TOPOLOGICAL INVARIANTS FOR SUBSTITUTION TILINGS AND THEIR ASSOCIATED C^*-ALGEBRAS

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Abstract: We consider the dynamical systems arising from substitution tilings. Under some hypotheses, we show that the dynamics of the substitution or inflation map on the space of tilings is topologically conjugate to a shift on a stationary inverse limit, *i.e.* one of R.F. William's generalized solenoids. The underlying space in the inverse limit construction is easily computed in most examples and frequently has the structure of a CW-complex. This allows us to compute the cohomology and K-theory of the space of tilings. This is done completely for several one and two dimensional tilings, including the Penrose tilings. This approach also allows computation of the zeta function for the substitution. We discuss C^*-algebras related to these dynamical systems and show how the above methods may be used to compute the K-theory of these.

§1. Introduction

We study the dynamics of substitution tiling systems, the most famous example being the Penrose tilings of the plane. These have been studied by many other authors [Ga,GS,Ken,Mo,Ra1,ERob2,Rud1,S]. We work in $\mathbb{R}^d$, d-dimensional Euclidean space. We begin with a finite set of prototiles, each of which is a compact subset of $\mathbb{R}^d$ homeomorphic to the unit ball. In all of our examples, each prototile is actually a CW-complex with a unique d-dimensional cell which is its interior. That is, for $d = 2$, these are polygons, with edges and vertices. We make this structure a hypothesis for much of the paper. A tiling of $\mathbb{R}^d$ is a cover of $\mathbb{R}^d$ by sets, each of which is a translation of one of the prototiles (we do not allow rotations), so that they overlap only on their boundaries. We also assume a substitution rule: we have a constant $\lambda > 1$ and, for each prototile, a rule for subdividing it into pieces, each of which is another prototile, scaled down by factor λ^{-1}. This allows us to define a space of tilings, Ω, and a map $\omega : \Omega \to \Omega$ via the substitution rule. We make three important hypotheses: a finite type condition, a primitivity condition and a recognizability condition. These are explained in Section 2. This is the setting for several authors [Ken,ERob1,S].

We begin analysing the dynamics of (Ω, ω) in Section 3. First, we define a metric on Ω, as in [RW,ERob1,Rud2,S], and observe that ω is a homeomorphism. The rest of this section is devoted to showing that (Ω, ω) is topologically mixing and that it has a certain hyperbolic structure — it is a Smale space. Although the term "Smale space" was not used, these facts have been shown already by Kenyon [Ken] and E. Robinson [ERob2]. We supply the relevant arguments here for completeness.

One of our motivations here is in the study of operator algebras associated with tilings. This was first discussed by A. Connes [Co]. More recently, Kellendonk [Kel1,Kel2,Kel3] has considered these C^*-algebras, their K-theory and implications in the study of quasi-crystals. C^*-algebras arising from tilings have also been constructed by Mingo [Mi]. We will describe some results for these C^*-algebras, although it is our intention that much of the paper can be read with no knowledge of C^*-algebra theory.

Section 4 is the heart of the paper. From our substitution tiling, we construct a

natural CW-complex Γ_0 and an expansive map $\gamma_0 : \Gamma_0 \to \Gamma_0$. In fact, Γ_0 is very much like a branched d-manifold as described by R.F. Williams [W1,W2], but we do not attempt to put any differential structure on it. We let Ω_0 denote the inverse limit of the system obtained by repeating Γ_0 and γ_0, which has a natural shift homeomorphism ω_0. In Williams's terminology, this is a d-solenoid. Under a technical condition we show that (Ω_0, ω_0) is topologically conjugate to (Ω, ω). In the absence of this condition, we construct another CW-complex Γ_1, map γ_1 and inverse limit system (Ω_1, ω_1). This system is shown to be topologically conjugate to (Ω, ω), in general. From a computational point of view, the space Ω_1 is substantially more complicated, however.

In Section 5, we construct a shift of finite type which maps onto our system (Ω, ω), and we show the factor map is right closing in the sense of Boyle, Marcus and Trow [BMT].

The results of Section 4 have an important application. They make possible the computation of Cech cohomology and K-theory of the space Ω. (It also allows for computation of the action ω^* on these groups, although we will not go into this.) The motivation for such computations again comes from C^*-algebraic considerations, but we also believe these are relevant invariants for the dynamics. In Section 6, we make some basic remarks and observations on this problem. Our attempts to describe the algebraic topology of the space Ω are not the first. Geller and Propp [GP] introduced a "projective fundamental group" for shift dynamics. This is certainly in the same spirit as our work, although, at the moment, it is not clear to us whether or not there is a precise relation. In Section 7, we discuss the C^*-algebras and their K-theory. We describe how our work provides an alternative approach to the problems considered by Kellendonk [Kel1,Kel2,Kel3]. In Section 8, we discuss the special case $d = 1$ and the relation between our systems and substitution dynamical systems. In Section 9, we provide a formula for the zeta function for the system (Ω, ω). Finally, in Section 10, we present several example, including the Penrose tilings, and our computations of their cohomological invariants.

§2. Substitution Tilings

We use a substitution rule to define a collection of tilings. This rule also defines an inflation map on the collection.

A *tile* is a subset of $\mathbb{R}^d$ homeomorphic to a closed ball in $\mathbb{R}^d$. A *partial tiling* is a collection of tiles in $\mathbb{R}^d$ with pairwise disjoint interiors, and its *support* is the union of its tiles. A *tiling* is a partial tiling with support $\mathbb{R}^d$. When we need different tiles that look alike, we associate a label with each tile; in such cases, a tile is formally an ordered pair consisting of the set and the label. We think of a tiling T as a multi-valued function: for $u \in \mathbb{R}^d$ and $U \subseteq \mathbb{R}^d$, let

$$T(u) = \{t \in T \mid u \in t\}$$

$$T(U) = \bigcup_{u \in U} T(u).$$

Tilings T and T' are said to *agree* on U if $T(U) = T'(U)$. For any partial tiling T, we define expansions and translations of T by

$$\lambda T = \{\lambda t \mid t \in T\} \qquad \text{for } \lambda \in \mathbb{R}^+$$
$$T + u = \{t + u \mid t \in T\} \quad \text{for } u \in \mathbb{R}^d.$$

A collection of tilings Ω is now defined by a substitution rule. We call these tilings *substitution tilings*. Let $\{p_i \mid i = 1, \cdots, n_{pro}\}$ be a finite set of tiles, which we call *prototiles*. Let $\hat{\Omega}$ be the collection of all partial tilings that only contain translations of these prototiles. We assume there is an inflation constant $\lambda > 1$ and a substitution rule that associates to each prototile p_i a partial tiling P_i with support p_i such that λP_i is in $\hat{\Omega}$. An inflation map $\hat{\omega} : \hat{\Omega} \to \hat{\Omega}$ is defined by $\hat{\omega}(T) = \lambda \bigcup_{p_i + u \in T}(P_i + u)$. Let Ω be the collection of tilings T in $\hat{\Omega}$ such that for any $P \subseteq T$ with bounded support, we have $P \subseteq \hat{\omega}^n(\{p_i + u\})$ for some n, i, u. Also let $\omega = \hat{\omega}|_{\Omega}$; this maps a tiling in Ω to another in Ω obtained by applying the substitution rule to each tile and expanding by the factor λ.

The above definition is adapted from a standard one for symbolic substitution dynamical systems; see [Mo] for example. C. Radin defines substitution tilings in a similar way. He [Ra1,Ra2] and others [Mo,RRob] add bumps and dents to the edges of two-dimensional tiles so that they can fit together in a tiling precisely when it has a hierarchical structure

similar to that defined above. An alternative approach is to start with one tiling and form its orbit closure under translation, using the metric defined in the next section; we later show that this is equivalent to our definition.

We thus have a collection of tilings Ω and an inflation map ω defined on it. At this point, one can show that Ω is not empty and that $\omega(\Omega) = \Omega$; the proofs are below. In the next section, a metric will be defined on Ω, and we will want ω to be a topologically mixing homeomorphism. We therefore make three assumptions about the substitution rule, which are satisfied by most of the interesting examples.

The first assumption is that ω is one-to-one, for we need an inverse of ω. This assumption comes under the broad idea of "recognizability." Substitution rules that admit periodic tilings do not satisfy this, an example being the rule defined by dividing a square into four equal squares. Indeed, it is easy to show that if ω is one-to-one then Ω contains no periodic tilings; this well-known proof is below.

The second assumption is that the substitution is *primitive*: there is a fixed positive integer N_0 such that for each pair of prototiles p_i and p_j, the partial tiling $\hat{\omega}^{N_0}(\{p_i\})$ contains a translation of p_j. We need this for ω to be topologically mixing.

The third assumption is that Ω satisfies a *finite type condition*: for each positive real number r, there are only finitely many partial tilings up to translation that are subsets of tilings in Ω and whose supports have diameters less than r.

Proposition 2.1. Ω *is not empty.*

Proof. Since there are only a finite number of prototiles, it is not hard to see that for some prototile p_i and positive integer n, the partial tiling $\hat{\omega}^n(\{p_i\})$ contains a translation of p_i that does not meet the boundary of $\lambda^n p_i$. It readily follows that Ω contains a fixed point for ω^n. ▐

Proposition 2.2. $\omega(\Omega) = \Omega$

Proof. This proof is essentially the one given by S. Mozes for two-dimensional symbolic substitution systems [Mo]. That $\omega(\Omega) \subseteq \Omega$ is clear. To show that $\Omega \subseteq \omega(\Omega)$, suppose T is in Ω. Since T maps one-to-one into $\mathbb{Q}^d$, we can order its tiles, say by $\{t_i\}_{i=1}^{\infty}$. Now let

$S_0 = \{(s_k, n_k)\}_{k=1}^{\infty}$ where each s_k is a tile and n_k a positive integer such that $\hat{\omega}^{n_k}(\{s_k\})$ agrees with T on the ball of radius k centered at the origin. We now inductively define a sequence $\{S_i\}_{i=1}^{\infty}$ of subsequences of S_0 and a sequence of tiles $\{t_i'\}_{i=1}^{\infty}$. For $i = 1, 2, \cdots$, we note that there are only finitely many tiles t_i' such that $t_i \in \hat{\omega}(\{t_i'\})$. Therefore, we can choose a subsequence S_i of S_{i-1} and a tile t_i' such that for each term (s, n) of S_i, we have $t_i \in \hat{\omega}(\{t_i'\}) \subseteq \hat{\omega}^{n-1}(\{s\})$. Let T' be the collection of tiles appearing in the sequence $\{t_i'\}_{i=1}^{\infty}$. One then verifies that T' is in Ω and that $T = \omega(T')$. ∎

Proposition 2.3. Ω *contains no periodic tilings.*

Proof. Suppose T is in Ω. We must show that $T \neq T + v$ for each nonzero v in $\mathbb{R}^d$. Given such a v, let n be a positive integer such that some ball of diameter $\lambda^{-n}\|v\|$ is contained in each prototile. Let t be any tile in $\omega^{-n}(T)$. Then the interiors of t and $t + \lambda^{-n}v$ meet, and, since these tiles are bounded, $t \neq t + \lambda^{-n}v$; therefore $\omega^{-n}(T) \neq \omega^{-n}(T) + \lambda^{-n}v$ so that $T \neq T + v$. ∎

For convenience in some of the later developments, we will also make the assumption that each prototile carries the structure of a d-dimensional CW-complex [Do,Ma] with a unique d-cell which is the interior of the prototile. Each tile, which is a translation of a prototile, obtains the structure of a CW-complex in an obvious way.

We require that our substitution rule respect the cellular structure in the following sense. Suppose T is a tiling in Ω. Declare a subset of $\mathbb{R}^d$ to be a k-cell if it is contained in some tile in T and is a k-cell in that tile. We assume that this yields a well-defined CW-structure on $\mathbb{R}^d$. Thus, the substitution rule must only allow two tiles to be adjacent if their intersection defines the same set of faces in each tile — that is, vertices match to vertices, edges match to edges, and so forth.

It seems to the authors that this condition implies the finite type condition. Unfortunately, the method of proof we have in mind is very tedious. It is more convenient to assume both conditions as hypotheses.

§3. Dynamics

We give Ω a topology and show that ω is a topologically mixing homeomorphism. Then we prove that the resulting dynamical system is a Smale space. Other authors [S,ERob2] study the different *tiling dynamical system* consisting of Ω and the action of $\mathbb{R}^d$ by translation. We shall see that these two dynamical systems are closely related in that the unstable equivalence classes in the Smale space are the orbits in the tiling dynamical system.

We define a metric on the space of tilings Ω, in which two tilings are close if they almost agree on a large ball about the origin. Several equivalent metrics and topologies arise from this notion of distance [RW,ERob1,Rud2,S], and our definition is almost identical to Solomyak's [S]. We define, for any tilings T and T' of $\mathbb{R}^d$,

$$d(T,T') = \inf(\{1/\sqrt{2}\} \cup \{\epsilon \mid T+u \text{ and } T'+v \text{ agree on } B_{1/\epsilon}(0) \text{ for some } \|u\|, \|v\| < \epsilon\}).$$

Here $\|.\|$ is the usual norm on $\mathbb{R}^d$ and $B_r(x)$ is the open ball of radius r centered at x in $\mathbb{R}^d$. One can easily show that d is a metric. One can also show that the metric space (Ω, d) is sequentially compact, by an argument very similar to that given in [RW]; thus, (Ω, d) is a compact metric space.

E.A. Robinson [ERob2] and R. Kenyon [Ken] have noted that the inflation map is hyperbolic: with each inflation, tilings that agree around the origin become exponentially closer together, while those that are translations of each other become exponentially farther apart. Robinson has used an algebraic theory of de Bruijn to prove some measure theoretic results about this for the Penrose tilings [ERob2]. We extend this idea by showing, for substitution tilings, that (Ω, d, ω) is a Smale space. First we prove that the inflation map is topologically mixing; Robinson proved this for the Penrose tilings in a different way [ERob2].

Proposition 3.1. *ω is a topologically mixing homeomorphism of (Ω, d).*

Proof. We have shown that $\omega(\Omega) = \Omega$ and assumed that ω is one-to-one. One can easily show that

$$d(\omega(T), \omega(T')) \leq \lambda d(T, T')$$

$$d(\omega^{-1}(T), \omega^{-1}(T')) \leq \lambda d(T, T'),$$

and it follows that ω is bicontinuous.

To see that ω is topologically mixing, suppose we are given non-empty open sets $U, V \subseteq \Omega$. We require a positive integer N such that $\omega^n(U) \cap V$ is not empty for all $n \geq N$. Choose tilings $T \in U$, $T' \in V$ and $0 < \epsilon < 1/\sqrt{2}$ such that we have open balls $B_\epsilon^\Omega(T) \subseteq U$ and $B_\epsilon^\Omega(T') \subseteq V$. Since the substitution rule is primitive, there is a positive integer N_0 such that for each prototile p_i, the partial tiling $\hat{\omega}^{N_0}(\{p_i\})$ contains a translation of every prototile. Let $P \subseteq T'$ be a partial tiling that covers $B_{1/\epsilon}(0)$ and has bounded support. By definition of Ω, we have $P \subseteq \hat{\omega}^{N_1}(\{p_j + v\})$ for some positive integer N_1 and translation of a prototile $p_j + v$. Since there are only finitely many prototiles, each of which is bounded, we can choose a positive integer N_2 such that, for all $m \geq N_2$, $\omega^m(T)$ contains a tile that is a subset of $B_{\epsilon\lambda^m}(0)$. Let $N = N_0 + N_1 + N_2$ and suppose $n \geq N$. Since $n - N_1 - N_0 \geq N_2$, we have that $\omega^{n-N_1-N_0}(T)$ contains a tile $t_n \subseteq B_{\epsilon\lambda^{n-N_1-N_0}}(0)$. As t_n is a translation of a prototile, we have, by definition of N_0, that $\omega^{n-N_1}(T)$ contains some translation $p_j + u_n$ of p_j such that $p_j + u_n \subseteq B_{\epsilon\lambda^{n-N_1}}(0)$. It then follows from the definition of N_1 that $P + v_n \subseteq \omega^n(T)$ for some $\|v_n\| < \epsilon\lambda^n$. Thus, $P \subseteq \omega^n(T - \lambda^{-n}v_n)$, so that $d(\omega^n(T-\lambda^{-n}v_n), T') < \epsilon$; therefore $\omega^n(T-\lambda^{-n}v_n) \in B_\epsilon^\Omega(T') \subseteq V$. Since $\|-\lambda^{-n}v_n\| < \epsilon$, we also have that $\omega^n(T - \lambda^{-n}v_n) \in \omega^n(B_\epsilon^\Omega(T)) \subseteq \omega^n(U)$. Hence $\omega^n(U) \cap V$ is not empty. ∎

A Smale space is a compact metric space with a homeomorphism, that has a certain hyperbolic structure: each point in the space is the intersection of a local stable set, on which the homeomorphism is contracting, and a local unstable set, on which the homeomorphism is expanding; moreover, the product of these sets is homeomorphic to a neighbourhood of the point. The local stable set for a tiling consists of tilings that agree with it on a large ball about the origin, while the local unstable set consists of tilings that are small translations of it.

The formal definition of a Smale space may be found in [Put] or [Rue]; it will also become apparent in the discussion to follow. This definition requires *local canonical coordinates*, given by a continuous function defined on pairs of sufficiently close points; this is

written

$$[\,,\,] : \{(T, T') \mid T, T' \in \Omega, \quad d(T, T') < \epsilon_0\} \to \Omega,$$

where ϵ_0 is a small positive parameter to be determined. For tilings T and T' in Ω with $d(T, T') < \epsilon_0$, we let $[T, T'] = T' + v - u$ where $T + u$ and $T' + v$ agree on $B_{1/\epsilon_0}(0)$ and $\|u\|, \|v\| < \epsilon_0$.

The proofs that this is well-defined and continuous — and, indeed, all parts of the proof that (Ω, d, ω) is a Smale space — rest on the identities

$$(T + u)(v) = T(v - u) + u$$
$$T(u + v) = (T - v)(u) + v$$
$$T(u) + v = (T + v)(u + v)$$

and this easy consequence of the finite type condition:

Lemma 3.2. *There is a positive constant r such that if $T(u) - u = T(v) - v$ where $T \in \Omega$ and $\|u\|, \|v\| < r$ then $u = v$.*

Here, and throughout this sketch of the proof, we omit the tedious details, but comment that for the several uses of the lemma and identities, it suffices to let ϵ_0 be smaller than both $r/3\lambda$ and $1/\sqrt{2}$.

We require that

$$[T, T] = T$$
$$[[T, T'], T''] = [T, T'']$$
$$[T, [T', T'']] = [T, T'']$$
$$[\omega(T), \omega(T')] = \omega([T, T']),$$

for T, T', T'' in Ω, whenever both sides of the equation are defined. The first of these is clear. The other three use the lemma and the identities.

Now, for any tiling T in Ω and for any $0 < \epsilon < \epsilon_0$, we define the local stable and unstable sets:

$$V^S(T, \epsilon) = \{T' \in \Omega \mid [T, T'] = T' \quad \text{and} \quad d(T, T') < \epsilon\}$$
$$V^U(T, \epsilon) = \{T' \in \Omega \mid [T', T] = T' \quad \text{and} \quad d(T, T') < \epsilon\}.$$

It is easy to show that

$$\{T' \in \Omega \mid T' \text{ and } T \text{ agree on } B_{\epsilon^{-1}}(0)\} \subseteq V^S(T, \epsilon)$$
$$V^S(T, \epsilon) \subseteq \{T' \in \Omega \mid T' \text{ and } T \text{ agree on } B_{\epsilon^{-1} - \epsilon}(0)\}$$
$$V^U(T, \epsilon) = \{T' \in \Omega \mid T' = T + w \text{ for some } \|w\| < 2\epsilon\}.$$

We require that ω be contracting on $V^S(T, \epsilon)$ and that ω^{-1} be contracting on $V^U(T, \epsilon)$; that is,

$$d\left(\omega(T'), \omega(T'')\right) \leq \lambda_0 \, d(T', T''), \qquad T', T'' \in V^S(T, \epsilon)$$

$$d(\omega^{-1}(T'), \omega^{-1}(T'')) \leq \lambda_0 \, d(T', T''), \qquad T', T'' \in V^U(T, \epsilon),$$

where λ_0 is a positive constant less than 1; we fix $\lambda^{-1} < \lambda_0 < 1$. These are also proved using the lemma and the identities, but, for the first one, we also need ϵ_0 to be less than $\sqrt{1 - \lambda^{-1}\lambda_0^{-1}}$. Putting this all together yields

Theorem 3.3. (Ω, d, ω) *is a Smale space.*

Now, given a Smale space, one defines three equivalence relations on it: stable equivalence, unstable equivalence, and asymptotic equivalence. See [Put] for the development. We consider unstable equivalence here. Two tilings T and T' are unstably equivalent if

$$\lim_{n \to -\infty} d\left(\omega^n(T), \omega^n(T')\right) = 0.$$

One shows [Put] that the equivalence classes are given by

$$V^U(T) = \bigcup_{n=0}^{\infty} \omega^n \left(V^U\left(\omega^{-n}(T), \epsilon_0\right)\right),$$

and considers the unstable equivalence relation G_u as a topological groupoid: let

$$G_u^0 = \left\{(T, T') \in \Omega \times \Omega \mid T' \in V^U(T, \epsilon_0)\right\}$$

$$G_u^n = (\omega \times \omega)^n \left(G_u^0\right)$$

for each $n = 1, 2, 3, \cdots$, with the relative topology from $\Omega \times \Omega$, and give $G_u = \bigcup_{n=1}^{\infty} G_u^n$ the inductive limit topology. For the Smale space (Ω, d, ω), the unstable equivalence relation has a simple characterization:

Proposition 3.4. *Two tilings are unstably equivalent if and only if they are translations of each other; indeed, there is a topological groupoid isomorphism from $\Omega \times \mathbb{R}^d$ to G_u.*

Proof. If tilings T and T' are unstably equivalent then $\omega^{-n}(T') \in V^U(\omega^{-n}(T), \epsilon_0)$ for some positive integer n, so that $\omega^{-n}(T') = \omega^{-n}(T) + w$ for some $\|w\| < 2\epsilon$, and

$T' = T + \lambda^n w$; therefore T and T' are translations of each other. Read backwards, this establishes the converse. Therefore the unstable equivalence relation G_u is the orbit equivalence relation of the action of $\mathbb{R}^d$ by translation. Thus, there is a natural map φ from $\Omega \times \mathbb{R}^d$ onto G_u which sends (T, u) to $(T, T+u)$. Since Ω contains no periodic tilings, $\mathbb{R}^d$ acts freely on it, and φ is one-to-one.

One defines the product of points (T, u) and (T', u') in $\Omega \times \mathbb{R}^d$ whenever $T' = T + u$,

$$(T, u)(T', u') = (T, u + u'),$$

and one defines an inverse,

$$(T, u)^{-1} = (T + u, -u).$$

It is easy to verify that these are the correct definitions for φ to be a groupoid isomorphism, where G_u has its usual partially defined product and inverse.

We previously defined a topology on G_u, and we give $\Omega \times \mathbb{R}^d$ the product topology, where Ω has the topology determined by its metric, and $\mathbb{R}^d$ has the usual topology. A basis for the topology on $\Omega \times \mathbb{R}^d$ consists of sets of the form $\{(T', u') \mid T' \in B_\epsilon^\Omega(T), u' \in B_\epsilon(u)\}$ where $\epsilon > 0$, $T \in \Omega$, and $u \in \mathbb{R}^d$. One can show that a basis for the topology on G_u consists of sets of the form $\{(T', T' + u') \mid T' \in B_\epsilon^\Omega(T), u' \in B_\epsilon(u)\}$ where $\epsilon > 0$, $T \in \Omega$, and $u \in \mathbb{R}^d$; note that the topology here is not the same as that induced by the product topology on $\Omega \times \Omega$. Then, it is easy to see that φ is a homeomorphism. ▌

Corollary 3.5. *The action of $\mathbb{R}^d$ on Ω by translation is minimal.*

Proof. By Proposition 3.4, the $\mathbb{R}^d$ orbit of any T in Ω is the same as its unstable equivalence class. It follows from [Rue] that for any mixing Smale space, each unstable equivalence class is dense. Hence the $\mathbb{R}^d$ orbit of any T in Ω is dense and the conclusion follows. ▌

This result has some important consequences. As mentioned earlier, some authors [ERob1,S] obtain a single tiling via inflation and look at the closure of its orbit under the $\mathbb{R}^d$ action. By Corollary 3.5, this space of tilings is the same as Ω.

As another consequence, let T and T' by any two tilings in Ω. Fix an $r > 0$. Since the $\mathbb{R}^d$ orbit of T is dense, we have $d(T + u, T') < 1/r$ for some u in $\mathbb{R}^d$. It follows that, for some u' in $\mathbb{R}^d$, $T + u'$ and T' agree on $B_r(0)$. That is, if we fix T and consider any bounded partial tiling contained in some T', this occurs somewhere in T. We have this result:

Corollary 3.6. *Any bounded partial tiling that occurs in a tiling in Ω occurs somewhere in every tiling in Ω.*

Some authors discuss the more general property that the partial tiling occurs infinitely often in every tiling and with bounded gaps between the occurrences; see [S] for example.

§4. Inverse Limit Spaces

We construct a compact Hausdorff topological space Γ_0 by gluing together the prototiles in all ways in which the substitution rule allows them to be adjacent. When the prototiles are d-cells, Γ_0 is a branched d-dimensional manifold that comes from a cell complex. The inflation map induces a continuous surjection γ_0 on the space, with respect to which we take the inverse limit to obtain the space Ω_0. Under an additional hypothesis on the substitution rule, this inverse limit space with the right shift map is topologically conjugate to (Ω, ω). As we shall see, the hypothesis may be dropped by using a larger complex, Γ_1.

We begin by defining Γ_k for $k = 0, 1$. If t is a tile in a tiling T, we let $T^{(0)}(t) = \{t\}$ and $T^{(1)}(t) = T(t)$. Thus $T^{(k)}(t)$ is the set of tiles in T that are within k tiles of t; in fact, everything we do works for every nonnegative integer k, but there is no need to go beyond $k = 1$. Now, consider the space $\Omega \times \mathbb{R}^d$ with the product topology, where Ω has the discrete topology and $\mathbb{R}^d$ has the usual topology. We let $\sim_k$ be the smallest equivalence relation on $\Omega \times \mathbb{R}^d$ that relates (T_1, u_1) to (T_2, u_2) whenever $T_1^{(k)}(t_1) - u_1 = T_2^{(k)}(t_2) - u_2$ for some tiles $u_1 \in t_1 \in T_1$ and $u_2 \in t_2 \in T_2$. The equivalence class of a point (T, u) is denoted $(T, u)_k$. Let $\Gamma_k = \Omega \times \mathbb{R}^d / \sim_k$ with the quotient topology.

We use Ω with the discrete topology in the above definition because this is the most convenient way to ensure that every possible small patch of tiles is represented. Alternatively, we could use a single tiling to define the equivalence relation just on $\mathbb{R}^d$, and then use Corollary 3.6 to show that the corresponding definition of Γ_k is independent of the tiling selected.

Let us see what this definition means when the prototiles are 2-cells. The cell complex can be described by drawing a picture of each cell and naming its vertices and edges to show how the cells are joined together. To do this, first give a name to each tile in each tiling in Ω so that two tiles $t_1 \in T_1$ and $t_2 \in T_2$ have the same name if and only if $T_1^{(k)}(t_1)$ and $T_2^{(k)}(t_2)$ are the same up to translation. Thus, for $k = 0$, we are just naming the different prototiles, and, for $k = 1$, we are naming 'collared tiles'; see, for example, Fig. 7. Of course, by Corollary 3.6 and the finite type condition, it suffices to do this just for those tiles that cover a large open ball in a single tiling; but the ball must be large enough that every possible adjacency of two named tiles is represented. Draw a picture of each named tile; these are the cells that make up the complex. Then name their vertices and edges in the obvious way: start with any edge of any cell, give it a name, and give this same name to all the edges of the other cells that may be adjacent to the given edge; keep repeating this for each edge that has been so named, until no more edges may be given this name. Repeat this for the other edges and vertices, using different names, until all have been named. It should be clear that this procedure gives the right identifications.

Formally, the cell complex is a collection of cells, and the space Γ_k is the set of points in these cells. As it is clear that Γ_k may be given this structure, it is convenient to also call Γ_k a cell complex. We also note that all the results in this section would still be true if the prototiles did not have the assumed cellular structure.

Proposition 4.1. Γ_k *is a compact Hausdorff space.*

Proof. (outline) By Corollary 3.6 and the finite type condition, Γ_k is the image of a compact subset $\{T\} \times \overline{B_r(0)}$ of $\Omega \times \mathbb{R}^d$ under the continuous quotient map that sends (T, u) to $(T, u)_k$; therefore Γ_k is compact. To show that for any distinct points $(T_1, u_1)_k$ and $(T_2, u_2)_k$ there are disjoint neighbourhoods of them, one considers separately the case

in which both u_1 and u_2 lie on boundaries of tiles in T_1 and T_2 respectively. ▮

Proposition 4.2. *The inflation map ω induces a continuous map γ_k from Γ_k onto Γ_k defined by $\gamma_k((T, u)_k) = (\omega(T), \lambda u)_k$.*

Proof. Observe that if $T_1^{(k)}(t_1) - u_1 = T_2^{(k)}(t_2) - u_2$ with $u_1 \in t_1 \in T_1$ and $u_2 \in t_2 \in T_2$ then $t_1 - u_1 = t_2 - u_2$ so that we can choose tiles $\lambda u_1 \in t_1' \in \hat{\omega}(\{t_1\})$ and $\lambda u_2 \in t_2' \in \hat{\omega}(\{t_2\})$ such that $\omega(T_1)^{(k)}(t_1') - \lambda_1 u_1 = \omega(T_2)^{(k)}(t_2') - \lambda_2 u_2$. That γ_k is well-defined follows by a finite induction. Since the map on $\Omega \times \mathbb{R}^d$ that sends (T, u) to $(\omega(T), \lambda u)$ is continuous, so also is the map γ_k, obtained by passing to the quotient. γ_k is surjective since $(\omega^{-1}(T), \lambda^{-1}u)_k$ is a preimage of $(T, u)_k$. ▮

We now define an inverse limit space, which is similar to the solenoids studied by R.F. Williams [W2]. Let $\Omega_k = \varprojlim \Gamma_k$, the inverse limit relative to the bonding map γ_k. Thus, Ω_k consists of all infinite sequences $\{x_i\}_{i=0}^{\infty}$ of points in Γ_k such that $\gamma_k(x_i) = x_{i-1}$ for $i = 1, 2, \cdots$, with the relative topology from the product topology. A basis for this topology is the collection of sets of the form $B_{U,n}^{\Omega_k} = \{x \in \Omega_k \mid x_i \in \gamma_k^{n-i}(U) \text{ for } i = 0, 1, \cdots, n\}$, where $U \subseteq \Gamma_k$ is open and n is a nonnegative integer. We define a right shift $\omega_k : \Omega_k \to \Omega_k$ by $\omega_k(x)_i = \gamma_k(x_i)$ for $i = 0, 1, 2, \cdots$. This map has an inverse, defined by $\omega_k^{-1}(x)_i = x_{i+1}$ for $i = 0, 1, 2, \cdots$. Clearly, ω_k is a homeomorphism, so that it generates a $\mathbb{Z}$ action on Ω_k. We thus have a dynamical system (Ω_k, ω_k). We will show that (Ω, ω) is topologically conjugate to (Ω_1, ω_1).

(Ω, ω) is also topologically conjugate to (Ω_0, ω_0) under the additional hypothesis of J. Kellendonk that the substitution *forces its border*. This means there is a fixed positive integer N such that for any tile t, and any two tilings T and T' containing t, $\omega^N(T)$ and $\omega^N(T')$ coincide, not just on $\hat{\omega}^N(\{t\})$, but also on all tiles that meet $\hat{\omega}^N(\{t\})$. This condition is satisfied by the Penrose tilings, for example [Kel1].

Theorem 4.3. *The dynamical systems (Ω, ω) and (Ω_1, ω_1) are topologically conjugate. If the substitution forces its border then (Ω, ω) and (Ω_0, ω_0) are topologically conjugate too.*

Proof. First, we show that (Ω, ω) is topologically conjugate to (Ω_1, ω_1). Define π :

$\Omega \to \Omega_1$ by $\pi(T) = x = \{x_i\}_{i=0}^{\infty}$, where $x_i = (\omega^{-i}(T), 0)_1$. This is well-defined since $\gamma_1(x_i) = x_{i-1}$. We claim π is a homeomorphism that conjugates the two actions.

Let us describe the construction for the inverse of π. This will not be rigourous – the precise proofs of surjectivity and injectivity of π will follow. Let $x = \{x_i\}_0^{\infty}$ be any element of Ω_0. We wish to find a tiling T such that $\pi(T) = x$. Since $x_0 = (T, 0)_0$, x_0 specifies the tile in T that contains the origin, which we denote t_0. Similarly x_1 specifies the tile in $\omega^{-1}(T)$ that contains the origin, which we denote t_1. This means T contains the partial tiling $\hat{\omega}(t_1)$. This partial tiling contains the tile t_0, since $\gamma_0(x_1) = x_0$. Continuing in this way, we obtain a nested sequence of partial tilings. If the union of these covers the plane, then this is T. Even if this is not the case, the forcing the border condition guarantees that there is a unique tiling T which contains this union. More generally, if we repeat this argument, replacing "tiles" with "collared tiles," then the union above does cover the plane.

To see that π is one-to-one, suppose $\pi(T_1) = \pi(T_2)$. Let $r = \inf \text{dist}(t, \partial \cup T^{(1)}(t))$, taken over all $t \in T \in \Omega$. This is positive since Ω satisfies a finite type condition. Suppose v is in $\mathbb{R}^d$. Let n be a positive integer such that $r\lambda^n > \|v\|$. Now, $\pi(T_1)_n = \pi(T_2)_n$, and a finite induction reduces to the case $\omega^{-n}(T_1)^{(1)}(t_1) = \omega^{-n}(T_2)^{(1)}(t_2)$ where t_1 and t_2 contain the origin. Since these are collared tiles, this implies that $\omega^{-n}(T_1)$ and $\omega^{-n}(T_2)$ agree on $B_r(0)$, and, consequently, T_1 and T_2 agree on $B_{r\lambda^n}(0)$, which contains v. Therefore T_1 and T_2 agree everywhere, and π is one-to-one.

To see that π is onto, suppose we are given $x = \{(T_i, u_i)_1\}_{i=0}^{\infty} \in \Omega_1$. Let $T = \bigcup_{i=0}^{\infty} \hat{\omega}^i(\bigcap_{u_i \in t \in T_i} T_i^{(1)}(t) - u_i)$. One verifies that this is a partial tiling, and, using the r in the previous paragraph, one sees that it is actually a tiling. It is then clear that T is in Ω, and one easily verifies that $\pi(T) = x$.

One can show that π is bicontinuous by standard methods, and it is easy to check that $\pi \circ \omega = \omega_1 \circ \pi$, as required. Therefore (Ω, ω) is topologically conjugate to (Ω_1, ω_1).

Second, we assume that the substitution rule forces its border, and show that (Ω_1, ω_1) is topologically conjugate to (Ω_0, ω_0). There is a natural map $f : \Omega_1 \to \Omega_0$ defined by $f((T, u)_1) = (T, u)_0$. Observe that if $T_1^{(1)}(t_1) - u_1 = T_2^{(1)}(t_2) - u_2$ then $T_1^{(0)}(t_1) - u_1 = T_2^{(0)}(t_2) - u_2$; a finite induction then shows that f is well-defined. Clearly, f is onto, but is

generally not one-to-one. A simple calculation shows that $\gamma_0 \circ f = f \circ \gamma_1$, so that the map f determines a map $F : \Omega_1 \to \Omega_0$, defined by $F(x)_i = f(x_i)$ for $i = 0, 1, \cdots$. We claim F is a homeomorphism that conjugates the actions of ω_1 and ω_0.

To show that F is one-to-one, we require the forcing the border condition. Suppose $F(\{x_i\}_{i=0}^\infty) = F(\{y_i\}_{i=0}^\infty)$ where $x_i = (T_i, u_i)_1$ and $y_i = (T_i', u_i')_1$. We must show that $x_j = y_j$ for each j. Observe that if $v_1 \in s_1 \in S_1$ and $v_2 \in s_2 \in S_2$, where S_1 and S_2 are any tilings, and $S_1^{(0)}(s_1) - v_1 = S_2^{(0)}(s_2) - v_2$, then the forcing the border condition implies that $(\omega^N(S_1), \lambda^N v_1)_1 = (\omega^N(S_2), \lambda^N v_2)_1$. Since $(T_{j+N}, u_{j+N})_0 = (T_{j+N}', u_{j+N}')_0$, it follows from this observation and a finite induction that $(\omega^N(T_{j+N}), \lambda^N u_{j+N})_1 = (\omega^N(T_{j+N}), \lambda^N u_{j+N})_1$; that is, $(T_j, u_j)_1 = (T_j', u_j')_1$, as required. Therefore F is one-to-one.

The compactness of Ω implies that F is onto, by an argument similar to one given in [Kel1]. Suppose we are given a point $\{(T_i, u_i)_0\}_{i=0}^\infty$ in Ω_0. Since Ω is compact, the sequence of tilings $\{\omega^n(T_n - u_n)\}_{n=0}^\infty$ has a subsequence $\{\omega^{n_k}(T_{n_k} - u_{n_k})\}_{k=0}^\infty$ that converges to some tiling T in Ω. Now, $\{(\omega^{-i}(T), 0)_1\}_{i=0}^\infty$ is in Ω_1, and we claim that it is a preimage of $\{(T_i, u_i)_0\}_{i=0}^\infty$ under F. We must show that $(\omega^{-i}(T), 0)_0 = (T_i, u_i)_0$ for $i = 0, 1, \cdots$; so fix such an i. A simple calculation shows that $(\omega^n(T_n - u_n), 0)_0 = (T_0, u_0)_0$ for all n. It follows by the finite type condition that in all the tilings $\omega^n(T_n - u_n)$, only finitely many different tiles contain the origin. Therefore, given any positive ϵ, there is a k sufficiently large so that $\omega^{n_k}(T_{n_k} - u_{n_k})$ agrees with T on $B_{1/\epsilon}(0)$. Let R be a constant greater than the diameter of each prototile, and choose k so that $n_k \geq i$ and $\omega^{n_k}(T_{n_k} - u_{n_k})$ agrees with T on $B_{\lambda^i R}(0)$. Then $(\omega^{-i}(T), 0)_0 = (\omega^{n_k-i}(T_{n_k} - u_{n_k}), 0)_0 = \gamma_0^{n_k-i}(T_{n_k}, u_{n_k})_0 = (T_i, u_i)_0$, as required.

Clearly F is bicontinuous, and a simple calculation shows that it conjugates the actions of ω_0 and ω_1. Hence (Ω_0, ω_0) is topologically conjugate to (Ω_1, ω_1) which, as we have already seen, is topologically conjugate to (Ω, ω). ∎

§5. Symbolic Dynamics

We have seen that the space of tilings with the inflation map is the same as the inverse limit of a cell complex with the right shift map; indeed, let k be 0 or 1 such that the topological conjugacy of the previous section holds. Now, since the d-cells partition the cell complex Γ_k, and the bonding map γ_k respects the cellular structure, one can use symbolic dynamics to study the dynamics of the inverse limit space, where each symbol corresponds to a d-cell. We will construct a shift of finite type (Σ, σ) that maps continuously onto (Ω_k, ω_k) by a semi-conjugacy α. This is not one-to-one because the cells overlap on their boundaries.

Observe that by partitioning the cell complex into its component cells, we have effectively constructed a Markov partition of Ω_k or Ω; see [Bow] for a definition. R. Kenyon has previously used Markov partitions in this context [Ken]. Each cell defines a rectangle in the Markov partition, consisting of those tilings in which a tile corresponding to the given cell contains the origin. We could begin by proving we have a Markov partition, and then use it to define the shift of finite type, as is done in [Bowen]; however, there is a minor difficulty in that we have not defined the local canonical coordinates for every pair of points in a rectangle. So we begin with the shift of finite type.

Let us number the d-cells in Γ_k by $1, 2, \cdots, n$. Define an n by n matrix $A = (A_{ij})$ by letting A_{ij} be the number of times cell i occurs in the inflation of cell j. This defines a shift of finite type (Σ, σ): Let G be the directed graph with vertices $\{1, 2, \cdots, n\}$ and A_{ij} edges from vertex i to vertex j. Let E be the set of edges and give it the discrete topology. One edge is said to *follow* another if the tail of the one is at the tip of the other. Let $\Sigma = \{e \in E^{\mathbb{Z}} \mid e_{i+1} \text{ follows } e_i \text{ for all } i\}$, with the product topology. Define the right shift map $\sigma : \Sigma \to \Sigma$ by letting $\sigma(e)_i = e_{i-1}$ for all i. Most of the notation here is borrowed from [BMT].

We comment here that the matrix A will show up in a different context later, when we calculate cohomology groups. Indeed, since the inflation map γ_k respects the cellular structure of Γ_k, it induces a natural homomorphism on the group of i-chains, for $i = 0, 1, \cdots, d$. Each homomorphism can be represented by a c_i by c_i matrix of integers, where

c_i is the number of i-cells in the complex. When $i = d$, this is just the matrix A. As we will look more at cohomology than homology, we will actually use the transposes of these matrices.

We now define the surjection $\alpha : \Sigma \to \Omega_k$. Suppose $\{e_i\}_{i=-\infty}^{\infty}$ is a point in Σ. For each $i = 0, 1, 2, \cdots$, the edge e_i determines the cell at its tail, and it is easy to see that the sequence $e_i, e_{i-1}, e_{i-2}, \cdots$ determines a point in this cell; we define $\alpha(e)_i$ to be the corresponding point in Γ_k. Clearly $\gamma_k(\alpha(e)_i) = \alpha(e)_{i-1}$ for $i = 1, 2, \cdots$ so that $\alpha(e)$ is in Ω_k.

To any point $x \in \Omega_k$ there corresponds a tiling T in Ω. It is easy to see that for any tile t_0 in T that contains the origin, there is a doubly infinite sequence of tiles $\{t_i\}_{i=-\infty}^{\infty}$ such that $0 \in t_i \in \hat{\omega}(\{t_{i+1}\}) \subseteq \omega^{-i}(T)$ for all i; this sequence determines a point in Σ which α sends to x. Note that there is at least one such sequence for each tile in T that contains the origin. Therefore α is onto, but is not one-to-one. One easily checks that α is continuous and that $\alpha \circ \sigma = \omega_k \circ \alpha$, as required.

Now, suppose two points e and e' in Σ are *left asymptotic*; that is, there is an integer M such that $e_m = e'_m$ for all $m \leq M$. Suppose, also, that $\alpha(e) = \alpha(e')$, so that both points correspond to the same tiling T. Then, one can easily show inductively that $e_m = e'_m$ for all $m > M$; therefore $e = e'$. Thus, α is a continuous shift-commuting map that never identifies two distinct left asymptotic points. Such a map is called *right closing* [BMT], although, technically, this term is only used when both the map's domain and range are shifts of finite type. Thus, we have shown:

Theorem 5.1. *The shift of finite type* (Σ, σ) *maps continuously onto* (Ω_k, ω_k) *by the right closing semi-conjugacy* α.

§6. Cohomology and K-theory

In this section, we make some remarks concerning various cohomology theories for the space Ω. Our interest will be in the K-theory of Ω, $K^i(\Omega)$, $i = 0, 1$, and the Cech cohomology with integer coefficients, $H^i(\Omega, \mathbb{Z})$, $i = 0, 1, 2, \cdots$. There are several motivations. First of all, these are invariants which may contain interesting dynamic information – we will have more to say about this at the end of the section. Second, these are readily computable for tilings of interest, as we explain here and demonstrate in Section 10. Finally, these are relevant to the study of some associated C^*-algebras. We will address this in the next section.

For the rest of this section we will assume the CW-complex structure on our prototiles as described in Section 2. The spaces Γ_0 and Γ_1 are both d-dimensional CW-complexes in the obvious way. The maps γ_0 and γ_1 of 4.2 are then cellular maps. The homology and cohomology of Γ_0 and Γ_1 are both readily computed as well as the homomorphisms induced by γ_0 and γ_1. If we now turn to the space Ω, which is homeomorphic to Ω_1 and to Ω_0, if the substitution forces its border, we may use 4.3 to compute its cohomology. Homology, on the other hand, is not as well-behaved under inverse limits and so we abandon it at this point.

Theorem 6.1. *For any substitution tiling system, we have that $H^i(\Omega, \mathbb{Z})$ is isomorphic to the direct limit of the system of abelian groups*

$$H^i(\Gamma_1, \mathbb{Z}) \xrightarrow[\gamma_1^*]{} H^i(\Gamma_1, \mathbb{Z}) \xrightarrow[\gamma_1^*]{} H^i(\Gamma_1, \mathbb{Z}) \longrightarrow \cdots,$$

for $i = 0, 1, 2, \cdots$.

Also, $K^i(\Omega)$ is isomorphic to the direct limit of the system

$$K^i(\Gamma_1) \xrightarrow[\gamma_1^*]{} K^i(\Gamma_1) \xrightarrow[\gamma_1^*]{} K^i(\Gamma_1) \longrightarrow \cdots,$$

for $i = 0, 1$.

If the substitution forces its border, then the same conclusions hold replacing Γ_1 and γ_1 by Γ_0 and γ_0.

Remark. The statement is also valid without the hypothesis of the CW-structure on the tilings, but we will not prove this. Given 4.3, these are fairly standard facts in topology, but we provide a proof (at least for the first statement) for completeness.

Proof. By 4.3, Ω is homeomorphic to Ω_k, for $k = 1$ and for $k = 0$ if the substitution forces its border. The definition of Cech cohomology of Ω_k is to consider all finite open covers of Ω_k, and to form the inductive system of the cohomology groups of their nerves and to take the direct limit of this system [Do]. Now Ω_k is defined as the inverse limit of Γ_k under the map γ_k. There is an obvious way to use the cellular structure on Γ_k to find a finite open cover of Γ_k whose nerve is homeomorphic to Γ_k. For each Γ_k in the inverse system, we may pull back this open cover to Ω_k. Since Ω_k is the inverse limit, these open covers generate the topology of Ω_k and so the direct limit of the cohomology groups is isomorphic to the cohomology of Ω_k.

The proof for the K-theory also relies on Ω_k being the inverse limit of the system with Γ_k and γ_k. For a proof that

$$K^*(\Omega_k) \cong \varinjlim_{\gamma_k^*} K^*(\Gamma_k),$$

we refer the reader to [Bla], although the proof given there is for the C^*-algebras of continuous functions $C(\Gamma_k)$ and $C(\Omega_k)$. ▊

It is also worth observing that the maps on cohomology and K-theory induced by the homeomorphism ω of Ω correspond, under the above isomorphisms, to the natural shift automorphism of the direct limit groups.

Since many examples of interest (and all the ones we consider in Section 10) occur in dimension $d = 1$ or $d = 2$, we also want to make note of the following.

Proposition 6.2. *If Γ is any one or two dimensional CW-complex, then there are natural isomorphisms*

$$K^0(\Gamma) \cong H^0(\Gamma) \oplus H^2(\Gamma)$$

$$K^1(\Gamma) \cong H^1(\Gamma).$$

Proof. We will consider the case when Γ is two-dimensional. Let Γ^0 and Γ^1 denote the 0-skeleton and 1-skeleton, respectively. Then Γ^0 consists of V points, Γ^1/Γ^0 is the wedge

of E circles and Γ/Γ^1 is the wedge of F two-spheres. Thus, we have [At]

$$K^0(\Gamma^0) \cong \mathbb{Z}^V \qquad\qquad K^1(\Gamma^0) = 0$$

$$K_r^0(\Gamma^1/\Gamma^0) = 0 \qquad\qquad K_r^1(\Gamma^1/\Gamma^0) \cong \mathbb{Z}^E$$

$$K_r^0(\Gamma/\Gamma^1) \cong \mathbb{Z}^F \qquad\qquad K_r^1(\Gamma/\Gamma^1) = 0.$$

The six-term exact sequence in K-theory [At] for the sequence

$$\Gamma^0 \longrightarrow \Gamma^1 \longrightarrow \Gamma^1/\Gamma^0$$

is

$$
\begin{array}{ccccc}
K^0(\Gamma^0) & \longleftarrow & K^0(\Gamma^1) & \longleftarrow & K_r^0(\Gamma^1/\Gamma^0) \\
\downarrow & & & & \uparrow \\
K_r^1(\Gamma^1/\Gamma^0) & \longrightarrow & K^1(\Gamma^1) & \longrightarrow & K^1(\Gamma^0).
\end{array}
$$

Under the isomorphisms above, this becomes

$$
\begin{array}{ccccc}
\mathbb{Z}^V & \longleftarrow & K^0(\Gamma^1) & \longleftarrow & 0 \\
\downarrow & & & & \uparrow \\
\mathbb{Z}^E & \longrightarrow & K^1(\Gamma^1) & \longleftarrow & 0.
\end{array}
$$

We leave it as an exercise to verify that, after the identifications $K^0(\Gamma^0) \cong \mathbb{Z}^V$ and $K^1(\Gamma^1) \cong \mathbb{Z}^E$, the connecting homomorphism coincides with the (co)boundary map

$$\partial_1' : \mathbb{Z}^V \longrightarrow \mathbb{Z}^E$$

in cellular cohomology. Hence, we have

$$K^0(\Gamma^1) \cong \ker(\partial_1')$$

$$K^1(\Gamma^1) \cong \operatorname{coker}(\partial_1').$$

In a similar way, from the sequence

$$\Gamma^1 \longrightarrow \Gamma \longrightarrow \Gamma/\Gamma^1$$

we obtain the six-term exact sequence

$$
\begin{array}{ccccc}
K^0(\Gamma^1) & \longleftarrow & K^0(\Gamma) & \longleftarrow & \mathbb{Z}^F \\
\downarrow & & & & \uparrow \\
0 & \longrightarrow & K^1(\Gamma) & \longrightarrow & K^1(\Gamma^1).
\end{array}
$$

From our earlier result $K^1(\Gamma^1) \cong \operatorname{coker}(\partial_1')$, we have a surjection $\mathbb{Z}^E \longrightarrow K^1(\Gamma^1)$, whose kernel is $\partial_1'(\mathbb{Z}^V)$. We leave it to the reader to verify that composing this surjection with the connecting map above yields the (co)boundary map $\partial_2' : \mathbb{Z}^E \longrightarrow \mathbb{Z}^F$. We have the commutative diagram (with top row exact)

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K^1(\Gamma) & \longrightarrow & K^1(\Gamma^1) & \longrightarrow & \mathbb{Z}^F \\
& & & & \uparrow & & \| \\
& & \mathbb{Z}^V & \xrightarrow{\ \partial_1'\ } & \mathbb{Z}^E & \xrightarrow{\ \partial_2'\ } & \mathbb{Z}^F.
\end{array}
$$

Now a standard diagram chase shows that

$$K^1(\Gamma) \cong \ker(\partial_2')/\operatorname{Im}(\partial_1') \cong H^1(\Gamma).$$

Turning now to $K^0(\Gamma)$, we use the fact that $K^0(\Gamma^1) \cong \ker(\partial_1') \cong H^0(\Gamma)$ which is free abelian and hence a summand of $K^0(\Gamma)$. From the computation we have just completed we also see that the image of $K^1(\Gamma^1)$ under the connecting homomorphism to $\mathbb{Z}^F$ is the same as the image of $\mathbb{Z}^E$ under ∂_2'. Hence, we have

$$K^0(\Gamma) \cong H^0(\Gamma) \oplus \left(\mathbb{Z}^F / \operatorname{Im}(\partial_2') \right)$$

$$\cong H^0(\Gamma) \oplus H^2(\Gamma). \quad \blacksquare$$

Theorem 6.3. *For a substitution tiling system in dimension $d = 1$ or 2, we have*

$$K^0(\Omega) \cong H^0(\Omega) \oplus H^2(\Omega)$$

$$K^1(\Omega) \cong H^1(\Omega).$$

Proof. This follows at once from 6.1 and 6.2. ∎

We close this section with some speculations on the rôle of the cohomology of Ω. Let us start with a simple example of a (non-substitution) space of tilings of the plane. Suppose our lone prototile is a unit square with sides parallel to the co-ordinate axes. Assume it has the obvious structure as a *CW*-complex and let Ω be the space of all tilings satisfying the *CW*-structure condition. Each such tiling is uniquely determined by the point in the square determined by the location of the origin. It is easy to see, in this way, that Ω is homeomorphic to the two-torus. Hence, we can compute the homology of Ω; $H_1(\Omega) \cong \mathbb{Z}^2$. In fact, in this example one can see that this is exactly the translational symmetry group of the tiling. In the case of our substitution tilings, one can show $H_1(\Omega) = 0$ as a consequence of aperiodicity. What about cohomology?

For a relatively simple space such as the two-torus, homology and cohomology are related fairly closely. This is not true for the solenoid-like spaces of substitution tilings. Our point of view here is that it is Cech cohomology which is really measuring the almost periodic structure of these tilings.

We will not make this remark precise, except to make the following observation. Let us make use of Bruchlinsky's Theorem (see [Hu]) which asserts that $H^1(\Omega, \mathbb{Z})$ is isomorphic to the group

$$C(\Omega, \mathbb{T}) / \left\{ \exp(2\pi i f) \mid f \in C(\Omega, \mathbb{R}) \right\},$$

where $\mathbb{T}$ is the unit circle in the complex plane and $C(\Omega, Y)$ denotes the set of continuous functions from Ω to Y. If f is any element of $C(\Omega, \mathbb{T})$ and T is any tiling in Ω, we may define $f^T : \mathbb{R}^d \to \mathbb{T}$ by $f^T(u) = f(T + u)$. It is a straightforward application of the Arzela-Ascoli theorem to see that f^T is an almost periodic function on $\mathbb{R}^d$ [HR].

§7. C^*-Algebras and Their K-Theory

Since (Ω, ω) is a mixing Smale space, we may associate several C^*-algebras to it [Put]. Our main interest here is in the C^*-algebra of the unstable equivalence relation, $U = C^*(G_u)$. This is partly because unstable equivalence is just translational equivalence (3.4). In fact, we obtain from 3.4 that

$$U \cong C(\Omega) \bowtie \mathbb{R}^d,$$

the crossed product C^*-algebra. As an immediate consequence of Connes's analogue of the Thom isomorphism, we have

Theorem 7.1. *For $i = 0, 1$, we have*

$$K_i(U) \cong K^{i+d}(\Omega).$$

We will not discuss the order structure on $K_0(U)$ here, except to say that it seems likely that one can make use of the map α of 5.1 to relate $K_0(U)$ with the K-theory of the unstable C^*-algebra of the shift of finite type (Σ, σ) to obtain some information. This is analogous to computations of Kellendonk [Kel2].

As we mentioned in the introduction, the computation of $K_0(U)$ by application of 7.1, 6.3 and 6.1 duplicates results obtained by Kellendonk [Kel2] using other methods.

We now want to describe a concrete relationship between our work and Kellendonk's. Let us begin with a substitution tiling as we have been considering.

For each prototile p, fix a point $x^{(p)}$ in its interior. This is referred to as a "puncture". For each tile $t = p + x_0$, its puncture is the point $x^{(p)} + x_0$, which we denote $x^{(t)}$. We define

$$\Omega^P = \Omega^{\text{punc}} = \left\{ T \in \Omega \mid 0 = x^{(T(0))} \right\}.$$

That is, Ω^P is the collection of all tilings such that 0 is a puncture. A word is in order about our notation $0 = x^{(T(0))}$. It may be that $T(0)$ has more than one value, but in this case, 0 lies on the boundary of the tiles containing it and hence is not a puncture, since the punctures are in the interior of the tiles.

Proposition 7.2. *In the groupoid G_u, Ω^P is an abstract transversal in the sense of Muhly et al. [MRW]. Equivalently, Ω^P is transverse to the $\mathbb{R}^d$-action on Ω by translation, each $\mathbb{R}^d$-orbit meets Ω^P and, for some $\epsilon > 0$,*

$$\left\{ T + u \mid T \in \Omega^P, \; \|u\| < \epsilon \right\} \quad \text{is open in } \Omega.$$

Proof. The equivalence of the two statements is fairly routine in the context of [MRW] and using 3.3. We leave this to the reader, and we prove the second part only. To do this, we take T_0 in Ω^P and find a neighbourhood W of T_0 in Ω^P and an $\epsilon > 0$ such that the map

$$W \times \left\{ r \in \mathbb{R}^d \mid \|r\| < \epsilon \right\} \longrightarrow \Omega$$

defined by

$$(T, r) \longrightarrow T + r$$

is a homeomorphism to its image. Let $W = \left\{ T \in \Omega^P \mid T(0) = T_0(0) \right\}$ and choose ϵ sufficiently small so that the ball of radius 2ϵ around $x^{T_0(0)}$ is contained in $T_0(0)$. We leave the verification of the desired conclusion to the reader. ∎

Let us take a few moments to explain the notion of abstract transversal to readers in dynamics who may be unfamiliar with this concept of groupoids. If one considers a flow with no periodic orbits (*i.e.* a free $\mathbb{R}$-action) on a space, one can hope to find a closed transversal to the flow and study the Poincaré first return map. The difficulties are well-known: first a point may fail to return to the transversal; second, for $\mathbb{R}^d$-actions, the notion of "first" return is not so clear. However, we may view the $\mathbb{R}^d$ action as a topological equivalence relation (*i.e.* a principal groupoid) whose equivalence classes are simply the orbits. One can reduce the equivalence relation on a closed transversal as in Muhly *et al.* [MRW] in an obvious way. The conditions of the theorem are needed to guarantee that this groupoid has the necessary structure.

In the case of the transversal Ω^P in Ω, the reduction of the groupoid G_u on Ω^P is just the translational equivalence relation on Ω^P. This is denoted by $\mathcal{R}$ in [Kel1]. By the results of Muhly *et al.*, our C^*-algebra $U = C^*(G_u)$ and Kellendonk's $A_T = C^*(\mathcal{R})$ are strongly Morita equivalent. In particular, they have isomorphic K-theory [Ri].

In the one-dimensional case, we will have more to say about this in the next section.

§8. One-dimensional Substitution Tilings

There is a general theory of symbolic substitution dynamical systems [Q], the definition of which parallels that for our substitution tilings. One begins with a finite alphabet and a substitution rule, which is used to define a space of symbolic sequences. Here we show how it also naturally defines a space of substitution tilings of the real line.

Suppose we are given a finite alphabet $\mathcal{A}$ and a substitution rule that associates a word to each letter in it. For each letter and positive integer n, one can construct its n-th iterated substitution in the obvious way: recursively use the substitution rule n times by applying it to every letter in the word at each step. We assume that the lengths of these substitutions go to infinity as n does. One could now define a symbolic substitution system by forming the space of all doubly infinite sequences of letters in which every subword is a subword of some iterated substitution of a letter; but we do not pursue this here — we want a tiling space.

Let $A = (A_{ij})$ be the square matrix that defines the substitution; A_{ij} is the number of times letter i occurs in the substitution for letter j. We assume that the substitution is primitive so that some power of A contains only positive entries. Then the transpose of A is also primitive, and, by the Perron-Frobenius Theorem [Q], it has a positive eigenvalue λ corresponding to a positive eigenvector $l = (l_j)$. This means that, for each letter i, we have $\sum_j A_{ji} l_j = \lambda l_i$. So, if we make a prototile of length l_i for each letter i, then we have a well-defined substitution rule for one-dimensional tiles, with inflation constant λ. Conversely, it is clear that any substitution rule for one-dimensional tilings defines one for symbolic sequences. We note here that one can also apply the Perron-Frobenius Theorem to A, instead of its transpose, to obtain the relative frequencies of the letters in the symbolic substitution [Q].

This procedure can be generalized to obtain higher dimensional substitution tilings. One essentially forms the Cartesian product of several one-dimensional systems, so that the prototiles are rectangles or blocks. Of course, the Penrose tilings and other higher dimensional tilings do not arise in this way. Let us describe the relation of our tiling system with the minimal substitutions of [Q,F,Ho]. Introduce punctures in our tiles as in

the last section and let Ω^P be as defined there. Define a map ρ from Ω^P to $\mathcal{A}^{\mathbb{Z}}$ as follows. Begin with a tiling T in Ω^P. Let $\{x_n^T \mid n \in \mathbb{Z}\}$ be the set of punctures in T arranged so that $x_n^T < x_{n+1}^T$, for all n, and $x_0^T = 0$. For each n, the tile $T(x_n^T)$ is associated with an element of our alphabet $\mathcal{A}$, which we call ρ_n. We define $\rho(T)_n = \rho_n$, for all n in $\mathbb{Z}$. One checks easily that ρ is one-to-one and continuous. Moreover, it is fairly routine to see that $\rho\left(\Omega^P\right)$ is exactly the substitution space X defined in [Q].

Now consider the $\mathbb{R}$-action on Ω defined by translation. As in the last section Ω^P is a closed transversal to the flow. Here, the situation is even better, for the Poincaré first return map is easily seen to be well-defined and continuous on Ω^P. Indeed, it is topologically conjugate to the shift map, σ, on X via ρ. In this case, we have

$$\mathbb{Z} \cong H^0(\Omega) \cong K^0(\Omega) \cong K_1\left(C(\Omega) \bowtie \mathbb{R}\right)$$

$$\cong K_1\left(C(X) \underset{\sigma}{\bowtie} \mathbb{Z}\right)$$

and

$$H^1(\Omega) \cong K^1(\Omega) \cong K_0\left(C(\Omega) \bowtie \mathbb{R}\right)$$

$$\cong K_0\left(C(X) \underset{\sigma}{\bowtie} \mathbb{Z}\right).$$

The computation of $K_0\left(C(X) \underset{\sigma}{\bowtie} \mathbb{Z}\right)$ has been carried out by A. Forrest [F]. An alternative method of computing this group has also been obtained by B. Host [Ho]. It would seem that, in terms of actual computations, Host's method amounts to the same thing as our construction of the space Γ_1, the computation of its first cohomology group and then of the inductive limit of Theorem 6.1. (We are indebted to A. Forrest and C. Skau for explanations of Host's method.)

§9. Zeta Functions

In this section, we compute the zeta function, $\zeta(z)$, for a substitution tiling system (Ω, ω). (We are grateful to D. Lind for bringing this question to our attention.) Our techniques are routine and the statement and proof of Theorem 9.1 is probably known to experts.

Let us first recall the definition. For each $m = 1, 2, 3, \cdots$, let N_m denote the number of fixed points of ω^m. Then, we have

$$\zeta(z) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} z^m\right),$$

at least formally [Fra,Fri,W3].

We assume throughout this section that our tiling has a *CW*-structure. Let $k = 1$, or $k = 0$ if the substitution forces its border. Now Γ_k is a *CW*-complex with, say, c_i cells of dimension i, $0 \leq i \leq d$. In the cellular cohomology complex for Γ_k, we may identify the group of cochains in dimension i with $\mathbb{Z}^{c_i}$. The cellular map γ_k induces a map on the cochains and in dimension i we may identify this map with a $c_i \times c_i$ integer matrix which we denote by A_i'. Note that A_d' is the transpose of the matrix A in Section 5.

Theorem 9.1. *For a substitution tiling system* (Ω, ω) *as above, we have*

$$\zeta(z) = \frac{\displaystyle\prod_{\substack{0 \leq i \leq d \\ i \text{ odd}}} \det\left(I - zA_{d-i}'\right)}{\displaystyle\prod_{\substack{0 \leq i \leq d \\ i \text{ even}}} \det\left(I - zA_{d-i}'\right)}$$

$$= \frac{\det\left(I - zA_{d-1}'\right) \det\left(I - zA_{d-3}'\right) \cdots}{\det\left(I - zA_d'\right) \det\left(I - zA_{d-2}'\right) \cdots}.$$

Proof. First, we appeal to 4.3, that (Ω, ω) is topologically conjugate to (Ω_k, ω_k) and so we may count the periodic points of (Ω_k, ω_k) instead. It is clear that a point $(x_1, x_2, x_3, \cdots)$ in Ω_k is a fixed point of ω_k^m if and only if $x_i = x_{i+m}$, for all i. The map sending x in Γ_k to $\left(x, \gamma_k^{m-1}(x), \cdots, \gamma_k(x), x, \cdots\right)$ is a bijection between $\{x \in \Gamma_k \mid \gamma_k^m(x) = x\}$ and $\{(x_i) \in \Omega_k \mid \omega_k^m((x_i)) = (x_i)\}$. That is, $N_m = |\{x \in \Gamma_k \mid \gamma_k^m(x) = x\}|$.

Let us keep m fixed. We say that an open set U in Γ_k is an I-set (I for "inflating") if it is contained in some cell in Γ_k and γ_k^m is a homeomorphism from U to that same cell. It is straightforward to see that the number of I-sets of dimension i is $\text{Tr}(A_i^m)$. It follows from the Brower fixed point theorem that each I-set has a fixed point of γ_k^m in its closure. Moreover, since the cells of Γ_k can be seen as subsets of $\mathbb{R}^d$ where γ_k is expanding, such a fixed point is unique. Thus, we have each fixed point contained in the closure of some I-set and each I-set has a unique fixed point in its closure. Of course, one fixed point may occur in the closure of several I-sets. We may partition the I-sets into classes according to their fixed points: if $\gamma_k^m(x) = x$, let $\mathcal{U}(x)$ denote the collection of I-sets with x in their closures.

Let x be a fixed point of γ_k^m. Let T be the tiling $\pi^{-1}\left(x, \ \gamma_k^{m-1}(x), \cdots\right)$. Recall that T induces a CW-structure on $\mathbb{R}^d$. Using the finite type condition, we may find $\delta > 0$ so that $B_\delta(0)$ meets only those cells with 0 in their closure. List these cells $\epsilon_1, \cdots, \epsilon_\ell$. It is easy to see that each ϵ_i corresponds to a unique I-set with x in its closure. Moreover, each such I-set arises in this way. This correspondence also preserves dimension.

The $\{\epsilon_1, \cdots, \epsilon_\ell\}$ form a CW-structure on $B_\delta(0)$. Adjoining one more cell of dimension zero gives a CW-structure on the d-sphere, S^d, seen here as the one-point compactification of $B_\delta(0)$. The Euler characteristic of the d-sphere is 2 for d even and 0 for d odd. We have

$$1 + (-1)^d = \chi(S^d) = \sum_{j=1}^{\ell} (-1)^{\dim \epsilon_j} + 1$$

so that

$$(-1)^d = \sum_{j=1}^{\ell} (-1)^{\dim \epsilon_j}$$

$$= \sum_{U \in \mathcal{U}(x)} (-1)^{\dim U}$$

and

$$1 = \sum_{U \in \mathcal{U}(x)} (-1)^{d - \dim U}.$$

Now, we sum over x such that $\gamma_k^m(x) = x$,

$$N_m = \sum_x \sum_{U \in \mathcal{U}(x)} (-1)^{d-\dim U}$$

$$= \sum_{I\text{-sets}} (-1)^{d-\dim U}$$

$$= \sum_{i=0}^{d} (-1)^{d-i} \operatorname{Tr}(A_i^m).$$

The formula of the conclusion of the theorem then follows by standard calculations — see 5.2 of [Fra]. ▊

Remark. To compute the number of fixed points of γ_k^m, we could also have appealed to the Lefschetz fixed point formula [Fra,Fri], showing, in the process, that each fixed point has index one. It seems easier to re-prove the special case of the Lefschetz formula as we have done above.

§10. Examples

Let us look at some examples. All of these can be shown to satisfy the assumptions that we made earlier. One can easily check that the substitutions below are primitive and satisfy a finite type condition. To show that the inflation map is invertible, one must do more work, but the result is known for the examples below: for the one-dimensional tilings, one uses the property of 'recognizability' [Q]; for the Penrose and Ammann tilings, we refer to [GS]; and for the triomino tiling, we refer to [S].

1. Fibonacci Tilings

The Fibonacci tilings are derived from the symbolic substitution rule, $0 \to 01, 1 \to 0$, as described in section 6. The matrix for the substitution is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ whose transpose has Perron-Frobenius eigenvalue $\lambda = (1 + \sqrt{5})/2$ and eigenvector $(\lambda + 1, \lambda)'$. Thus, the substitution rule for the Fibonacci tilings is
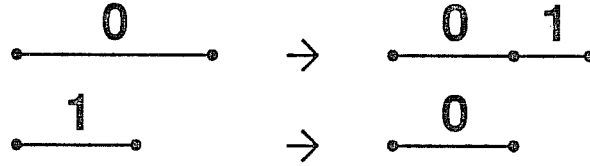
and a typical tiling looks like
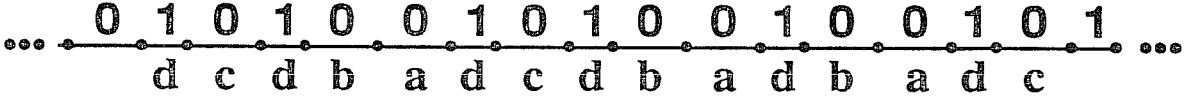
Below it, we have named the collared tiles, which are needed because the substitution does not force its border. These are the edges in the one-dimensional complex. There are 4 edges and 3 vertices: $E = \{a, b, c, d\}$, $V = \{\alpha, \beta, \gamma\}$. The left side of Fig. 1 shows the complex Γ_1, and the right side shows what the inflation map γ_1 does. All of the matrices ∂_1', A_1', A_2' may be read off of this diagram. They are

$$\partial_1' = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad A_1' = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A_0' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

with

$$\mathrm{rank}(\partial_1') = 2$$
$$\mathrm{rank}(A_1') = 3$$
$$\mathrm{rank}(A_0') = 2.$$

One calculates

$$H^0(\Omega) \cong \mathbb{Z}$$
$$H^1(\Omega) \cong \mathbb{Z}^2$$

and

$$\zeta(z) = \frac{1 - z}{1 - z - z^2}.$$
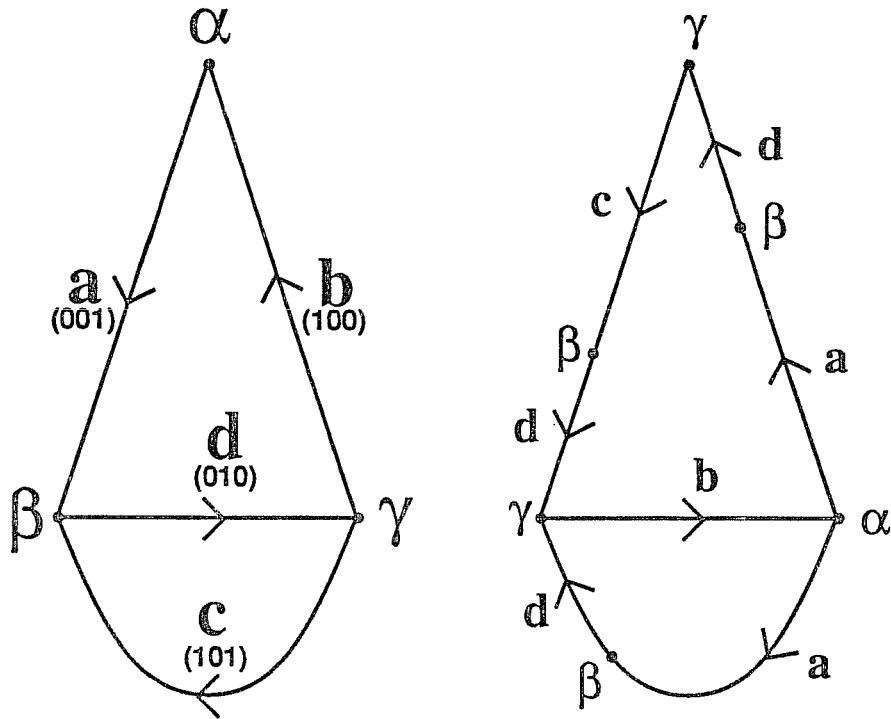
Figure 1: Cell complex for Fibonacci tilings
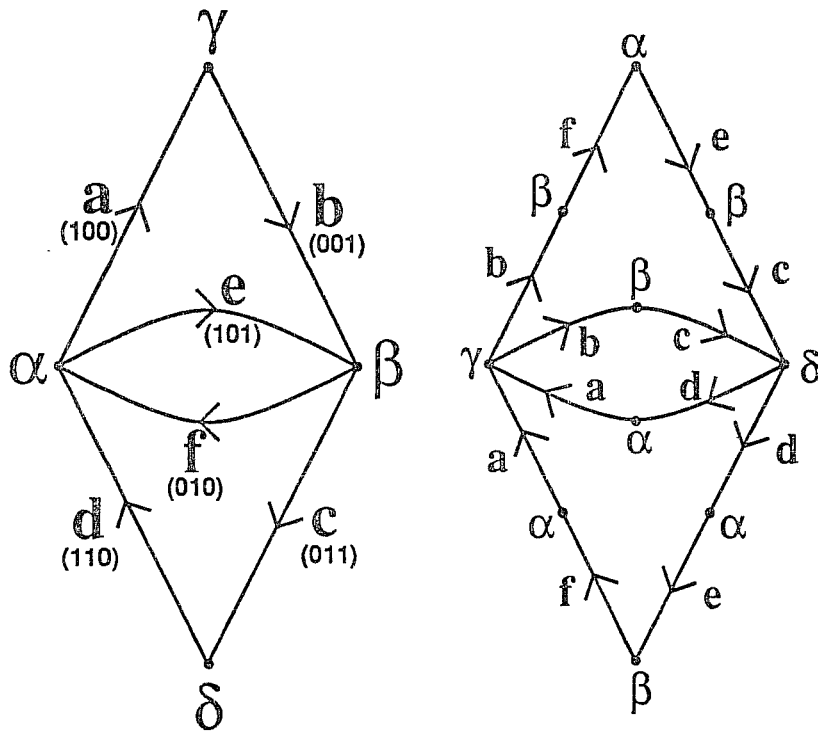
Figure 2: Cell complex for Thue Morse tilings

2. Thue Morse Tilings

The Thue Morse tilings are derived from the rule $0 \to 01, 1 \to 10$. The matrix for the substitution is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ whose transpose has Perron-Frobenius eigenvalue $\lambda = 2$ and eigenvector $(1,1)'$. Note that the two prototiles here have the same length, so that, formally, labels are needed to distinguish them. The substitution rule is
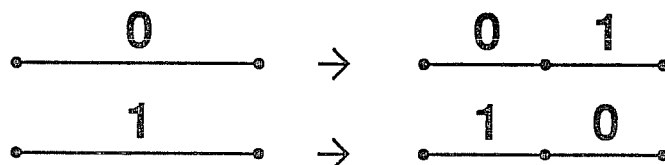
and a typical tiling looks like
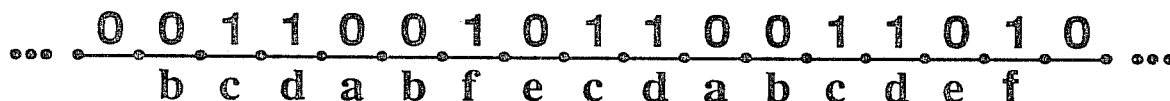
Again, the collared tiles are named below it because the substitution rule does not force its border. The complex has 6 edges and 4 vertices: $E = \{a, b, c, d, e, f\}$, $V = \{\alpha, \beta, \gamma, \delta\}$. Both it and the inflation map are shown in Fig. 2. The matrices are

$$\partial_1' = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \quad A_1' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad A_0' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

with

$$\text{rank}(\partial_1') = 3$$
$$\text{rank}(A_1') = 5$$
$$\text{rank}(A_0') = 4.$$

One calculates

$$H^0(\Omega) \cong \mathbb{Z}$$
$$H^1(\Omega) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}$$

and

$$\zeta(z) = \frac{(1-z)}{(1+z)(1-2z)}.$$

3. Ammann Tilings

We consider one of the aperiodic tilings discovered by R. Ammann. These tilings are described in [GS], and we use a variant of his set of tiles 'A2'. Actually, the substitution rule below is the one in [GS] iterated twice, so that it is primitive. Then there are only eight prototiles, four of each of two similar hexagons. Part of a tiling is shown in Fig. 3. The darker lines illustrate that the substitution forces its border with $N = 4$.

The cell complex has 8 faces, 8 edges, and 3 vertices: $F = \{A, B, C, D, E, F, G, H\}, E = \{a, b, c, d, e, f, g, h\}, V = \{\alpha, \beta, \gamma\}$. The left side of Fig. 4 shows its faces and how they are glued together, and the right side illustrates the substitution rule and the inflation map. One way to partially see what the two-dimensional complex looks like is to realize that each group of four tiles shown forms the space $\mathbb{T}^2 \# \mathbb{T}^2$ (a donut with two holes) when just those edges on the boundary of the group are identified. Of course, many more identifications are then made to form the complex. The calculations are easy because each matrix A_i' has determinant ± 1 and therefore defines a group isomorphism. The matrices are

$$\partial_1' = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \quad \partial_2' = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$A_2' = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad A_1' = \begin{bmatrix} 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_0' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

with

$$\text{rank}(\partial_2') = 2$$
$$\text{rank}(\partial_1') = 2$$
$$\text{rank}(A_2') = 8$$
$$\text{rank}(A_1') = 8$$
$$\text{rank}(A_0') = 3,$$

and one calculates

$$H^0(\Omega) \cong \mathbb{Z}$$
$$H^1(\Omega) \cong \mathbb{Z}^4$$
$$H^2(\Omega) \cong \mathbb{Z}^6$$

and

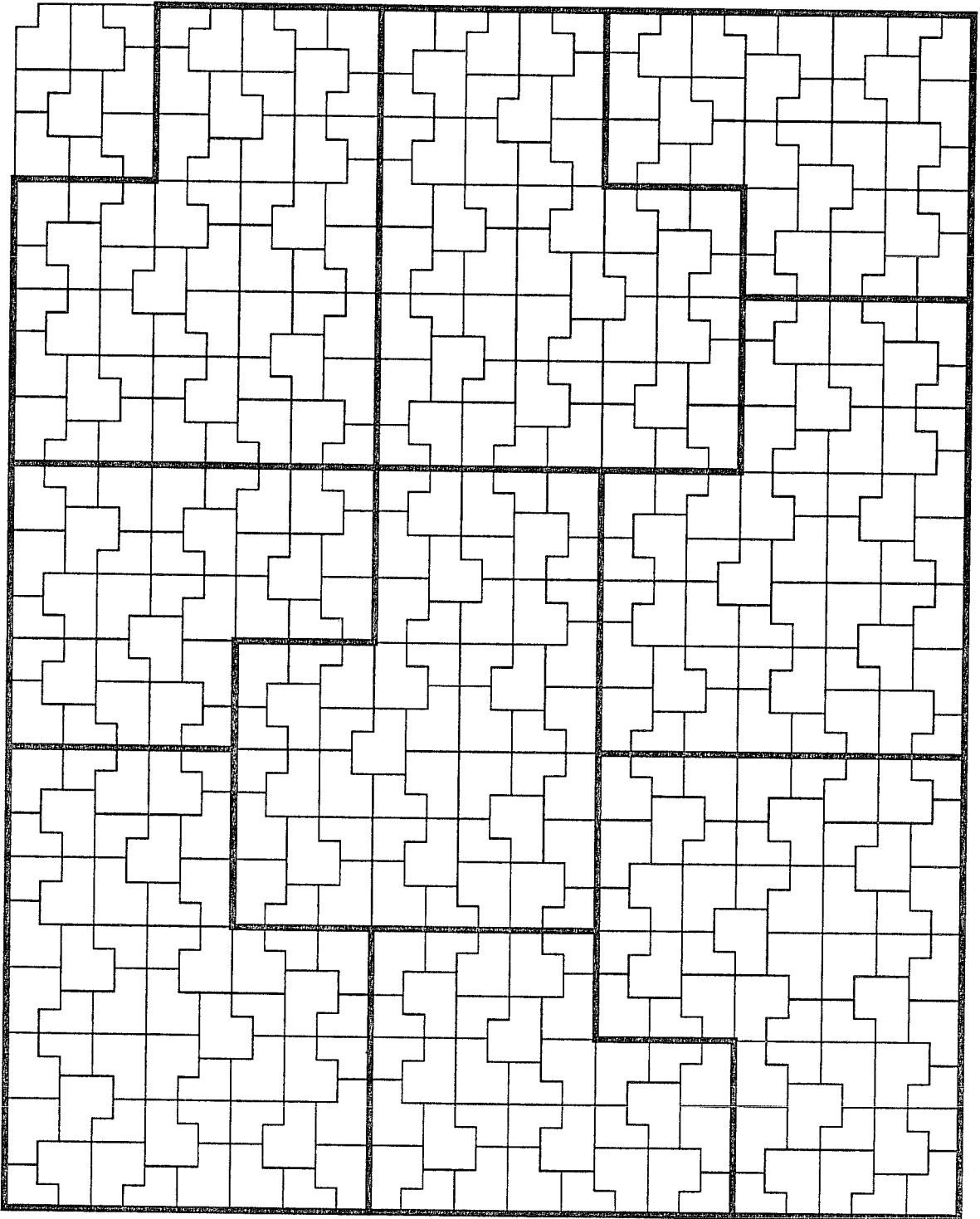$$\zeta(z) = \frac{(z^2 + z - 1)^2}{(z^2 - 3z + 1)(z^2 - z - 1)^2(1 - z)}.$$
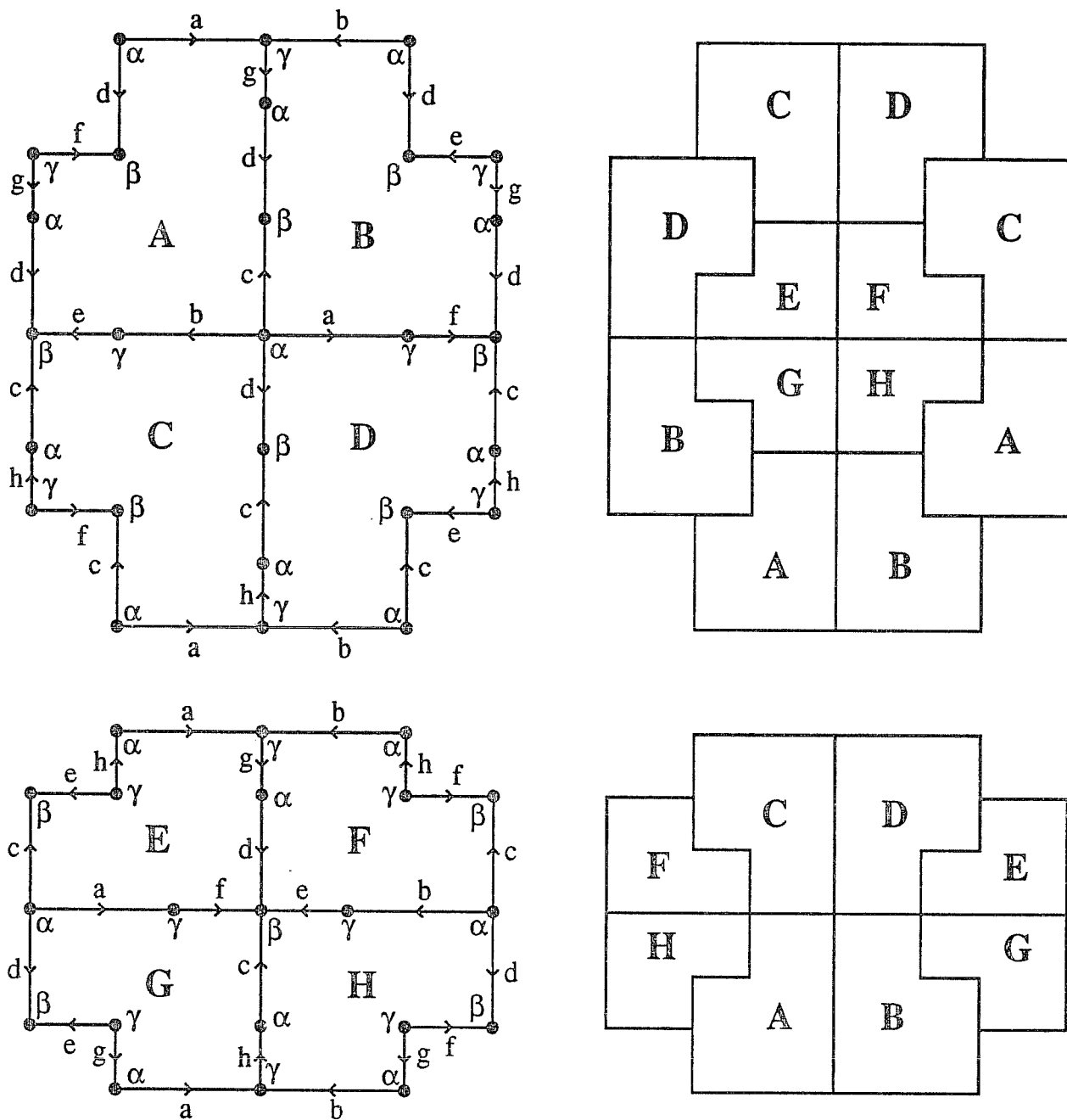
Figure 3: An Ammann Tiling

Figure 4: Cell complex for Ammann tilings

4. Penrose Tilings

The most famous substitution tilings were discovered by R. Penrose. There are several variants of these Penrose tilings [Ga,GS,Kel2,P], and we use the one with forty triangular prototiles. Part of such a tiling is shown in Fig. 5. The darker lines illustrate that the substitution forces its border with $N = 4$.

The cell complex has 40 faces, 40 edges, and 4 vertices: $F = \{1,2,3,\cdots,40\}, E = \{1,2,3,\cdots,40\}, V = \{\alpha,\beta,\gamma,\delta\}$. The left side of Fig. 6 shows its faces and how they are glued together, and the right side illustrates the substitution rule and the inflation map. As with the Ammann tilings, one obtains the space $\mathbb{T}^2 \# \mathbb{T}^2$ when just those edges at the boundary of each group of ten tiles are identified. The calculations are again easy because each matrix A_i' has determinant ± 1 and therefore defines a group isomorphism. The matrices are too large to list here, but may be read off of Fig. 6; their ranks are

$$\mathrm{rank}(\partial_2') = 32$$
$$\mathrm{rank}(\partial_1') = 3$$
$$\mathrm{rank}(A_2') = 40$$
$$\mathrm{rank}(A_1') = 40$$
$$\mathrm{rank}(A_0') = 4.$$

One calculates

$$H^0(\Omega) \cong \mathbb{Z}$$
$$H^1(\Omega) \cong \mathbb{Z}^5$$
$$H^2(\Omega) \cong \mathbb{Z}^8$$

and

$$\zeta(z) = \frac{(z^2 + z - 1)^2 (z + 1)}{(z^2 - 3z + 1)(z^2 - z - 1)^3(z - 1)}.$$
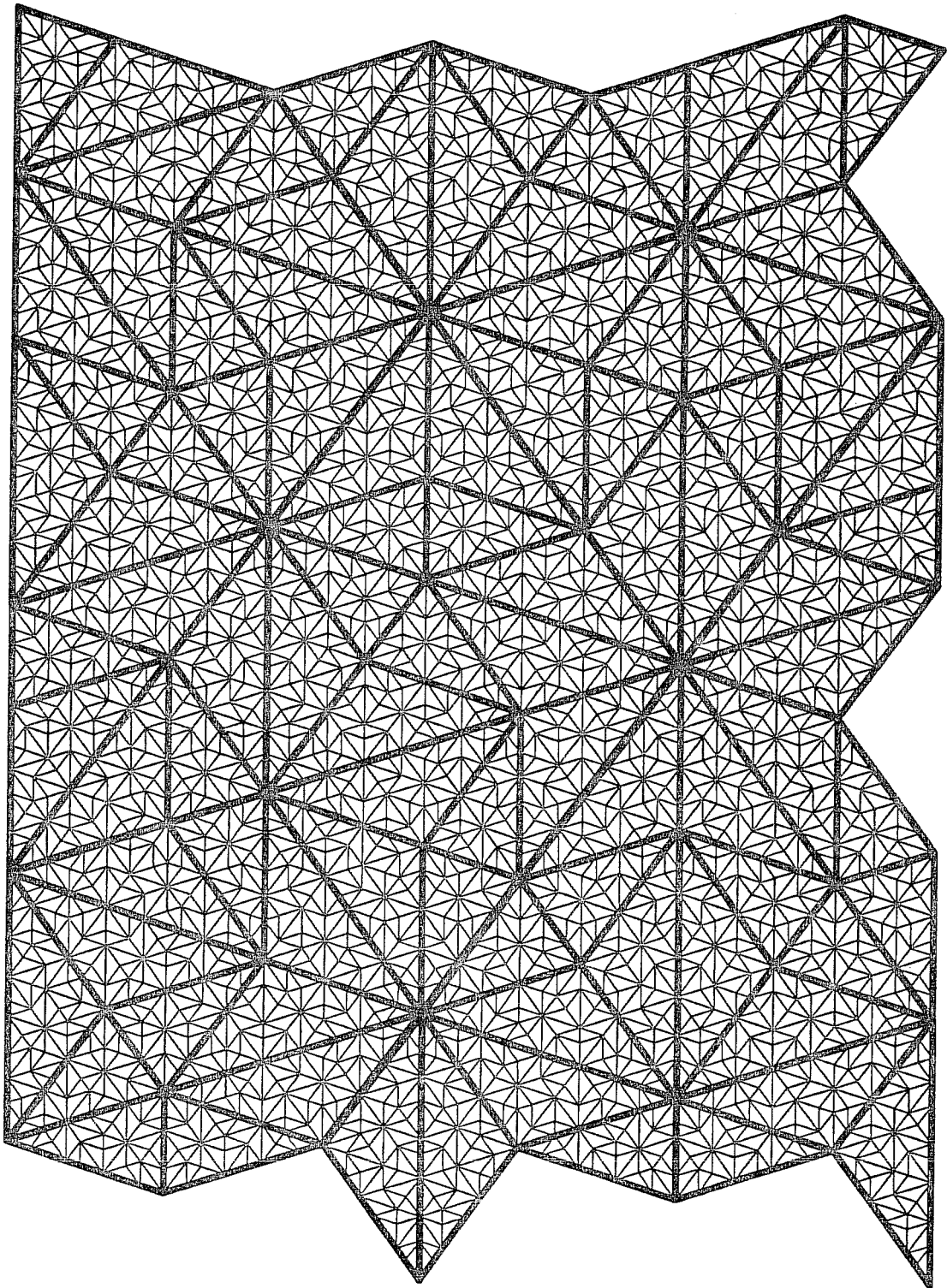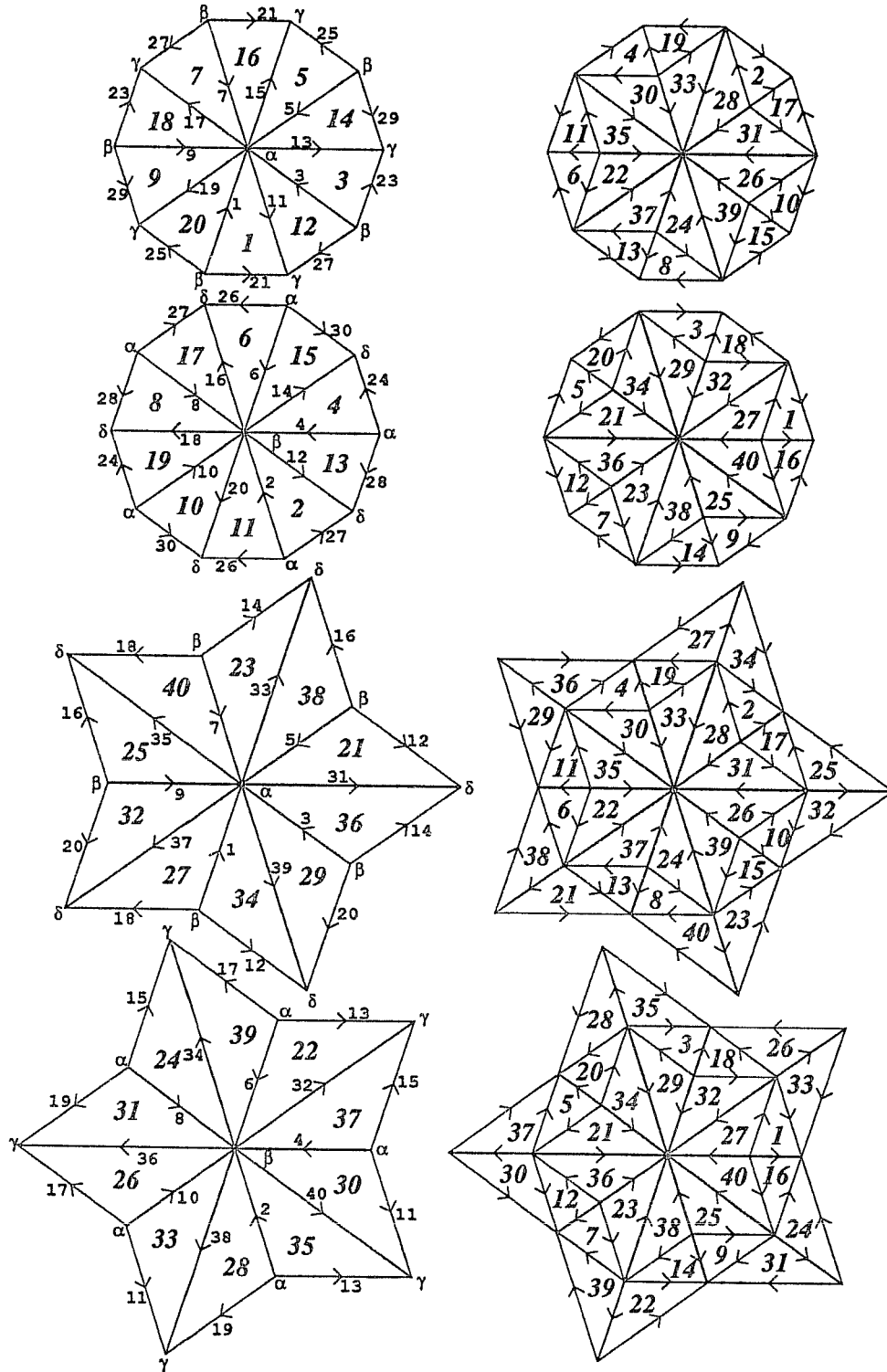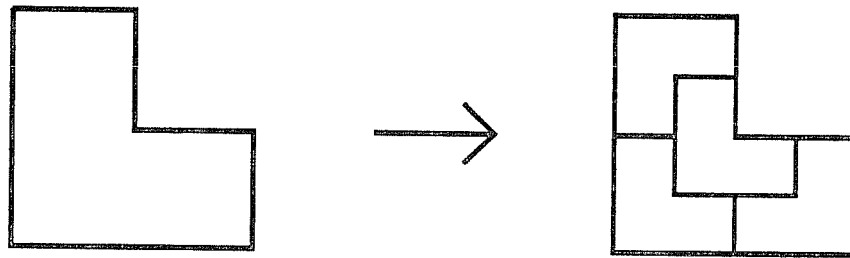
Figure 5: A Penrose Tiling

Figure 6: Cell complex for Penrose tilings

5. Triomino Tilings

We end with an example of a two-dimensional substitution which does not force its border, but we do not carry out the computations. There are four L-shaped prototiles, and the substitution rule is illustrated here for one of them:

We show part of a triomino tiling below. The letter on each tile is the name of the corresponding collared tile. Thus, the cell complex would have $4 \times 14 = 56$ faces.
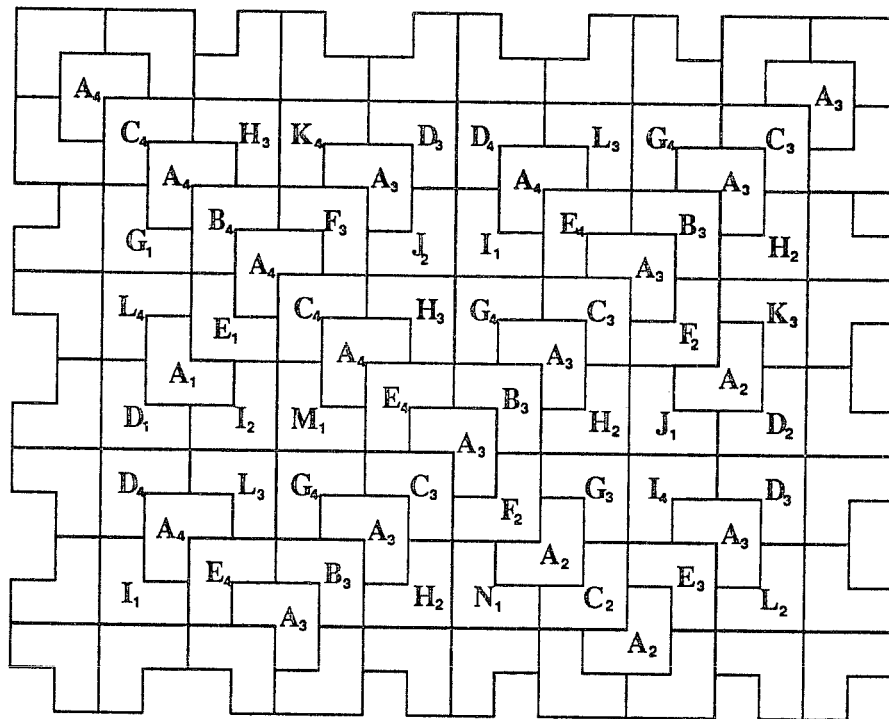

Figure 7: Collared tiles for the Triomino tiling

There are many other substitutions that do not force their borders; see [GS,S] for some examples. We do not do the computations for these, because they are too long to be done easily by hand. This is, of course, because the cell complex is much larger when we must use collared tiles. We note, however, that this procedure could be programmed into a

computer; the algorithm would involve beginning with one tile and repeatedly inflating it until, at some stage, no new collared tiles are formed.

References

[At] M.F. Atiyah, *K-theory*, Addison-Wesley, New York, 1964.

[Bla] B. Blackadar, *K-theory for Operator Algebras*, MSRI Publications 5, Springer-Verlag, Berlin-Heidelberg-New York, 1986.

[Bow] R. Bowen, Markov partitions for Axiom A diffeomorphisms, *Amer. J. Math.* 92(1970), 725–747.

[BMT] M. Boyle, B. Marcus, and P. Trow, Resolving maps and the dimension group for shifts of finite type, *Memoirs AMS* 377(1987).

[Co] A. Connes, *Non-commutative Geometry*, Academic Press, San Diego, 1994.

[Do] A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, Berlin-Heidelberg-New York, 1995.

[F] A.H. Forrest, *K*-groups associated with substitution minimal systems. Preprint.

[Fra] J.M. Franks, Homology and Dynamical Systems, CBMS Lecture Notes 49, AMS, Providence, 1982.

[Fri] D. Fried, Rationality for isolated expansive sets, *Adv. in Math.* 65(1987), 35–38.

[Ga] M. Gardner, Extraordinary nonperiodic tiling that enriches the theory of tiles, *Scientific American* (1977), 110–121.

[GP] W. Geller and J. Propp, The projective fundamental group of a $\mathbb{Z}^2$-shift. Preprint.

[GS] B. Grümbaum and G.C. Shephard, *Tilings and Patterns*, Freeman and Company, New York, 1987.

[HR] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[Ho] B. Host. Preprint.

[Hu] S.T. Hu, *Homotopy Theory*, Academic Press, New York, 1959.

[Kel1] J. Kellendonk, Non-commutative geometry of tilings and gap labelling. Preprint.

[Kel2] J. Kellendonk, On K_0-groups for substitution tilings. Preprint.

[Kel3] J. Kellendonk, The local structure of tilings and their integer group of coinvariants. Preprint.

[Ken] R. Kenyon, Inflationary tilings with a similarity structure. Preprint.

[Ma] W.S. Massey, *A Basic Course in Algebraic Topology*, Graduate Texts in Mathematics **127**, Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[Mi] J.A. Mingo, C^*-algebras associated with one dimensional almost periodic tilings. Preprint.

[Mo] S. Mozes, Tilings, substitution systems and dynamical systems generated by them, *J. d'Analyse Math.* **53**(1989), 139–186.

[MRW] P.S. Muhly, J.N. Renault and D.P. Williams, Equivalence and isomorphism for groupoid C^*-algebras, *J. Operator Th.* **17**(1987), 3–22.

[P] R. Penrose, Pentaplexity — a class of non-periodic tillings of the plane, *Math. Intelligencer* **2**(1979), 32–37.

[Put] I.F. Putnam, C^*-algebras from Smale spaces, *Canad. J. Math.*, to appear.

[Q] M. Queffelec, Substitution dynamical systems – spectral analysis, *Lecture Notes in Math.* **1294**, Springer-Verlag, 1987.

[Ra1] C. Radin, Symmetry of tilings of the plane, *Bull. AMS* **29**(1993), no. 2, 213–217.

[Ra2] C. Radin, The pinwheel tilings of the plane, *Annals of Math.* **139**(1994), 661–702.

[RW] C. Radin and M. Wolff, Space tilings and local isomorphism, *Geom. Ded.* **42**(1992), 355–360.

[Ri] M.A. Rieffel, Morita equivalence for operator algebras, *Proc Symp. Pure Math.* **93** (1981), 415–429.

[ERob1] E.A. Robinson, Jr., The dynamical theory of tilings and quasicrystallography. Preprint.

[ERob2] E.A. Robinson, Jr., The dynamical properties of Penrose tilings. Preprint.

[RRob] R.M. Robinson, Undecidability and nonperiodicity for tilings of the plane, *Inventiones Math.* **12**(1971), 177–209.

[Rud1] D.J. Rudolph, Rectangular tilings of $\mathbb{R}^n$ and free $\mathbb{R}^n$-actions, *Lecture Notes in Math.* **1342**, Springer-Verlag, 1988.

[Rud2] D.J. Rudolph, Markov tilings of $\mathbb{R}^n$ and representations of $\mathbb{R}^n$ actions, *Contemp. Math.* **94**(1989), 271–289.

[Rue] D. Ruelle, Thermodynamics Formalism, *Encyclopedia of Math. and its Appl.*, Vol. 5, Mass: Addison-Wesley, Reading, 1978.

[S] B. Solomyak, Dynamics of self-similar tilings. Preprint.

[W1] R.F. Williams, One-dimensional non-wandering sets, *Topology* 6(1967), 473–487.

[W2] R.F. Williams, Expanding attractors, *Publ. I.H.E.S.* 43(1974), 169–203.

[W3] R.F. Williams, Zeta functions in global analysis, *Proc. Symp. Pure Math.* 14, A.M.S., Providence, 1970.