


Evolution of populations with strategy-dependent time delaysJacek Miękiśz^{*} and Marek Bodnar[†]*Institute of Applied Mathematics and Mechanics, University of Warsaw, Warsaw, Poland* (Received 16 June 2020; revised 18 December 2020; accepted 4 January 2021; published 22 January 2021)

We study the effects of strategy-dependent time delays on the equilibria of evolving populations. It is well known that time delays may cause oscillations in dynamical systems. Here we report a novel behavior. We show that microscopic models of evolutionary games with strategy-dependent time delays lead to a new type of replicator dynamics. It describes the time evolution of fractions of the population playing given strategies and the size of the population. Unlike in all previous models, the stationary states of such dynamics depend continuously on time delays. We show that in games with an interior stationary state (a globally asymptotically stable equilibrium in the standard replicator dynamics), at certain time delays it may disappear or there may appear another interior stationary state. In the Prisoner's Dilemma game, for time delays of cooperation smaller than time delays of defection, there appears an unstable interior equilibrium, and therefore for some initial conditions the population converges to the homogeneous state with just cooperators.

DOI: [10.1103/PhysRevE.103.012414](https://doi.org/10.1103/PhysRevE.103.012414)**I. INTRODUCTION**

Many social and biological processes can be modeled as systems of interacting individuals within the framework of evolutionary game theory [1–9]. The evolution of very large (infinite) populations can then be given by differential replicator equations that describe time changes of fractions of populations playing different strategies [3,5,10,11]. It is usually assumed (as in the replicator dynamics) that interactions between individuals take place instantaneously and their effects are immediate. In reality, all social and biological processes take a certain amount of time. The results of biological interactions between individuals may appear in the future. In social models, individuals or players may act, i.e., choose appropriate strategies, on the basis of information concerning events in the past. It is natural, therefore, to introduce time delays into evolutionary game models.

It is well known that time delays may cause oscillations in dynamical systems [12–15]. One usually expects that interior equilibria of evolving populations, describing coexisting strategies or behaviors, are asymptotically stable for small time delays. Above a critical time delay, where Hopf bifurcation appears, they become unstable, and evolutionary dynamics exhibits oscillations and cycles. Here we report a novel behavior, namely the continuous dependence of equilibria on time delays.

The effects of time delays on replicator dynamics were discussed in [16–27] for games with an interior stable equilibrium (an evolutionarily stable strategy [1,2]). In [16], the authors discussed the model in which individuals at time t imitate a strategy with a higher average payoff at time $t - \tau$ for some time delay τ . They showed that the interior stationary

state of the resulting time-delayed differential equation is locally asymptotically stable for small time delays, while for big ones it becomes unstable and there appear oscillations. In [17] we constructed a different type of model, in which individuals are born τ units of time after their parents played. Such a model leads to a system of equations for the frequency of the first strategy and the size of the population. We showed the absence of oscillations—the original stationary point is globally asymptotically stable for any time delay. In both models, the position of the equilibrium is not affected by time delays.

Here we modify the second of the above models by allowing time delays to depend on strategies played by individuals, and we observe a new behavior of a dependence of equilibria on delays. Recently there were studied models with strategy-dependent time delays. In particular, Moreira *et al.* [21] discussed a multiplayer Stag-hunt game, Ben Khalifa *et al.* [27] investigated asymmetric games in interacting communities, and Wesson and Rand [24] studied Hopf bifurcations in two-strategy delayed replicator dynamics. The authors generalized the model presented in [16], and they studied the asymptotic stability of equilibria and the presence of bifurcations. Some very specific examples of three-player games with strategy-dependent time delays were studied in [28], in which a shift of interior equilibria was observed.

Here we present a systematic study of the effects of strategy-dependent time delays on the long-run behavior of two-player games with two strategies in models that are generalizations of the one in [17]. We consider Stag-hunt-type games with two basins of attraction of pure strategies, Snowdrift-type games with a stable interior equilibrium (a coexistence of two strategies), and the Prisoner's Dilemma game. We report a novel behavior, i.e., we show that stationary states depend continuously on time delays. Moreover, at certain time delays an interior stationary state may disappear or there may appear another interior stationary state.

^{*}miekisz@mimuw.edu.pl[†]mbodnar@mimuw.edu.pl

Below we present a general theory and particular examples. In the Appendix, we provide proofs and additional theorems, in particular general conditions for the existence and uniqueness of interior states in Snowdrift-type games (Theorems A.3 and A.6).

II. METHODS

A. Replicator dynamics

We assume that our populations are haploid, that is, the offspring have identical phenotypic strategies to those of their parents. We consider symmetric two-player games with two strategies, C and D , given by the following payoff matrix:

$$U = \begin{matrix} & \begin{matrix} C & D \end{matrix} \\ \begin{matrix} C \\ D \end{matrix} & \begin{matrix} a & b \\ c & d \end{matrix} \end{matrix},$$

where the ij entry, $i, j = C, D$, is the payoff of the first (row) player when it plays the strategy i and the second (column) player plays the strategy j . We assume that both players are the same and hence payoffs of the column player are given by the matrix transposed to U ; such games are called symmetric.

Below we will consider all three main types of two-player games with two strategies: Stag-hunt, Snowdrift, and Prisoner's Dilemma. They serve as simple models of social dilemmas. Strategies C and D may be interpreted as cooperation and defection.

Let us assume that during a time interval of length ε , only an ε -fraction of the population takes part in pairwise competitions, that is, plays games. Let $p_i(t)$, $i = C, D$, be the number of individuals playing at time t the strategy C and D , respectively; $p(t) = p_C(t) + p_D(t)$ is the total number

of players; and $x(t) = \frac{p_C(t)}{p(t)}$ is the fraction of the population playing C . Let

$$\begin{aligned} U_C(t) &= ax(t) + b[1 - x(t)] \\ U_D(t) &= cx(t) + d[1 - x(t)] \end{aligned} \tag{1}$$

be average payoffs of individuals playing C and D , respectively.

Now we would like to take into account that individuals are born some units of time after their parents played. We assume that time delays depend on strategies and are equal to $t - \tau_C$ or $t - \tau_D$.

We propose the following equations:

$$p_i(t + \varepsilon) = (1 - \varepsilon)p_i(t) + \varepsilon p_i(t - \tau_i)U_i(t - \tau_i); \quad i = C, D, \tag{2}$$

$$\begin{aligned} p(t + \varepsilon) &= (1 - \varepsilon)p(t) + \varepsilon(p_C(t - \tau_C)U_C(t - \tau_C) \\ &+ p_D(t - \tau_D)U_D(t - \tau_D)). \end{aligned} \tag{3}$$

In the above, we assume that populations are large in order to justify a continuous description of population sizes, but they are not infinite as in classical replicator equations. The parameter ε represents the time length of game interactions and therefore it multiplies the second terms in the above equations. We assume the replacement of parents or equivalently a death rate 1, and hence $p_i(t)$ and $p(t)$ are multiplied by $1 - \varepsilon$.

We divide (2) by (3) for $i = A$, obtain the equation for $x(t + \varepsilon) \equiv x_C(t + \varepsilon)$, subtract $x(t)$, divide the difference by ε , take the limit $\varepsilon \rightarrow 0$, and get an equation for the frequency of the first strategy,

$$\frac{dx}{dt} = \frac{p_C(t - \tau_C)U_C(t - \tau_C)[1 - x(t)] - p_D(t - \tau_D)U_D(t - \tau_D)x(t)}{p(t)}, \tag{4}$$

which can also be written as

$$\frac{dx}{dt} = \frac{x(t - \tau_C)p(t - \tau_C)U_C(t - \tau_C)[1 - x(t)] - [1 - x(t - \tau_D)]p(t - \tau_D)U_D(t - \tau_D)x(t)}{p(t)}. \tag{5}$$

Let us notice that unlike in the standard replicator dynamics, the above equation for the frequency of the first strategy is not closed; there appears in it a variable describing the size of the population at various times. One needs equations for populations sizes. From (2) and (3) we get

$$\frac{dp_i(t)}{dt} = -p_i(t) + p_i(t - \tau_i)U_i(t - \tau_i); \quad i = C, D, \tag{6}$$

$$\begin{aligned} \frac{dp(t)}{dt} &= -p(t) + (p_C(t - \tau_C)U_C(t - \tau_C) \\ &+ p_D(t - \tau_D)U_D(t - \tau_D)). \end{aligned} \tag{7}$$

To trace the evolution of the population, we have to solve the system of equations (5) and (7) together with initial conditions on the interval $[-\tau_M, 0]$, where $\tau_M = \max\{\tau_C, \tau_D\}$. We

assume that

$$x(t) = \varphi_x(t), \quad p(t) = \varphi_p(t) \quad \text{for } t \in [-\tau_M, 0]. \tag{8}$$

We have the following proposition concerning the existence of non-negative solutions.

Proposition 1. If the initial functions φ_x and φ_p are continuous on $[-\tau_M, 0)$ and non-negative, then there exists a unique non-negative solution of the system [(5) and (7)] with initial conditions (8) well defined on the interval $[0, +\infty)$.

Proof. The local existence of the solution follows immediately from a standard theory of delay differential equations, [14]. The non-negativity follows from [29]. To prove the global existence, it is enough to use the step method and to observe that on the interval $[0, \tau_M]$ the system [(5) and (7)] becomes a system of nonautonomous ordinary differential equations. The equation for p becomes linear; it can be

solved and then the equation for x becomes linear with respect to $x(t)$.

B. Stationary states

We derive here an equation for an interior stationary state (the stationary frequency of the first strategy). Let us assume that there exists a stationary frequency \bar{x} such that $x(t) = \bar{x}$ for all $t \geq 0$ and for some suitably chosen function $p(t)$. Then average payoffs of each strategy are constant and are equal to

$$\bar{U}_C = a\bar{x} + b(1 - \bar{x}) \quad \text{and} \quad \bar{U}_D = c\bar{x} + d(1 - \bar{x}). \quad (9)$$

Thus, Eq. (7) becomes a linear delay differential equation,

$$\begin{aligned} \frac{dp}{dt} = & \bar{x}(a\bar{x} + b(1 - \bar{x}))p(t - \tau_C) \\ & + (1 - \bar{x})(c\bar{x} + d(1 - \bar{x}))p(t - \tau_D) - p(t). \end{aligned} \quad (10)$$

Note that solutions of (10) with non-negative initial conditions are non-negative. This implies that the leading eigenvalue of this equation is real. The eigenvalues λ of (10) satisfy

$$\lambda + 1 = \bar{x}\bar{U}_C e^{-\lambda\tau_C} + (1 - \bar{x})\bar{U}_D e^{-\lambda\tau_D}. \quad (11)$$

Assume now that λ is a solution of (11) (of course λ depends on \bar{x}) and $p(t) = p_0 \exp(\lambda t)$ for some p_0 .

We plug such p into (4) and we get two stationary solutions of (4) [i.e., such that the right-hand side of (4) is equal to 0], $\bar{x} = 0, 1$, and possibly interior ones—solutions to

$$\bar{U}_C e^{-\lambda\tau_C} = \bar{U}_D e^{-\lambda\tau_D}. \quad (12)$$

If $\tau_C = \tau_D$ and $d < b < a < c$, then (12) gives us a mixed Nash equilibrium (an evolutionarily stable strategy [1,2]) of the game,

$$x^* = \frac{b - d}{b - d + c - a},$$

which represents the equilibrium fraction of an infinite population playing C [3,5]. In the nondelayed replicator dynamics, x^* is globally asymptotically stable and there are two unstable stationary states: $x = 0$ and 1.

If $\tau_C \neq \tau_D$, then $\lambda = \ln(\bar{U}_C/\bar{U}_D)/(\tau_C - \tau_D)$. We plug it into (11) and we conclude that \bar{x} satisfies an equation $F(\bar{x}) = 0$, where

$$\begin{aligned} F(x) = & \frac{1}{\tau_C - \tau_D} \ln \left(\frac{ax + b(1 - x)}{cx + d(1 - x)} \right) \\ & + 1 - \frac{(cx + d(1 - x))^{\tau_C/(\tau_C - \tau_D)}}{(ax + b(1 - x))^{\tau_D/(\tau_C - \tau_D)}}. \end{aligned} \quad (13)$$

From the above we get the following proposition that explains in what sense \bar{x} is a stationary state of the replicator dynamics [(5) and (7)].

Proposition 2. Assume that $\tau_C \neq \tau_D$. Let \bar{x} be a solution to $F(x) = 0$, where F is defined by (13), and let

$$\bar{\lambda} = \frac{1}{\tau_C - \tau_D} \ln \left(\frac{a\bar{x} + b(1 - \bar{x})}{c\bar{x} + d(1 - \bar{x})} \right).$$

Then the functions

$$x(t) = \bar{x}, \quad p(t) = p_0 e^{\bar{\lambda} t}, \quad t \geq 0$$

are solutions of the system [(5) and (7)] with the initial conditions

$$\varphi_x(t) = \bar{x}, \quad \varphi_p(t) = p_0 e^{\bar{\lambda} t} \quad \text{for} \quad t \in [-\tau_M, 0].$$

In this paper, we consider only examples of games with a positive λ , that is, our populations grow exponentially with time, hence the use of differential equations to describe time evolution is appropriate. For a negative λ , the population becomes extinct.

Further properties of the function F related to the existence of zeros of this function in the interval (0,1) are analyzed in Appendix A 1. In particular, we provide in Theorems A.3 and A.6 general conditions for the existence and uniqueness of the interior stationary state in Snowdrift-type games.

III. RESULTS

A. Stag-hunt games

Here we consider games with a unique unstable interior equilibrium. We begin with a general Stag-hunt game given by the following payoff matrix:

$$U_1 = \begin{bmatrix} a & 0 \\ \beta a & \beta a \end{bmatrix},$$

where $a > 0$ and $\beta \in (0, 1)$. In this case, the function F takes the form

$$F(x) = 1 + \frac{\alpha}{\tau_C} \ln \frac{x}{\beta} - a\beta^\alpha x^{1-\alpha}, \quad \alpha = \frac{\tau_C}{\tau_C - \tau_D}. \quad (14)$$

Note that if $\bar{x} \in (0, 1)$ is an interior stationary frequency, then the leading eigenvalue is given by the following formula (see Proposition 2):

$$\bar{\lambda} = \frac{1}{\tau_C - \tau_D} \ln \frac{\bar{x}}{\beta}. \quad (15)$$

Thus $\bar{\lambda} > 0$ if and only if $\tau_C > \tau_D$ and $\bar{x} \in (\beta, 1)$ or $\tau_C < \tau_D$ and $\bar{x} \in (0, \beta)$.

For $\tau_C > \tau_D$ it is easy to see that F is increasing from $-\infty$ to $F(1) = 1 - a\beta^\alpha$. Thus, the interior stationary frequency $\bar{x} \in (0, 1)$ exists if and only if $F(1) > 0$, which is equivalent to the inequality $\beta < a^{-1/\alpha}$. The leading eigenvalue is positive if $\bar{x} > \beta$, which is equivalent to the inequality $F(\beta) < 0$. This gives a condition $a\beta > 1$. Finally, we obtain a necessary condition for the existence of an interior stationary frequency with a positive leading eigenvalue:

$$\frac{1}{a} < \beta < \frac{1}{a^{1/\alpha}} \implies a > 1. \quad (16)$$

For $\tau_C < \tau_D$ it is easy to see that F is decreasing from ∞ to $F(1) = 1 - a\beta^{-|\alpha|}$. Thus, the interior stationary frequency $\bar{x} \in (0, 1)$ exists if and only if $F(1) < 0$, which is equivalent to the inequality $\beta < a^{1/|\alpha|}$. Similarly, the leading eigenvalue is positive if $\bar{x} < \beta$, which is equivalent to the inequality $F(\beta) < 0$. This gives a condition $a\beta > 1$. Again we see that a necessary condition for the existence of an interior stationary frequency with a positive leading eigenvalue is $a > 1$.

For general Stag-hunt games, we can derive an explicit formula for the interior stationary state \bar{x} using the Lambert W function W_p . Namely, we solve $F(x) = 0$, where F is given

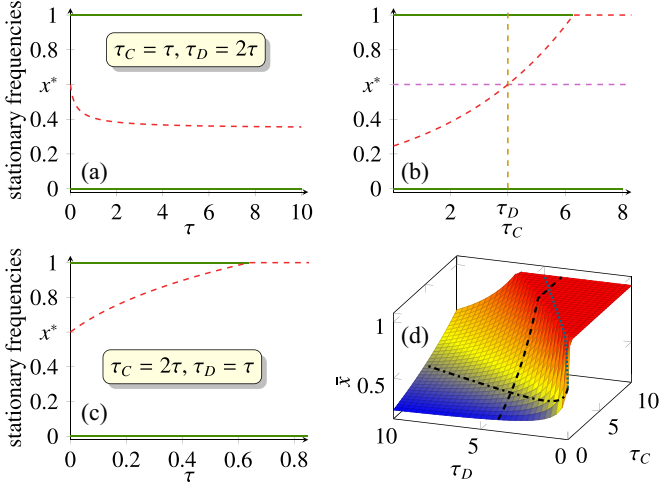


FIG. 1. Numerical solutions of $F(x) = 0$ for the matrix U_1 for the classical Stag-hunt game, that is, for $a = 3$ and $\beta = 3/5$. (a) The interior stationary state as a function of τ , when $\tau_C = \tau$, $\tau_D = 2\tau$. (b) The interior stationary state as a function of τ_C , while $\tau_D = 4$ is fixed. (c) The interior stationary state as a function of τ , when $\tau_C = 2\tau$, $\tau_D = \tau$. (d) The interior stationary state as a function of τ_C and τ_D . The dash-dotted line indicates the cross-section presented in panel (a), the black dotted line indicates the cross-section presented in panel (b), while the dotted line indicates the cross-section presented in (c).

by (14), and after some algebraic calculations we get

$$\bar{x} = \left(\frac{\tau_D a \beta^{\frac{\tau_C}{\tau_C - \tau_D}}}{W_p(\tau_D \beta a e^{\tau_D})} \right)^{\frac{\tau_C - \tau_D}{\tau_D}}. \quad (17)$$

For $a = 5$ and $\beta = 3/5$, we get the classical Stag-hunt game.

In Fig. 1 we present the dependence of the unstable interior stationary state on time delays. We see that the bigger a time delay of a given strategy is, the smaller is its basin of attraction. Moreover, if the time delay of the strategy C increases, at a certain point the interior stationary state ceases to exist and therefore the strategy D becomes globally asymptotically stable (the internal equilibrium hits $x = 1$, destabilizing this solution). For $\tau_C = \tau$, $\tau_D = 2\tau$ we get $\alpha = -1$ and therefore (14) simplifies and in the limit $\tau \rightarrow \infty$ one obtains $\bar{x} = \sqrt{\frac{\beta}{a}}$. For the classical Stag-hunt game, $\bar{x} = \frac{\sqrt{3}}{5} \approx 0.346$ as it is seen in panel (a).

Figure 1(d) indicates the existence of a curve $\tau_D^*(\tau_C^*)$ such that for $\tau_C > \tau_C^*$ and fixed $\tau_D = \tau_D^*$ there is no interior stationary frequency, and for $\tau_C < \tau_C^*$ there is a unique one. The curve $\tau_D^*(\tau_C^*)$ that splits the plane (τ_C, τ_D) into two regions (below this curve there is no interior stationary frequency, above it there is a unique one) is given by the following formula:

$$\tau_D^* = \tau_C^* \left(1 + \frac{\ln \beta}{W_p(a \tau_C^* e^{\tau_C^*}) - \tau_C^*} \right), \quad \tau_C^* \geq \frac{-\ln \beta}{a\beta - 1}. \quad (18)$$

The formula can be easily obtained from $F(1) = 0$ (detailed calculations can be found in Appendix A 2). For the classical Stag-hunt game, the curve given by (18) is almost a straight

line. Its derivative with respect to τ_C^* changes from

$$\frac{a\beta - 1}{a\beta - 1 - a\beta \ln \beta} \approx 0.566$$

for $\tau_C^* = \frac{-\ln \beta}{a\beta - 1} \approx 0.255$ to

$$1 + \frac{\ln \beta}{\ln a} \approx 0.683$$

for $\tau_C^* \rightarrow \infty$.

Note also that if τ_C is fixed, and we take $\tau_D \rightarrow +\infty$, then we get $\alpha \rightarrow 0$ and the formula (14) simplifies to $1 - \alpha x$, which implies that the interior stationary frequency for sufficiently large τ_D is close to $\frac{1}{a}$.

Let us summarize the results. When we parametrize both delays by τ , if the delay of defection is bigger than that of cooperation, then in the limit of infinite τ the interior unstable equilibrium tends to some value. Both strategies have some basin of attraction; see Fig. 1(a). However, if the delay of cooperation is bigger than that of defection, then above some critical τ the unstable interior equilibrium ceases to exist and the defection becomes globally asymptotically stable; see Fig. 1(c). Under no circumstances does cooperation become globally asymptotically stable.

B. Snowdrift-type games

We consider here two examples of games with a unique stable interior equilibrium state. Because of the nonlinearity of our equations, we could not treat general cases analytically.

Example 1

Here we discuss a classical Snowdrift game (with the reward equal to 5 and the cost equal to 2) with the following payoff matrix:

$$U_2 = \begin{bmatrix} 4 & 3 \\ 5 & 0 \end{bmatrix}.$$

We have that $x^* = 0.75$ is the stable stationary state of the nondelayed replicator dynamics.

When delays are parametrized in the form $\tau_C = \tau$, $\tau_D = 2\tau$ and τ increases, then the interior stationary state increases until it disappears at some value of τ . Above that point, $x = 1$, the equilibrium when all players cooperate, becomes globally asymptotically stable [see Fig. 2(a)]. In the case of $\tau_C = 2\tau$, $\tau_D = \tau$, the interior stationary state decreases and approaches the value $\frac{1 + \sqrt{301}}{50} \approx 0.37$ [see Fig. 2(c)].

We see in Fig. 2(b) that for fixed $\tau_D = 2$, there is a value of τ_C below which there is no interior stationary point; $x = 1$ is globally asymptotically stable. Above this point, the stable interior stationary state decreases.

In Fig. 2(d) we present the unstable interior stationary state as a function of τ_C and τ_D . If $\tau_C = \tau_D$, then the stable stationary frequency does not depend on time delay and is equal to $x^* = 0.75$. For fixed τ_D , if $\tau_C \gg \tau_D$, the stationary state decreases approaching the value 0.2 [that can be easily calculated taking a limit $\tau_C \rightarrow +\infty$ in (13)]. Similarly, analyzing the graph of the function F for $\tau_D = 0$, one can easily find that the stationary state decreases in this case from 0.75 for $\tau_C = 0$ to 0.2 for $\tau_C \rightarrow +\infty$. On the other hand, if we

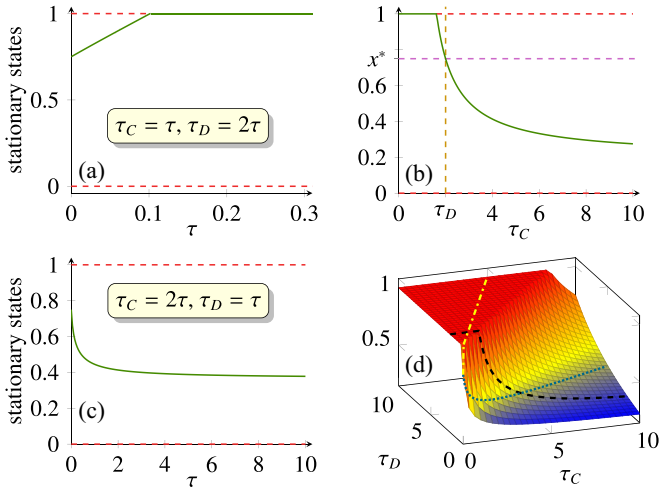


FIG. 2. Numerical solutions of $F(x) = 0$ for the matrix U_2 for the classical Snowdrift game. (a) The unstable interior stationary state as a function of τ , when $\tau_C = \tau$, $\tau_D = 2\tau$. (b) The interior stationary state as a function of τ_C , while $\tau_D = 2$ is fixed. (c) The interior stationary state as a function of τ , when $\tau_C = 2\tau$, $\tau_D = \tau$. (d) The interior stationary state as a function of τ_C and τ_D . The yellow dash-dotted line indicates the cross-section presented in panel (a), the black dotted line indicates the cross-section presented in panel (b), while the dotted line indicates the cross-section presented in panel (c).

fix τ_D , then the stationary state is a decreasing function of τ_C [see Fig. 2(b)]. Additionally, if τ_D is large enough [greater than $\ln(5/4)/3$, see Remark A.10 in the Appendix], then there exists a value τ_C^* of τ_C , such that for $\tau_C < \tau_C^*$ there exists no interior stable frequency and all individuals are playing C. The critical value $\tau_D^* = \frac{1}{3} \ln \frac{5}{4}$ for $\tau_C = 0$ depends on τ_C almost linearly [see the implicit formula (A5) in the Appendix for the relation between τ_C^* and τ_D^*].

We see that the situation here is in a sense a reverse one to that in the Stag-hunt game. When we parametrize both delays by τ , if the delay of defection is bigger than that of cooperation, then above some critical τ the stable interior equilibrium ceases to exist and the cooperation becomes a globally asymptotically stable equilibrium; see panel (a). However, if the delay of cooperation is bigger than that of defection, then in the limit of infinite τ the stable interior equilibrium tends to some value. Both strategies coexist; see panel (c). Under no circumstances does defection become globally asymptotically stable.

Example 2

Here we study another particular example that illustrates a complex behavior of games with asymmetric time delays. The game has the following payoff matrix:

$$U_3 = \begin{bmatrix} 2 & 0.5 \\ 2.95 & 0 \end{bmatrix}.$$

Now, $x^* \approx 0.345$ is the stable stationary state of the non-delayed replicator dynamics.

In Fig. 3(a), we set $\tau_C = \tau$, $\tau_D = 2\tau$. We see that there exists a threshold $\tau^* \approx 1.1$ such that for $\tau < \tau^*$ there exists

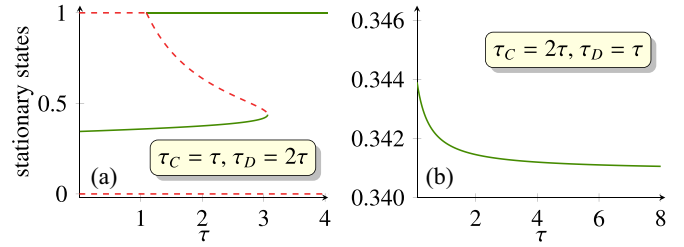


FIG. 3. Numerical solutions of $F(x) = 0$ for the matrix U_3 : the stable interior stationary state as a function of τ , when $\tau_C = \tau$, $\tau_D = 2\tau$ (a) and when $\tau_C = 2\tau$, $\tau_D = \tau$ (b).

a unique interior stationary state. Numerical simulations suggest that this state is stable. For $\tau > \tau^*$ and $\tau < \tau^*$ there are two interior stationary states, one of them stable and the other unstable. Notice that $x = 1$ is another stable equilibrium. Numerical simulations suggest that if the initial frequency of the strategy C is large enough, then the D strategy is eliminated. Let us also observe that there are two asymptotically stable equilibria in the system. Cooperation becomes an alternative equilibrium, and then the only equilibrium when both delays increase.

In Fig. 3(b), we present the stable interior stationary state as a function of τ , $\tau_C = 2\tau$, $\tau_D = \tau$. In this case, the stationary state is almost constant. In fact, for this set of parameters one can easily see that the function F is decreasing to 1 as both terms of F decrease. Because only the first term depends on τ (and it decreases with increasing τ), one can deduce that the stationary state is a decreasing function of τ . For $\tau \rightarrow +\infty$ it converges to the positive solution of the quadratic equation $(2.95x)^2 - 1.5x - 0.5 = 0$, so it is approximately equal to 0.341. Thus, $\bar{x} \in [0.341, 0.345]$, so it is almost constant. Here the states 1 and 0 are unstable.

C. Prisoner’s Dilemma

Here we consider the Prisoner’s Dilemma game with the following payoff matrix:

$$U_4 = \begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}.$$

The function F takes the following form:

$$F(x) = \frac{1}{\tau_C - \tau_D} \ln \left(\frac{3x}{4x + 1} \right) + 1 - 3x \left(\frac{4x + 1}{3x} \right)^{\frac{\tau_C}{\tau_C - \tau_D}}. \tag{19}$$

It is easy to see that $\lim_{x \rightarrow 0^+} F(x) < 0$ for $\tau_C > \tau_D$ and $\lim_{x \rightarrow 0^+} F(x) > 0$ for $\tau_C < \tau_D$. However, the sign of

$$F(1) = 1 - \frac{1}{\tau_C - \tau_D} \ln \frac{5}{3} - 3 \left(\frac{5}{3} \right)^{\frac{\tau_C}{\tau_C - \tau_D}} \tag{20}$$

cannot be easily determined. Plots of the function F suggest that it has no zeros inside the interval $[0,1]$ for $\tau_C > \tau_D$ and it can have one for some set of delays $\tau_C < \tau_D$. However, we can deduce that $F(1) > 0$ if τ_D is sufficiently large, thus then there exists an interior stationary frequency. In Fig. 4(a), we plotted the value of this interior stationary frequency for $\tau_D = 1$. The limiting values of τ_C^* and τ_D^* are given by the implicit formula

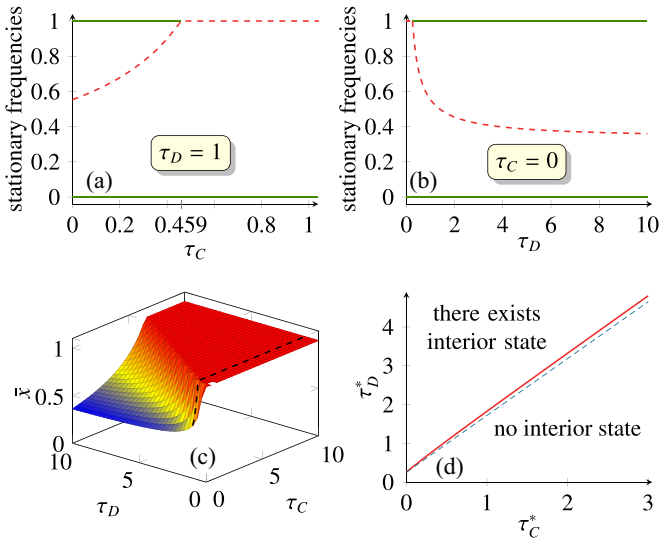


FIG. 4. Numerical solutions of $F(x) = 0$ for the matrix U_4 for the classical Prisoner's Dilemma game. (a) Interior stationary state as a function of τ_C , when $\tau_D = 1$ is fixed. The interior stationary state exists for $\tau_C < 0.459$. (b) The interior stationary state as a function of τ_D , while $\tau_C = 0$ is fixed. An interior stationary state exists for $\tau_D > 0.2554$. (c) The interior stationary state as a function of τ_C and τ_D . The dotted line indicates the cross-section presented in panel (a). (d) The region of delays for which there exists the interior stationary state and the region where there is no interior stationary state. The solid curve denotes the border between these two regions. The curve is almost a straight line (the dashed line), plotted for comparison, and it has the same slope as the solid curve for $\tau_C^* \rightarrow +\infty$.

$F(1) = 0$, which can be solved,

$$\tau_D^* = \tau_C^* \left(1 - \frac{\ln \frac{5}{3}}{\ln \left(\frac{1}{3\tau_C^*} W_L(3\tau_C^* e^{\tau_C^*}) \right)} \right). \quad (21)$$

The curve is plotted in Fig. 4(d) (solid line). It is almost a straight line with the slope given by

$$1 - \frac{\ln \frac{5}{3}}{\ln \left(\frac{1}{3\tau_C^*} W_L(3\tau_C^* e^{\tau_C^*}) \right)} \rightarrow \frac{\ln 5}{\ln 3} \approx 1.465 \text{ as } \tau_C^* \rightarrow \infty. \quad (22)$$

On the other hand, $\tau_D^* \rightarrow \frac{1}{2} \ln \frac{5}{3} \approx 0.255$ as $\tau_C^* \rightarrow 0$. If a point (τ_C, τ_D) is above the curve (τ_C^*, τ_D^*) , then there exists an interior stationary state (which is unstable as suggested by numerical simulations). This means that the system is bistable and that it goes to all cooperation or all defection depending on the initial condition.

In Fig. 4(c), we presented the interior stationary frequency for various τ_C 's and τ_D 's.

If $\tau_C = 0$, then we have

$$F(x) = \frac{1}{\tau_D} \ln \frac{4x+1}{3x} + 1 - 3x, \quad (23)$$

which is a decreasing function that has a zero inside the interval $[0, 1]$ for $\tau_D > \frac{1}{2} \ln \frac{5}{3}$. A numerical solution of the equation

$$\frac{1}{\tau_D} \ln \frac{4x+1}{3x} + 1 - 3x = 0 \quad (24)$$

is plotted in Fig. 4(b).

We see that when one introduces asymmetric delays with a delay of cooperation smaller than a delay of defection, then there appears an interior unstable equilibrium. Hence the state with just cooperators becomes locally asymptotically stable.

IV. DISCUSSION

We studied the effects of strategy-dependent time delays on stationary states of evolutionary games. Recently, the effects of the duration of interactions between two players on their payoffs, and therefore on evolutionary outcomes, were discussed by Křivan and Cressman [30]. In their models, the duration of interactions depends on the strategies involved. This can be interpreted as strategy-dependent time delays. They showed that interaction times change stationary states of the system.

Another approach is to consider ordinary differential equations with time delays. It was pointed out in [17] that in the so-called biological model, where it is assumed that the number of players born in a given time is proportional to payoffs received by their parents at a certain moment in the past, the interior state is asymptotically stable for any time delay. In such models, microscopic interactions lead to a new type of replicator dynamics that describes the time evolution of fractions of the population playing given strategies and the size of the population.

Here we studied biological-type models with strategy-dependent time delays. We observed a novel behavior. We showed that interior stationary states depend continuously on time delays. In particular, in games with two pure Nash equilibria (Stag-hunt-type games), the bigger the time delay of a given strategy is, the smaller is its basin of attraction.

When we parametrize both delays by τ , if the delay of defection is bigger than that of cooperation, then in the limit of infinite τ the interior unstable equilibrium tends to some value. Both strategies have some basin of attraction. However, if the delay of cooperation is bigger than that of defection, then above some critical τ the unstable interior equilibrium ceases to exist and the defection becomes globally asymptotically stable. Under no circumstances does cooperation become globally asymptotically stable.

In games with a stable interior equilibrium, the bigger the time delay of a strategy is, the less frequent it is in the population. Moreover, at certain time delays the interior stationary state ceases to exist, or there may appear another interior stationary state.

We see that the situation here is in a sense a reverse one to that in the Stag-hunt game. When we parametrize both delays by τ , if the delay of defection is bigger than that of cooperation, then above some critical τ the stable interior equilibrium ceases to exist and the cooperation becomes a globally asymptotically stable equilibrium. However, if the delay of cooperation is bigger than that of defection, then in the limit of infinite τ the stable interior equilibrium tends to some value. Both strategies coexist. Under no circumstances does defection become globally asymptotically stable.

In the Prisoner's Dilemma game, for time delays of cooperation smaller than the time delay of defection, there appears an unstable interior equilibrium. Therefore, for some initial conditions, the population converges to the homogeneous

state with just cooperators. This shows an asymptotic stability of cooperation in a simple model of a social dilemma.

It would be interesting to analyze strategy-dependent time delays in stochastic dynamics of finite populations. The work in this direction is in progress.

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APPENDIX

We provide here propositions, theorems, and their proofs, which support the results presented in the paper.

The main results for Snowdrift-type games are included in Theorem A.3 and following Corollary A.4. We present there a set of conditions that guarantee the existence of at least one interior stationary state. In Theorem A.6 we give a condition for the monotonicity of a function which zeros give us stationary states. Finally, in Proposition A.9 we state conditions for the absence of stationary states for small time delays.

In Remark A.1 we give an explicit formula for the leading eigenvalue λ for pure stationary states, $\bar{x} = 0$ and 1. This eigenvalue determines the growth rate of the whole population. In Remark A.10 we state some results about the existence of stationary states for $d = 0$.

We provide here also additional calculations concerning Stag-hunt type games.

1. General results for Snowdrift-type games

Interior stationary states are given by zeros of the function $F(x)$ defined by (13). Let $\bar{U}_C = a\bar{x} + b(1 - \bar{x})$ and $\bar{U}_D = c\bar{x} + d(1 - \bar{x})$.

Remark A.1. We see that for $\bar{x} = 0$ we have

$$\lambda + 1 = de^{-\lambda\tau_D} \implies \lambda = \frac{W_p(d\tau_De^{\tau_D})}{\tau_D}, \tag{A1}$$

while for $\bar{x} = 1$ we have

$$\lambda + 1 = ae^{-\lambda\tau_C} \implies \lambda = \frac{W_p(a\tau_Ce^{\tau_C})}{\tau_C}, \tag{A2}$$

where W_p is the Lambert W function, which is a principal branch of the relation $W_p(x)\exp(W_p(x)) = x$.

We show that if $\tau_C \neq \tau_D$, then the value of an interior stationary state depends on time delays. Moreover, for some payoff matrices and some values of a delay, multiple interior stationary states exist. We would like to point out that these relations are not linear. Thus, contrary to the case without time delays or with equal delays (i.e., $\tau_C = \tau_D$), adding a constant to a column of a payoff matrix or multiplying the matrix by a constant changes interior stationary states or even may change the number of interior stationary states. Thus, the intuition from nondelayed games about the asymptotics of the dynamics of replicator equations is not necessarily valid for the case of nonequal delays.

It turns out that if the eigenvalue $\lambda(\bar{x})$ corresponding to the stationary state \bar{x} is positive, then the frequency of the given strategy is smaller than the frequency of this strategy in a nondelayed case if the delay corresponding to this strategy is larger than the time delay of the other strategy. We have the following proposition.

Proposition A.2. Let $a < c$ and $d < b$. Assume that $\bar{x} \in (0, 1)$ and let λ be the leading eigenvalue that corresponds to \bar{x} . Then if $\lambda > 0$ we have the following:

- (i) If $\tau_C > \tau_D$, then $\bar{x} < x^*$.
- (ii) If $\tau_C < \tau_D$, then $\bar{x} > x^*$.

Proof. Assume that $\tau_C > \tau_D$. Then the sign of λ is the same as the sign of

$$\phi(x) = \ln \left(\frac{ax + b(1 - x)}{cx + d(1 - x)} \right).$$

Note that $\phi(x^*) = 0$ and

$$\phi'(x) = \frac{ad - bc}{(ax + b(1 - x))(cx + d(1 - x))} < 0$$

because $ad < bc$. Thus, $\phi(x) > 0$ for $x < x^*$, so $\bar{x} < x^*$. Similarly, if $\tau_C < \tau_D$, then the sign of λ is opposite to the sign of $\phi(x)$ and therefore $\bar{x} > x^*$.

We will study general properties of F , which will help us to determine a number of stationary states of our replicator dynamics. First, we determine conditions that would imply the sign of F at $x = 0$ and 1. Let us define

$$\begin{aligned} c^*(a) &= a^{-\tau_D/\tau_C} \left(\frac{W_p(a\tau_Ce^{\tau_C})}{\tau_C} \right)^{1-\tau_D/\tau_C}, \\ d^*(b) &= b^{-\tau_D/\tau_C} \left(\frac{W_p(b\tau_Ce^{\tau_C})}{\tau_C} \right)^{1-\tau_D/\tau_C}, \end{aligned} \tag{A3}$$

where W_p is the Lambert W function, which is a principal branch of the relation $W_p(x)\exp(W_p(x)) = x$.

Theorem A.3. (A) If $a < c$, then $F(1) > 0$ if and only if one of the following conditions holds:

- (i) $\tau_C > \tau_D$, $a < 1$, and $c < c^*(a)$.
- (ii) $\tau_C < \tau_D$, $a < 1$, or $c > c^*(a)$.

(B) If $b > d$, then $F(0) > 0$ if and only if one of the following conditions holds:

- (i) $\tau_C > \tau_D$, $b < 1$ or $d < b^*(b)$.
- (ii) $\tau_C < \tau_D$, $b < 1$ and $d > b^*(b)$.

Proof. First we study the sign of $F(1)$. It is easy to see that

$$F(1) = \frac{1}{\tau_C - \tau_D} \ln \left(\frac{a}{c} \right) + 1 - a \left(\frac{c}{a} \right)^{\tau_C/(\tau_C - \tau_D)}.$$

Assume that $\tau_C > \tau_D$. Due to the assumption $a < c$, there exists $z \in (0, 1)$ such that $a = zc$. We plug this into the expression for $F(1)$ and we obtain

$$\frac{1}{\tau_C - \tau_D} \ln z + 1 - a \left(\frac{1}{z} \right)^{\tau_C/(\tau_C - \tau_D)}. \tag{A4}$$

Let us introduce $u = z^{\tau_C/(\tau_C - \tau_D)}$. Then the above expression simplifies to

$$f_a(u) = \frac{1}{\tau_C} \ln u + 1 - \frac{a}{u}.$$

It is easy to see that f_a is a continuous, strictly increasing function of u and $\lim_{u \rightarrow 0^+} f_a(u) = -\infty$. Thus we have two possibilities. Either $f_a(1) < 0$ and $f_a(u) < 0$ for all $u \in (0, 1)$ [thus $F(1) < 0$ for all $a < c$] or there exists a unique $u^* \in (0, 1)$ such that $f_a(u^*) = 0$ [$f_a(u) < 0$ for $0 < u < u^*$ and $f_a(u) > 0$ for $u^* < u < 1$] due to monotonicity [and this implies an appropriate sign of $F(1)$ depending on the value of a with respect to c].

Note that $f_a(1) \leq 0$ is equivalent to $a \geq 1$. Assume now that $a < 1$ and we find u^* . We have

$$\begin{aligned} \frac{1}{\tau_C} \ln u^* + 1 &= \frac{a}{u^*} \iff \tau_C - a\tau_C \frac{1}{u^*} \\ &= \ln \frac{1}{u^*} \iff a\tau_C \frac{1}{u^*} e^{a\tau_C \frac{1}{u^*}} \\ &= a\tau_C e^{\tau_C}. \end{aligned}$$

Thus,

$$u^* = \frac{a\tau_C}{W_p(a\tau_C e^{\tau_C})},$$

where W_p is the Lambert function. Because $\frac{a}{c} = u^{(\tau_C - \tau_D)/\tau_C}$, then $F(1) \leq 0$ if $a \geq 1$ or $c > c^*(a)$, where $c^*(a)$ is given by (A3). Thus, if $a > 1$ and $c < c^*(a)$, then $F(1) > 0$ and point (i) is proven. Finally, note that if $\tau_C < \tau_D$, then the variable u changes from 1 to $+\infty$ instead of from 0 to 1, and very similar arguments lead to the assertion of point (ii).

Let us calculate the value

$$F(0) = \frac{1}{\tau_C - \tau_D} \ln \left(\frac{d}{b} \right) + 1 - b \left(\frac{b}{d} \right)^{\tau_C/(\tau_C - \tau_D)}.$$

Note that this situation is analogous to the previous one, and analogous arguments prove this part of the theorem.

From Theorem A.3 and its proof, the following corollary can be easily deduced.

Corollary A.4. (A) If $a < c$, then $F(1) < 0$ if and only if one of the following conditions holds:

- (i) $\tau_C > \tau_D$, $a \geq 1$ or $c > c^*(a)$.
- (ii) $\tau_C < \tau_D$, $a \geq 1$ and $c < c^*(a)$.

(B) If $b > d$, then $F(0) > 0$ if and only if one of the following conditions holds:

- (i) $\tau_C > \tau_D$, $b \geq 1$ and $d > b^*(b)$.
- (ii) $\tau_C < \tau_D$, $b \geq 1$ or $d < b^*(b)$.

Corollary A.5. Theorem A.3 and Corollary A.4 give conditions that guarantee the existence of at least one interior stationary state. Namely, if $a < c$ and $b > d$, then we need $F(1) > 0$ [Theorem A.3(A)] and $F(0) < 0$ [Corollary A.4(B)] or, reversely, $F(1) < 0$ [Corollary A.4(A)] and $F(0) > 0$ [Theorem A.3(B)].

If the function F is monotonic, then the previous theorem gives us a condition guaranteeing the existence of a solution of $F(x) = 0$ on $(0, 1)$, that is, the existence of an interior stationary state.

Theorem A.6. If $a < c$, $d < b$, and additionally $a < d < c$ or $d < a < b$, then the function F is monotonic. Moreover, if additionally $\tau_C > \tau_D$, then it is decreasing and if $\tau_C < \tau_D$, then it is increasing.

Proof. It is enough to calculate the first derivative of F . It reads

$$\begin{aligned} F'(x) &= \frac{1}{\tau_C - \tau_D} \left(\frac{ad - bc}{(ax + b(1-x))(cx + d(1-x))} \right. \\ &\quad + \left(\tau_C(d - c) + \tau_D(a - b) \frac{cx + d(1-x)}{ax + b(1-x)} \right) \\ &\quad \times \left. \left(\frac{cx + d(1-x)}{ax + b(1-x)} \right)^{\tau_D/(\tau_C - \tau_D)} \right). \end{aligned}$$

The assumptions guarantee that $ad - bc < 0$, $d - c < 0$, $a - b < 0$, and the thesis follows.

Corollary A.7. If $a = b$ or $c = d$, then the function F is monotonic and there is at most one solution of $F(x) = 0$ in the interval $[0, 1]$.

Remark A.8. Theorems A.3 and A.6 give a complete description of the existence of the unique stationary state \bar{x} inside the interval $(0, 1)$ if $a < d < c$ or $d < a < b$ (under the assumption that $a < c$, $d < b$).

Now, we give a condition that guarantees the existence of the interior stationary point \bar{x} when τ_D is fixed and τ_C converges to zero.

Proposition A.9. Assume that $0 < b < a < c$ and $d = 0$. If one of the following conditions holds:

- (a) $\tau_D \leq \frac{b}{a(a-b)}$, $a > 1$, and $\tau_D > \frac{\ln(\frac{c}{a})}{a-1}$.
- (b) $\tau_D > \frac{b}{a(a-b)}$ and $b > 1$.

then for $\tau_C < \tau_D$ close enough to 0, there exists no interior equilibrium state $\bar{x} \in (0, 1)$.

Proof. We check if the equation $F(x) = 0$, where F is given by (13), has a solution $\bar{x} \in (0, 1)$ for fixed τ_D and small τ_C . It is easy to see that the function F is continuous with respect to τ_C and x for $\tau_C < \tau_D$. Thus, it is enough to check the existence of a solution to $F(x) = 0$ for $\tau_C = 0$. In this case, the function F simplifies to

$$F_0(x) = \frac{1}{\tau_D} \ln \left(\frac{cx}{ax + b(1-x)} \right) + 1 - (ax + b(1-x)).$$

We calculate the derivative of F_0 ,

$$F'_0(x) = \frac{b}{\tau_D} \frac{1}{x((a-b)x + b)} - (a-b).$$

We see that it is a decreasing function of $x \in (0, 1)$ (as $a > b$) and therefore F_0 is concave. Moreover, $\lim_{x \rightarrow 0^+} F'_0(x) = +\infty$ thus

- (i) If $\tau_D \leq \frac{b}{a(a-b)}$, then F_0 is increasing in $(0, 1)$.
- (ii) If $\tau_D > \frac{b}{a(a-b)}$, then F_0 has exactly one maximum at $(0, 1)$ at the point

$$x_{\max} = \frac{b}{2(a-b)} \left(\sqrt{1 + \frac{4}{b\tau_D}} - 1 \right).$$

Let us consider the first case. Here $F'_0(x) > 0$ for all $x \in (0, 1)$ because it is decreasing and $F'_0(1) > 0$. Thus F_0 is an increasing function of x and as $\lim_{x \rightarrow 0^+} F_0(x) = -\infty$ it has a zero in the interval $(0, 1)$ if and only if $F(1) > 1$. An easy calculation allows us to derive the condition (a).

Now let us consider the second case. Some algebraic manipulations lead to

$$F_0(x_{\max}) = \frac{1}{\tau_D} \left[\ln \frac{c}{a-b} + \ln \left(\frac{4}{b\tau_D} \right) - 2 \ln \left(1 + \sqrt{1 + \frac{4}{b\tau_D}} \right) \right] + 1 - \frac{b}{2} \left(1 + \sqrt{1 + \frac{4}{b\tau_D}} \right).$$

We show that $F_0(x_{\max})$ approaches its supremum either for $\tau_D \rightarrow +\infty$ or for $\tau_D \rightarrow \frac{b}{a(b-b)}$. Let us introduce a new variable $z = \frac{4}{b\tau_D}$ and denote $\tilde{c} = \frac{c}{a-b}$. Now, writing $F_0(x_{\max})$ in variable z we have

$$F_0(x_{\max}) = h(z) = \frac{b}{4} z (\ln c + \ln z - 2 \ln(1 + \sqrt{1+z})) + 1 - \frac{b}{2} (1 + \sqrt{1+z}),$$

where $0 < z < \frac{4(a-b)b}{b^2}$ as $\tau_D > \frac{b}{(a-b)b}$. We calculate the derivative of h with respect to z and obtain

$$h'(z) = \frac{b}{4} \ln \left(\frac{cz}{(1 + \sqrt{1+z})^2} \right).$$

Now it is easy to see that h' has at most one zero for $z > 0$ and it is negative for z close to 0. Hence, the function h is decreasing for small z and it may increase for large z having at most one minimum for $z > 0$. Thus it approaches its supremum either for $z \rightarrow 0$ (i.e., $\tau_D \rightarrow +\infty$) or for $z \rightarrow \frac{4(a-b)b}{b^2}$. In the latter case, we have

$$\tau_D \rightarrow \frac{b}{a(a-b)} \implies x_{\max} \rightarrow 1,$$

and we arrive at the case considered earlier. On the other hand,

$$\lim_{z \rightarrow 0+} h(z) = 1 - b < 0$$

if the condition (b) holds. Thus, $F_0(x) < 0$ for all $x \in (0, 1)$ and no interior stationary state exists.

Remark A.10. Note that if $\tau_D \leq \frac{b}{a(a-b)}$ and either $a < 1$ or $\tau_D < \frac{\ln(\frac{c}{a})}{a-1}$, then there exists exactly one interior stationary state $\bar{x} \in (0, 1)$ for $\tau_C < \tau_D$ close enough to 0. On the other hand, if $\tau_D > \frac{b}{a(a-b)}$ and $b < 1$, then there exist one or two interior stationary states $\bar{x} \in (0, 1)$ for $\tau_C < \tau_D$ close enough to 0 if τ_D is large enough. In fact, if $a < 1$ for a sufficiently large τ_D , then there exist two stationary states $\bar{x} \in (0, 1)$.

If τ_C is small enough (for a fixed τ_D) or, reversely, if τ_D is large enough (for a fixed τ_C), there exists no interior stationary state. Thus, it is possible to calculate these threshold values τ_C^* and τ_D^* . It can be seen that the interior stationary state disappears when it merges with the stationary state 1. Thus, looking for τ_C^* and τ_D^* such that $F(1) = 0$, after some algebraic calculations we obtain the implicit formula

$$\tau_C^* = - \frac{(\tau_D^* - \tau_C^*) \ln \left(\frac{\ln(c/a)}{4(\tau_D^* - \tau_C^*)} + \frac{1}{a} \right)}{\ln(c/a)}. \tag{A5}$$

We have that for $\tau_C < \tau_C^*$ or for $\tau_D > \tau_D^*$ there exists no interior stationary state.

2. Calculations concerning Stag-hunt-type game

In this part of the Appendix, we provide some details of calculations that allowed us to formulate statements about the slope of the curve (τ_C^*, τ_D^*) given by formula (18) in the paper. Let us recall this formula here,

$$\tau_D^* = \tau_C^* \left(1 + \frac{\ln \beta}{W_p(a\tau_C^* e^{\tau_C^*}) - \tau_C^*} \right). \tag{A6}$$

First note that $\tau_D^* > 0$ for $\tau_C^* > \tau_{A,0}^*$, where

$$\tau_{A,0}^* = \frac{-\ln \beta}{a\beta - 1}. \tag{A7}$$

We also have $\tau_D^* = 0$ for $\tau_C^* = \tau_{A,0}^*$. We have then

$$W_p(a\tau_{A,0}^* e^{\tau_{A,0}^*}) = \tau_{A,0}^* - \ln \beta. \tag{A8}$$

We calculate two quantities:

(i) The derivative of the right-hand side of (A6) with respect to τ_C^* at the point $\tau_{A,0}^*$.

(ii) The slope of the right-hand side of (A6) for $\tau_C^* \rightarrow +\infty$. Here we calculate $g'(\tau_{A,0}^*)$, where

$$g(\tau_C^*) = \tau_C^* \left(1 + \frac{\ln \beta}{W_p(y(\tau_C^*)) - \tau_C^*} \right), \quad y(\tau_C^*) = a\tau_C^* e^{\tau_C^*}.$$

We have

$$g'(\tau_{A,0}^*) = \left(1 + \frac{\ln \beta}{W_p(y(\tau_{A,0}^*)) - \tau_{A,0}^*} \right) - \frac{\tau_{A,0}^* \ln \beta}{[W_p(y(\tau_{A,0}^*)) - \tau_{A,0}^*]^2} (W_p'(y(\tau_{A,0}^*))) y'(\tau_{A,0}^*) - 1). \tag{A9}$$

Because of (A8), the first parenthesis of (A9) is equal to 0. Moreover, $W_p(y(\tau_{A,0}^*)) - \tau_{A,0}^* = -\ln \beta$. Thus, the formula (A9) simplifies to

$$g'(\tau_{A,0}^*) = - \frac{\tau_{A,0}^*}{\ln \beta} (W_p'(y(\tau_{A,0}^*))) y'(\tau_{A,0}^*) - 1). \tag{A10}$$

Now we calculate the expression in the parentheses. To shorten the notation, we write y and y' omitting the dependence of y on $\tau_{A,0}$. Using (A8) and (A7), we get

$$\begin{aligned} W_p'(y) &= \frac{W_p(y)}{y[1 + W_p(y)]} \\ &= \frac{-\frac{\ln \beta}{a\beta - 1} - \ln \beta}{y \frac{a\beta - 1 - a\beta \ln \beta}{a\beta - 1}} \\ &= \frac{-a\beta \ln \beta}{y(a\beta - 1 - a\beta \ln \beta)} \\ &= \frac{a\beta - 1}{a\beta - 1 - a\beta \ln \beta} \beta^{a\beta} \end{aligned}$$

because

$$y = a\tau_{A,0} e^{\tau_{A,0}^*} = \frac{-a \ln \beta}{a\beta - 1} \beta^{1-a\beta}.$$

On the other hand, we have

$$y'(\tau_{A,0}^*) = (1 + \tau_{A,0}^*) e^{\tau_{A,0}^*} = \frac{\beta - 1 - \ln \beta}{a\beta - 1} \beta^{1-a\beta}.$$

Plugging formulas for $W_p'(y)$ and y' into (A10), we get the final formula

$$g'(\tau_{A,0}^*) = \frac{a\beta - 1}{a\beta - 1 - a\beta \ln \beta}. \quad (\text{A11})$$

To calculate the slope of the right-hand side of (A6) for $\tau_C^* \rightarrow +\infty$, we note (see Ref. [31], Eq. 4.13.10) that

$$W_p(y) = \ln y - \ln \ln y + r(y), \quad \text{where } r(y) \rightarrow 0 \text{ as } y \rightarrow +\infty.$$

We have then

$$\begin{aligned} & W_p(a\tau_C^* e^{\tau_C^*}) - \tau_C^* \\ &= \ln(a\tau_C^* e^{\tau_C^*}) - \ln \ln(a\tau_C^* e^{\tau_C^*}) + r(a\tau_C^* e^{\tau_C^*}) - \tau_C^* \\ &= \ln a + \ln \frac{\tau_C^*}{\tau_C^* + \ln a + \ln \tau_C^*} + r(a\tau_C^* e^{\tau_C^*}) \\ &\rightarrow \ln a \end{aligned}$$

as $\tau_C^* \rightarrow +\infty$. Thus, the slope of the right-hand side of (A6) for $\tau_C^* \rightarrow +\infty$ is equal to $1 + \frac{\ln \beta}{\ln a}$.

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