Midterm Methods of Applied Mathematics 10 December 2013

All problems are worth 10 points.

1. Construct the system of differential equations for the probability mass function (master equations) for the number of particles of A and B types for the following reactions:

$$\emptyset \to^{k_A} A, \quad \emptyset \to^{k_B} B, \quad A \to^{k_A} B, \quad B \to^{\gamma} \emptyset.$$

2. Find the stationary state for the self-repressed gene with the maximal number of protein molecules (present in the system) equal to 2. Assume that proteins are produced directly from DNA.

3. Find all periodic ground-state configurations for the following one-dimensional Ising model of interacting spins, $s_i = \pm 1$:

$$H = -\sum_{i \in Z} (s_i s_{i+1} - s_i s_{i+2} - 4 s_i s_{i+3}).$$

4. Find the expected value of the magnetization in the Gibbs measure for the following spin model: $s_i = \pm 1, 0$,

$$H = -\sum_{i=1}^{N} (s_i^2 - s_i).$$

What is the magnetization in the ground state that is at zero temperature?

BONUS

Prove that every nearest-neighbor and next-nearest-neighbor interaction on the square lattice $\mathbf{Z}^2$ which is symmetric (it depends only on types of particles and distances between them) has at least one periodic ground-state configuration.