS

We begin by describing more formally classical lattice-gas models, introducing necessary definitions and notation.

A classical lattice-gas model is a system in which every site of a lattice $\mathbf{Z}^d$ can be occupied by one of n different particles. An infinite-lattice configuration is an assignment of particles to lattice sites, i.e., an element of $\Omega = \{1,..., n\}^{\mathbf{Z}^d}$. If $X \in \Omega$ and $A \subset \mathbf{Z}^d$, then we denote by $X(A)$ a restriction of X to A. Particles at lattice sites $\mathbf{a}$ and $\mathbf{b}$ interact through a two-body, translation-invariant potential $\Phi(\mathbf{a}-\mathbf{b})$, which is a function on $\{1,..., n\}^{\{\mathbf{a}, \mathbf{b}\}}$—the space of all possible assignments of particles to lattice sites $\mathbf{a}$ and $\mathbf{b}$, and we assume that Φ is of finite range, that is $\Phi(\mathbf{a}-\mathbf{b})=0$ if $\text{dist}(\mathbf{a}, \mathbf{b})>r$ for some $r>0$. The Hamiltonian in a bounded region Λ can be then written as $H_\Lambda = \sum_{\mathbf{a}, \mathbf{b} \in \Lambda} \Phi(\mathbf{a}-\mathbf{b})$.

Y is a *local excitation* of X, $Y \sim X$, $Y, X \in \Omega$, if there exists a bounded $\Lambda \subset \mathbf{Z}^d$ such that $X = Y$ outside Λ.

For $Y \sim X$, the *relative Hamiltonian* is defined by

$$H(Y, X) = \sum_{\{\mathbf{a}, \mathbf{b}\} \cap \Lambda \neq \varnothing} (\Phi(\mathbf{a}-\mathbf{b})(Y) - \Phi(\mathbf{a}-\mathbf{b})(X))$$

$X \in \Omega$ is a *ground-state configuration* of H if

$$H(Y, X) \geqslant 0 \qquad \text{for any} \quad Y \sim X \tag{1}$$

That is, we cannot lower the energy of a ground-state configuration by changing it locally.

The energy density $e(X)$ of a configuration X is

$$e(X) = \liminf_{\Lambda \to \mathbf{Z}^d} \frac{H_\Lambda(X)}{|\Lambda|}$$

where $|\Lambda|$ is the number of lattice sites in Λ. It can be shown that any ground-state configuration has the minimal energy density. It means that local conditions present in the definition of a ground-state configuration force the global minimization of the energy density.

We will introduce now two conditions which are equivalent to the zero-temperature stability of nonperiodic ground states. They generalize the so-called Peierls condition[26–28] for models without periodic ground-state configurations.

For clarity of presentation we assume that our models are two-dimensional. The important assumption is that our models are nonfrustrated; they have a unique ground state supported by ground-state configurations for which all interactions attain simultaneously their minima (we choose them to be equal to zero). Therefore, if Y is not a ground-state configuration, it contains pairs of particles (at least one) with nonminimal energies (we choose them to be equal to 1), the so-called *broken bonds*. Denote by $B(Y)$ the number of broken bonds in Y. Thus $H(Y, X) = B(Y)$ if $Y \sim X$ and X is a ground-state configuration.

Condition 1. The strict boundary condition for local ground-state configurations.

Let $X(A)$ be a configuration on a bounded region A of the lattice such that all interactions between particles in $X(A)$ attain their minimal values. $X(A)$ is called a local ground-state configuration; it might not be extendable to any infinite lattice ground-state configuration. Let $|A|$ be the area of A (the number of sites in A) and $P(A)$ the length of the boundary of A (the length of the path enclosing A on the dual lattice).

We assume that our models have unique ground states. In particular, every ground-state configuration in their supports has well defined densities of all local particle arrangements. Let *ar* be a local arrangement of particles and ω_{ar} its density in the unique ground state (which is equal to the density in every ground-state configuration in its support).

We say that a model satisfies the strict boundary condition for local ground-state configurations and an arrangement *ar* if the number of appearances of *ar* in each local ground-state configuration $X(A)$, $n_{ar}(X(A))$, satisfies the following inequality:

$$\left|n_{ar}(X(A)) - \omega_{ar}\,|A|\right| < C_{ar} P(A) \tag{2}$$

where $C_{ar} > 0$ is a constant which depends only on the arrangement *ar*.

Let us notice that Condition 1 can be also applied for tilings without any reference to lattice-gas models. We simply replace the requirement that all interactions between particles in $X(A)$ attain their minimal values by the one stating that all matching rules are satisfied in A. For example, it was proven in ref. 29 that Penrose matching rules satisfy the strict boundary condition for all patterns in bounded regions of perfect infinite tilings. It is not known if this is also true for local tilings—regions without any matching rule violations which cannot be extended to any infinite tiling; compare Open Problem 3 in Section 4.

Condition 2. The strict boundary condition for local excitations.[16]

Let X be a ground-state configuration and Y a local excitation of X; $Y \sim X$. Let $n_{ar}(Y, X)$ denote the difference of the number of appearances of an arrangement *ar* of particles in Y and the number of its appearances in X.

We say that a model satisfies the strict boundary condition for local excitations and an arrangement *ar* if there exists a $C_{ar} > 0$ such that for every ground-state configuration X and every $Y \sim X$

$$|n_{ar}(Y, X)| < C_{ar} B(Y) \tag{3}$$

We will prove the equivalence of these two conditions in the Appendix.

The following theorem—the equivalence of Condition 2 and the zero-temperature stability of ground states—was proven in ref. 16.

Theorem 1. A unique ground state of a finite-range, nonfrustrated Hamiltonian is stable against small perturbations of chemical potentials and two-body interactions of range smaller than r if and only if the strict boundary condition for local excitations is satisfied for particles and pairs of particles at a distance smaller than r.

One can easily generalize this theorem for any perturbation (infinite-range, many-body).

Our first lattice-gas model does not satisfy the strict boundary conditions for particles and therefore is unstable with respect to arbitrarily small perturbations of chemical potentials.[8] In ref. 16, there was constructed a classical lattice-gas model (based on modified Robinson's tilings) with translation-invariant, finite-range interactions, whose unique nonperiodic ground state satisfies the strict boundary conditions for particles and pairs of particles and therefore is stable against small perturbations of chemical potentials and two-body interactions. It means that there is an open set in a space of two-body interactions without periodic ground-state configurations. This constitutes the first generic counterexample to the so-called crystal problem.

The frustrated model discussed is Section 2 satisfies the strict boundary conditions for all local arrangements of particles. However, it is not stable; frustration being the reason of the instability. The ground-state density of particles changes continuously as a function of their chemical potentials (compare ref. 8). It is a novel feature; such a situation cannot happen in systems with periodic ground-state configurations. Continuous changes of stoichiometry have been observed in real quasicrystals.

4. LOW-TEMPERATURE EQUILIBRIUM STATES; PARTIAL RESULTS AND HYPOTHESES

Now we would like to show that nonperiodic order present in our lattice models survive at nonzero temperatures. Low-temperature behavior of models of quasicrystals based on the Penrose tilings was investigated in refs. 30–34. In positive temperatures, equilibrium behavior of a system of many interacting particles can be described by a grand-canonical ensemble. An infinite-volume limit of this ensemble is called an equilibrium state or a (translation-invariant) Gibbs state. Our goal is then to construct a nonperiodic Gibbs state for models without periodic ground states.

Open Problem 1. To construct a translation-invariant, finite-range interaction without periodic ground-state configurations and with a nonperiodic Gibbs state.

In principle we have to control the entropy such that it does not destroy the non-periodicity of the unique ground state, forced by the minimization of the energy. More precisely, let $X \in \Omega$ be a nonperiodic ground-state configuration of one of our models, Λ a finite subset of $\mathbf{Z}^d$ and ρ_Λ^X a finite volume Gibbs state:

$$\rho_\Lambda^X(Y) = (1/Z_\Lambda^X) \exp(-H(Y', X)/T) \tag{4}$$

for every $Y \in \Omega_\Lambda$, where $Y' \sim X$, $Y' = Y$ on Λ and $Y' = X$ on $\mathbf{Z}^d - \Lambda$ and

$$Z_\Lambda^X = \sum_{Y \in \Omega_\Lambda} \exp(-H(Y', X)/T) \tag{5}$$

We would like to prove that,

$$\rho_\Lambda^X(\{Y: Y(0) \neq X(0)\} < \varepsilon(T) \tag{6}$$

where $\varepsilon(T) \to 0$ uniformly in Λ as $T \to 0$. Then we would pass to the thermodynamic limit, $\Lambda \to Z^d$, constructing in this way a nonperiodic Gibbs state, ρ^x, which would satisfy (6) for every $Y \in \Omega$ and therefore be a small perturbation of the ground-state configuration X. At the same time we would prove the existence of infinitely many extremal Gibbs states, translates of ρ^X. Our models were so far two-dimensional. There is a hypothesis stating that in two-dimensional classical lattice-gas models with finite-range interactions, there is always a finite number of extremal Gibbs states.[28, 38] However, it is supported mainly by lack of a counterexample and negative results in special families of models, like instability of an interface in the two-dimensional ferromagnetic Ising model[39, 40] and finiteness of the number of a certain type of Gibbs states in general ferromagnetic models.[41] One of the problems dealing with two-dimensional nonperiodic models is their nonregularity. We say that a lattice model is *regular* if the excitation energy increases with the support of the excitation. Let X be a ground-state configuration. We say that X is regular when, if card($a \in Z^d : Y_n(a) \neq X(a)$; $Y_n \sim X) \to \infty$, then $H(Y_n, X) \to \infty$.

Open Problem 2. To construct a two-dimensional regular lattice-gas model with a unique nonperiodic ground state.

Now we pass to three-dimensional versions of our models. In addition to interactions in the plane, we introduce a ferromagnetic interaction along the axis perpendicular to the plane such that two vertically-neighboring particles interact with the zero energy if they are of the same type; otherwise the energy is positive, say 1. Such three-dimensional stacked models are obviously regular. Their ground-state configurations are layered two-dimensional ground-state configurations which are translation-invariant in the third direction. Now the main goal is to construct a model which would satisfy the strict boundary condition for all finite arrangements of particles.

Open Problem 3. To construct a nonfrustrated lattice model with a unique nonperiodic ground state which satisfies the strict boundary condition for all finite arrangements of particles.

One can formulate an analogous problem for tilings. We would like to find a tiling satisfying Condition 1, where instead of interactions between particles we have matching rules.

The strict boundary conditions imply, as we discussed in Chapter 3, the stability of ground states with respect to small perturbations of interactions. Now let us pass to small but nonzero temperatures. Consider a region of a two-dimensional lattice with a local ground-state configuration on it and bounded by a line of broken bonds. Condition 1 tells us that a cluster expansion of the free energy of low-energy excitations is the same, up to boundary terms, for all local ground-state configurations. In this way excitations can be thought of as small perturbations of interactions.(25) This means that the strict boundary condition should in principle guarantee the low-temperature stability of ground states and therefore the existence of nonperiodic Gibbs states. One may consider this as an attempt to generalize the Pirogov–Sinai theory(26–28) to systems without periodic ground states and therefore not satisfying the standard Peierls condition. The work in this direction is in progress.

Low-temperature behavior of a lattice-gas model based on modified Robinson's tilings was investigated in ref. 12. It was proven that there exists a sequence of temperatures, T_n, such that if $T < T_n$, then there exists a Gibbs state with the period at least 2^n. Although nonperiodic Gibbs states were not ruled out in this model, we conjecture that the sequence converges to the zero temperature, giving rise to a period doubling of Gibbs states.

Finally, let us say few words about systems with interactions which are not of finite range.

Nonperiodic long-range order can occur at nonzero temperatures for slowly decaying (summable) interactions in arbitrary dimensions.(35, 36) Recently a nonperiodic Gibbs state was constructed for twicely-stacked three-dimensional models with exponentially decaying interaction along one axis and the nearest-neighbor ferromagnetic interaction along two other axes.(37)

APPENDIX

For clarity of presentation, we assume that interactions are nearest-neighbor only and *ar* is a pair of particles at a distance *D*.

Condition 1 Implies Condition 2

Proof by Contraposition. Fix $C>0$ and assume that Condition 2 is not satisfied. Hence, for every N there exists a local excitation, $Y_N \sim X$, of a ground-state configuration X, such that $|n_{ar}(Y_N, X)| > NCB(Y_N)$. Assume that $n_{ar}(Y_N, X) > NCB(Y_N)$; the case $n_{ar}(Y_N, X) < -NCB(Y_N)$ can be treated in an analogous way. Let $B = \{\mathbf{a} : Y_N(\mathbf{a}) \neq X(\mathbf{a})\}$. Now we will use the fact that X has a uniformly defined frequency of any finite arrangement of particles. It means, in particular, that there exist $\mathbf{n}_{ij} = (n_i, n_j) \in \mathbf{Z}^2$, $1 \leqslant i, j \leqslant k$ such that $X(\tau_{\mathbf{n}_{ij}} B) = X(B)$ and $\tau_{\mathbf{n}_{ij}} B$ are pairwise disjoint for different i and j. Therefore, we can excite X on every $\tau_{\mathbf{n}_{ij}} B$ in the same way as on B, creating a local excitation $Y'_N \sim X$. Let A be a set of lattice sites inside a square of size L containing $\bigcup_{1 \leqslant i, j \leqslant k} (\tau_{\mathbf{n}_{ij}} B)$ and such that all interactions of particles in $Y'_N(A)$ attain their minimal values. We choose k such that $4L < k^2 B(Y_N) = B(Y'_N)$. It follows that $P(A) < 6B(Y'_N) + 4L < 7B(Y'_N)$, where 6 comes from the worst case of an isolated broken bond and the boundary of A enclosing two nearest-neighbor lattice sites. Now, either

$$\left| n_{ar}(X(A)) - \omega_{ar} \, |A| \right| > C(7B(Y'_N)) > CP(A) \tag{7}$$

so Condition 1 is not satisfied or

$$\left| n_{ar}(X(A)) - \omega_{ar} \, |A| \right| \leqslant C(7B(Y'_N)) \tag{8}$$

In the second case we have

$$\begin{aligned} n_{ar}(Y'_N(A)) - \omega_{ar} \, |A| &> NCB(Y'_N) - C(7B(Y'_N)) - 8B(Y'_N) \\ &> C(7B(Y'_N)) > CP(A) \end{aligned} \tag{9}$$

if

$$N > 14 + 8/C \tag{10}$$

Condition 2 Implies Condition 1

Proof by Contraposition. Fix $C>0$ and assume first that Condition 1 is not satisfied for ground-state configurations. Hence, for every N there exists a bounded $A \subset \mathbf{Z}^2$ and a ground-state configuration X such that $\left| n_{ar}(X(A)) - \omega_{ar} \, |A| \right| > NCP(A)$. Assume now that $n_{ar}(X(A)) - \omega_{ar} \, |A| > NCP(A)$; the other case can be treated in an analogous way. Now we will show that there must exist an $\mathbf{a} \in \mathbf{Z}^2$ such that $n_{ar}(X(\tau_{\mathbf{a}} A)) - \omega_{ar} \, |A| \leqslant 0$, where $\tau_{\mathbf{a}} A$ is a translation of A by $\mathbf{a}$. Let m be a number of translates of A which contain the support of the arrangement ar, $m \leqslant |A|$ and if N is sufficiently big, then it follows from our assumption that $m \neq 0$. If $n_{ar}(X(\tau_{\mathbf{a}} A)) > \omega_{ar} |A|$ for every $\mathbf{a} \in \mathbf{Z}^2$, then $\omega_{ar} = \lim_{A \to \mathbf{Z}^2} 1/(m|A|) \sum_{\mathbf{a} \in A} n_{ar}(X(\tau_{\mathbf{a}} A)) > \omega_{ar}$, which is a contradiction.

Now put $X(\tau_{\mathbf{a}} A)$ on A and X outside A and construct $Y \sim X$. Obviously, $B(Y) \leqslant P(A)$ so

$$n_{ar}(Y, X) < -(NC - D)\, B(Y) < -CB(Y) \tag{11}$$

if

$$N > 1 + D/C \tag{12}$$

Now assume that there exists a bounded $A \subset \mathbf{Z}^2$ and $Y(A)$ which is not necessarily a piece of a ground-state configuration but a local ground-state configuration (all interactions between particles in $Y(A)$ attain their minimum values) such that

$$n_{ar}(Y(A)) - \omega_{ar} |A| > NCP(A) \tag{13}$$

Put now a ground-state configuration X outside A and construct $Y \sim X$. Then either

$$n_{ar}(X(A)) - \omega_{ar} |A| > NCP(A)/2 \tag{14}$$

and then we follow the first part of the proof to conclude that Condition 2 is not satisfied, or else

$$n_{ar}(Y, X) > (NC/2 - D)\, B(Y) > CB(Y) \tag{15}$$

if

$$N > 2 + 2D/C \tag{16}$$

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