Comment on "Stochastic dynamics of the prisoner's dilemma with cooperation facilitators"

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In a recent paper [Phys. Rev. E **86**, 011134 (2012)], Mobilia introduces cooperation facilitators in the standard prisoner's dilemma game. He claims that natural selection favors the replacement of defection by cooperation in the weak-selection case if and only if their frequency satisfies a certain inequality. We show that this is not true, and we point out an error in the author's proof which follows from the improper handling of the large-population limit. In addition, we prove a stronger result that cooperation is favored for any selection strength if and only if the average payoff of cooperation is bigger than the average payoff of defection (which is a weaker condition than the author's inequality). We also show that, if we include self-interaction, then the presence of a fixed number of facilitators causes a rescaling of the payoff matrix, and for their certain frequency, cooperation becomes a dominant strategy, and the prisoner's dilemma simply disappears.

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In a recent paper [1], the author addresses a very important problem of the origin of cooperative behavior in animal and human populations. In the Darwinian survival of the fittest world, cooperation or, generally, an altruistic behavior which incurs cost should not be present in the long-run evolution [2,3]. However, it is observed that such a behavior is ubiquitous in nature. One may address this issue in the framework of evolutionary game theory [4–8].

It is well known that, in all classical models of evolutionary game theory, the cooperative behavior cannot be supported in the long run. One has to introduce additional features of the players and their interactions to obtain a satisfactory cooperation level. In Ref. [1], a fixed number of the so-called cooperation facilitators was introduced to classic evolutionary models of the prisoner's dilemma game. Cooperators receive the same benefit when they interact with cooperators or facilitators, and defectors receive the benefit just from cooperators. The author then shows that, for a certain range of game parameters (the number of facilitators being one of them), cooperation is evolutionarily favored in the standard Fermi and Moran stochastic dynamics of the finite population of players.

In the standard prisoner's dilemma game, two players may choose to cooperate (C) or to defect (D). The payoff matrix is given by

$$\begin{array}{cc} & \begin{array}{cc} \mathsf{C} & \mathsf{D} \end{array} \\ \begin{array}{c} \mathsf{C} \\ \mathsf{D} \end{array} & \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix}, \end{array}$$

where b and c, respectively, are the benefit and the cost of cooperation. It is easy to see that no matter what your opponent does, your best reply (a strategy giving the highest payoff) is defection. We say that D dominates C. However, the mutual cooperation gives both of us a reward which is higher than a payoff, resulting from the mutual defection, and hence, we face the dilemma. In Ref. [1], the population of N players and l facilitators was considered. Let j be the number of cooperators, and let k be the number of defectors, then the expected payoffs of both strategies are [1] as follows:

$$f_{\mathsf{C}} = (b-c)\frac{j+l-1}{N-1} - c\frac{k}{N-1}, \quad f_{\mathsf{D}} = b\frac{j}{N-1}. \tag{1}$$

Let $z = l/N$ be a fraction of cooperation facilitators, and let $r = c/b$ be a cost-to-benefit ratio. We have that $f_{\mathsf{C}} > f_{\mathsf{D}}$ if and only if

$$z - r > \frac{1-r}{N}. \tag{2}$$

One of the main results in Ref. [1] is that, under a weak-selection pressure ($|v_N| \equiv |f_{\mathsf{D}} - f_{\mathsf{C}}| \ll 1$) and in the large-population limit ($N|v_N| \gg 1$), invasion and replacement of defection by cooperation is favored by selection if and only if

$$b(z-r) > \frac{1}{N(1-z)}. \tag{3}$$

We will show that this is incorrect. We will prove that invasion and replacement of defection by cooperation is favored by selection if and only if $f_{\mathsf{C}} > f_{\mathsf{D}}$, which is a weaker condition—one can easily show that (2) follows from (3), and we can construct various examples which satisfy (2) but not (3).

We have the following formula for the probability of cooperation fixation, starting from a single cooperator in the Fermi process (Eq. (16) in Ref. [1]):

$$\phi_1^{\mathsf{C}} = \frac{e^{v_N} - 1}{e^{N(1-z)v_N} - 1}. \tag{4}$$

If v_N is negative, so $z - r > (1-r)/N$, that is, C dominates D, and if N is large ($N|v_N| \gg 1$), it follows that $\phi_1^{\mathsf{C}} \simeq 1 - e^{-|v_N|}$ [1]. In the weak-selection case, we get $|v_N|$ for the fixation probability of cooperation. Then the author performs the limit $|v_N| \to |v| = b(z-r)$, which is inappropriate because his result concerns finite populations, and $1/N$ terms are important as we see in the following. We should keep $|v_N| = b[(z-r) - (1-z)/(N-1)]$, but then the author's proof of the if part does not work. In fact, one can easily find an example such that $b(z-r) > 1/N(1-z)$ but $b[(z-r) - (1-z)/(N-1)] < 1/N(1-z)$, which shows that the basic author's inequality is satisfied, but nevertheless, the fixation probability given by $|v_N|$ is not bigger than the one in the neutral case. This is a very subtle situation, we have $1/N$ on the right-hand side, so we should be very careful with performing the weak-selection and infinite N limits on the left-hand side. The only if part of the result claimed by the author [that is, that the inequality (3) is a necessary condition for the cooperation selection] is not true as can be shown in the following counterexample. Let $b = 0.8$, $c = 0.1$, hence, $r = 0.125$, and let $N = 200$ and $z = 0.130$ (we can generate other examples with an arbitrarily weak-selection strength and an arbitrarily large N, but then z should be close to 1). One can verify that (3) is not satisfied, but (2) is satisfied, that is, $f_{\mathsf{C}} > f_{\mathsf{D}}$, so cooperation is dominant, and hence, replacement of defection by cooperation is favored by selection under any selection pressure, not just a weak one, as we demonstrate in the following theorem.

Theorem 1. In the Fermi process, the fixation probability of C starting with a single cooperator is bigger than $1/N(1-z)$ if and only if $f_{\mathsf{C}} > f_{\mathsf{D}}$.

Proof. If $f_{\mathsf{C}} > f_{\mathsf{D}}$, then $v_N < 0$. Let $e^{v_N} = y < 1$, and to avoid technicalities, let us assume that $N(1-z) = M$ is a natural number, then it follows from (4) that

$$\phi_1^{\mathsf{C}} = \frac{y-1}{y^M - 1} = \frac{1}{y^{M-1} + y^{M-2} + \cdots + 1} > \frac{1}{M}. \quad (5)$$

If $f_{\mathsf{C}} < f_{\mathsf{D}}$, then $v_N > 0$, hence, $y > 1$, so $\phi_1^{\mathsf{C}} < 1/M$, and this ends the proof.

Actually, we can see in the more general way that the above result also holds in the Moran and other processes. First we again prove the result for the Fermi process. Let x be the frequency of cooperators. Hence, $x(1-z-x)$ is the probability that a cooperator and a defector are chosen for the comparison in the Fermi process. In the neutral case, the probability of one step towards the C or D population is $x(1-x)/2$. Now if $f_{\mathsf{C}} > f_{\mathsf{D}}$ and the population is in state x, then the probability of a step towards the C population is $x(1-x)p_x$, where $p_x > 1/2$, so the fixation probability of C is bigger than $1/N(1-z)$, which is the fixation probability in the neutral case.

Theorem 2. In the Moran process, the fixation probability of C, starting with a single cooperator, is bigger than $1/N(1-z)$ if and only if $f_{\mathsf{C}} > f_{\mathsf{D}}$.

Proof. In the Moran process, in the neutral case, $x(1-z-x)$ is the probability that the offspring of an i strategist replaces a j strategist. If $f_{\mathsf{C}} > f_{\mathsf{D}}$, then $x(1-z-x)f_{\mathsf{C}}/\bar{f} > x(1-z-x)f_{\mathsf{D}}/\bar{f}$, where $\bar{f}$ is the population average payoff $\bar{f} = xf_{\mathsf{C}} + (1-z-x)f_{\mathsf{D}}$. Therefore, the probability of one step towards the C population is bigger than the probability of one step towards the D population, and hence, the fixation probability of C is bigger than $1/N(1-z)$.

Finally, to make things very simple, let us introduce self-interaction. One more cooperation facilitator should not change the global picture, and from the practical mathematical point of view, we do not have to deal with inconvenient $N-1$ denominators. Now it is easy to see that $f_{\mathsf{C}} > f_{\mathsf{D}}$ if and only if $z > r$. It follows that C becomes a dominant strategy if and only if $z > r$, and then the prisoner's dilemma disappears.

From another perspective, when one includes self-interaction, then the presence of a fixed number of facilitators causes the rescaling of the payoff matrix—all entries are multiplied by $(N-l)/N$, and then $(b-c)z$ is added to the C row of payoffs. It means that f_{C} and f_{D} can be computed using an N-dependent payoff matrix with two strategies. Because dynamics discussed in the paper depend only on f_{C} and f_{D}, then the evolution of the population and the disappearance of the prisoner's dilemma can be understood on the basis of the rescaled matrix.

To summarize, we have shown that, contrary to claims in Ref. [1], in the presence of cooperation facilitators, cooperation is favored for any selection strength if and only if the average payoff of cooperation is bigger than the average payoff of defection.

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