Midterm Models of Applied Mathematics 16 December 2014

All problems are worth 10 points.

1. Construct a system of differential equations for the probability mass function of the number of protein molecules for the self-repressing gene with two binding sites of the promoter.

2. Find the stationary state for the following birth and death process:

$$\emptyset \to^{k_A} A, \;\; \emptyset \to^{k_B} B, \;\; A + B \to^{\gamma} \emptyset,$$

where the number of molecules of type A and B is at most one.

3. Find all ground-state configurations for the following one-dimensional Ising model of interacting spins, $s_i = \pm 1, 0$:

$$H(s_1, s_2) = -(s_1 - s_2)^2.$$

4. Find the expected value of the magnetization in the Gibbs measure for the following spin model: $s_i = \pm 1$,

$$H = -\sum_{i=1}^{N} (s_i^2 - s_i).$$

What is the magnetization in the ground state that is at zero temperature?

BONUS (10 points)

Prove that every nearest-neighbor and next-nearest-neighbor interaction on the square lattice $\mathbf{Z}^2$ which is symmetric (it depends only on types of particles and distances between them) has at least one periodic ground-state configuration.

All the best in 2015