Czy fizyk udowodni Hipotezę Riemanna ?

Marek Wolf

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“Między duchem a materią pośredniczy matematyka.”

Hugo Dyonizy Steinhaus (1887 — 1972)

S.P.
HUGO DYONIZY STEINHAUS
UR. 14. I. 1887 W JAŚLE
ZM. 25. II. 1972 WE WROCŁAWIU


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PROFESOR ZWYCZAJNY TWÓRCA SZKÓŁ
MATEMATYCZNYCH WE LWOWIE I WROCŁAWIU

1. Układ van der Pola.

2. Hipoteza Poly'a – Hilberta:

3. Macierze losowe.

4. Hamiltoniany: Okubo, Berry–Keatinga, Bendera–Brody–Müllera.

5. Ciepło właściwe ciał stałych.

6. Ruch losowy Schlesingera.

7. Bilardy.

8. Twierdzenie Lee–Yanga i układy spinowe.

9. Pułapkowane jony.

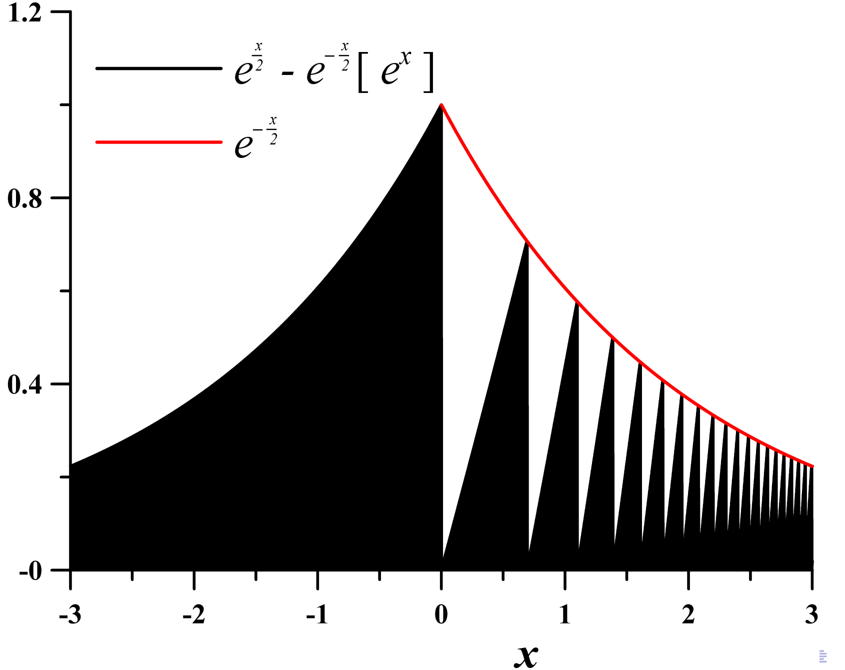
B. van der Pol

An Electro-Mechanical Investigation of The Riemann Zeta Function In The Critical Strip

Bulletin of the AMS **53** (1947), strony 976-981

$$\frac{\zeta(\frac{1}{2}+it)}{\frac{1}{2}+it} = \int_{-\infty}^{\infty} \left(e^{-x/2} \lfloor e^{x} \rfloor - e^{x/2} \right) e^{-ixt} dx$$

papier, nożyczki, silnik, światło, fotokomórka, prąd sinusoidalny o zmiennej częstotliwości.



1.2
0.8
0.4
-0
-3
-2
-1
0
1
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3
x
$e^{\frac{x}{2}} - e^{-\frac{x}{2}}[\,e^{x}\,]$
$e^{-\frac{x}{2}}$

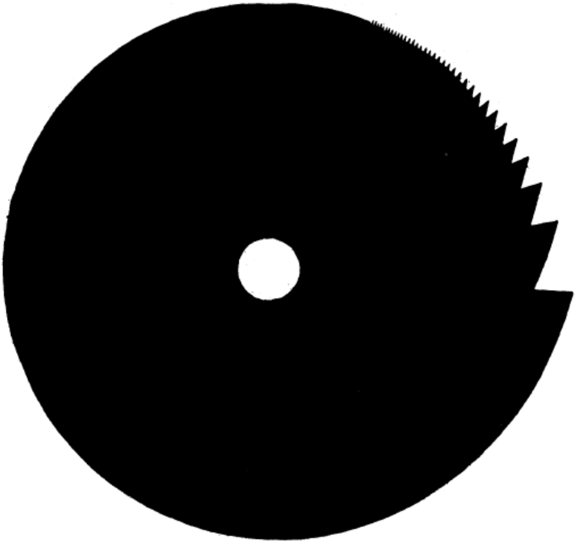


FIG. 2. Paper disc, showing the sawtooth function $y(x)$ for the range $-9 < x < +9$.

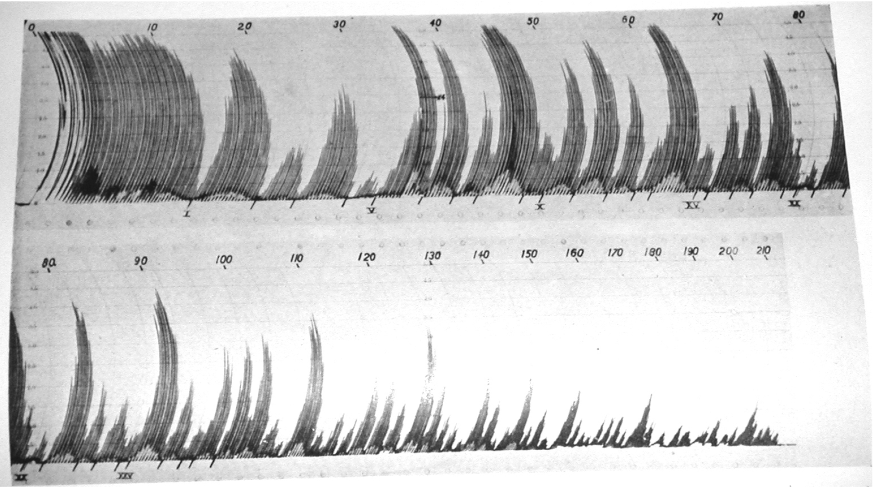


Fig. 3. Record of $|\zeta(1/2+it)/(1/2+it)|$ as produced electromechanically showing o.a. minima, the first 29 of which *(marked* $|$*) correspond* to the 29 known zeros of the zeta-function on the critical line.

D. Schumayer and D. A. W. Hutchinson, “Physics of the Riemann hypothesis,” Rev. Mod. Phys., vol. 83, pp. 307–330, str.326:

This construction, despite its limited achievement, deserves to be treated as a gem in the history of the natural sciences.

Hipoteza Poly'a– Hilberta:

$$``\zeta(\tfrac{1}{2} + i\hat{H}) = 0"$$

RH jest prawdziwa, ponieważ części zespolone nietrywialnych zer odpowiadają wartościom własnym operatora samosprzężonego. Idea powstała około 1910 roku, opublikowana po raz pierwszy w 1973.

List Polya (1887-1985) do Odłyżki:

STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305

DEPARTMENT OF MATHEMATICS

Jan. 3, 82

Dear Mr. Odlyzko,

Many thanks for your letter of Dec, 8, I can only tell you what happened to me.

I spent 2 years in Göttingen ending around the begin of 1914. I tried to learn analytic number theory from Landau. He asked me one day: "You know some physics. Do you know a physical reason that the Riemann hypothesis should be true."

This would be the case, I answered, if the nontrivial zeros of the ξ-function were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.

I never published this remark, but somehow it became known and it is still remembered.

With best regards
Yours sincerely
George Polya

Montgomery (1973): Załóżmy RH:

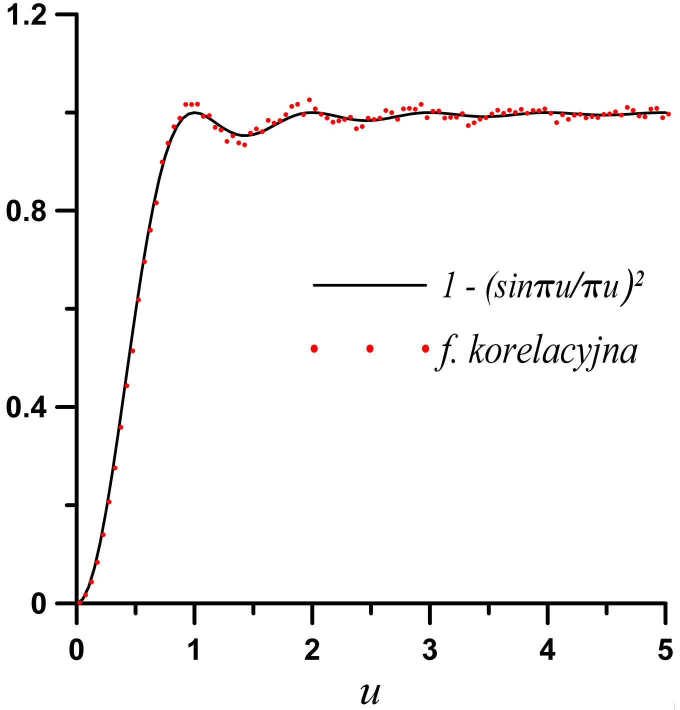
$\rho = \frac{1}{2} + i\gamma$.

$$\sum_{\substack{0<\gamma,\gamma' \leq T \\ \frac{2\pi\alpha}{\ln T} \leq \gamma - \gamma' \leq \frac{2\pi\beta}{\ln T}}} 1 = \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 \right) du$$

Odlyzko (1987): $\delta_n = (\gamma_{n+1} - \gamma_n) \frac{\ln(\gamma_n/(2\pi))}{2\pi}$

$$\frac{1}{N} \sum_{\substack{1 \leq n \leq N, \\ k \geq 0 \\ \delta_n + \delta_{n+1} + \ldots + \delta_{n+k} \in [\alpha, \beta]}} 1 \sim \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 \right) du$$

Przez kilka lat ujawniony w czasie krótkiej rozmowy Montgomery'ego z Dysonem związek nietrywialnych zer $\zeta(s)$ z wartościami własnymi macierzy GUE nie wzbudzał dużego zainteresowania. W latach osiemdziesiątych Andrew Odlyzko rozpoczął sprawdzanie przypuszczenia za pomocą superkomputerów Cray-1 i Cray X-MP.



1.2
0.8
0.4
0
0
1
2
3
4
5
u
1 - (sinπu/πu)²
f. korelacyjna

A. Odlyzko:

The 10^{20}-th zero of the Riemann zeta function and 70 million of its neighbors, tysiące godzin na Cray-1 i Cray X-MP

The 10^{20}-th zero of the Riemann zeta function and 175 million of its neighbors, 1992 revision of 1989

The 10^{21}-st zero of the Riemann zeta function,

The 10^{22}-nd zero of the Riemann zeta function, ($\sim 10^9$ zer)

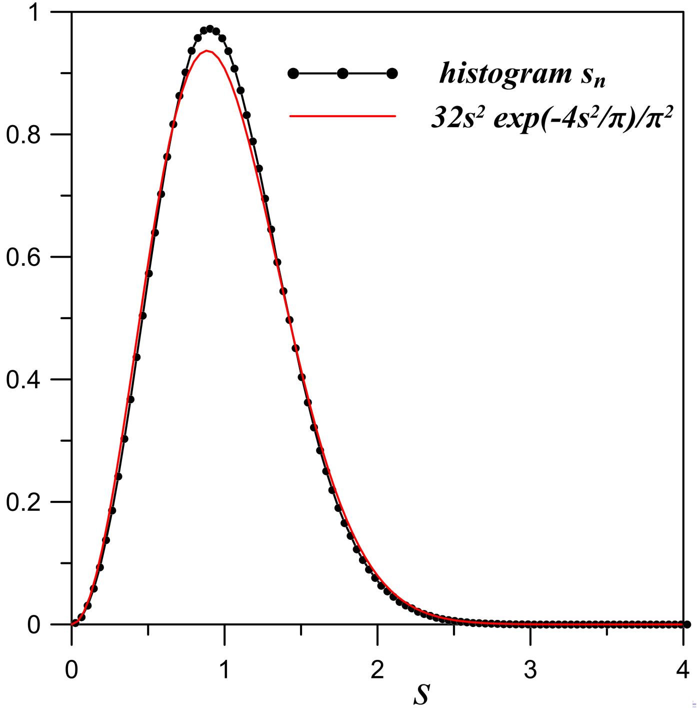
P. Sarnak napisał: „Na poziomie fenomenologicznym to jest chyba najbardziej zdumiewające odkrycie dotyczące dzety od czasów Riemanna.”

E. Wigner, F.J. Dyson, M.L. Mehta: ciężkie jądra atomowe. Macierze losowe.

Gaussian Unitary Ensemble (GUE): układy bez symetrii względem odwrócenia czasu.

Inne zespoły: GOE, GSE.

Histogram odległości δ_n.



histogram s_n
$32s^2 \exp(-4s^2/\pi)/\pi^2$
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4
S

J. Phys. A: Math. Gen. **31** (1998) 1049–1057. Printed in the UK PII: S0305-4470(98)86210-0

Lorentz-invariant Hamiltonian and Riemann hypothesis

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Received 22 July 1997, in final form 15 September 1997

Abstract. We have given some arguments that a two-dimensional Lorentz-invariant Hamiltonian may be relevant to the Riemann hypothesis concerning zero points of the Riemann zeta function. Some eigenfunction of the Hamiltonian corresponding to infinite-dimensional representation of the Lorentz group have many interesting properties. Especially, a relationship exists between the zero zeta-function condition and the absence of trivial representations in the wavefunction. We also give a heuristic argument for the validity of the hypothesis.

The Riemann hypothesis (RH) [1–3] is one of the long-standing problems in the number theory. The Riemann's zeta function $\zeta(z)$ for a complex variable z is defined for $\mathrm{Re}\, z > 1$ by

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^\infty \frac{u^{s-1}du}{1+e^u}, \ \ \Re(s) > 0$$

$$\zeta(s) = 0 \Leftrightarrow \int_0^\infty \frac{u^{s-1}du}{1+e^u} = 0$$

Jeżeli $1 > \ \Re(s) > \frac{1}{2}$ dla $\zeta(s) = 0$ prowadzi do sprzeczności $\Rightarrow$ dowód RH.

$$H = \frac{\partial^2}{\partial x \partial y} + i\beta y \frac{\partial}{\partial y} + i(1-\beta) x \frac{\partial}{\partial x} + \frac{i}{2}, \quad \beta \in \mathbb{R}$$

$$< \phi|\psi >= \int_{-\infty}^{\infty} dx \int_0^{\infty} dy \; \overline{\phi}(x,y)\psi(x,y)$$

Aby $H = H^\dagger$: $\psi(x,0) = \psi'(x,0) = 0$ i $\psi(x)$ oraz $\psi'(x)$ muszą szybko zanikać do zera dla $x \to \pm\infty$ i $y \to \infty$.

$$H\phi = \lambda\phi, \qquad z = \tfrac{1}{2} + i\lambda$$

$$\left\{\frac{\partial^2}{\partial x \partial y} + i\beta y \frac{\partial}{\partial y} + i(1-\beta) x \frac{\partial}{\partial x}\right\}\phi = -iz\phi$$

rozwiązania dla $\Re(z) > 0$:

$$\phi(x,y) = \int_0^\infty t^{z-1} e^{ixt^{1-\beta}} g(t + yt^\beta)\,dt$$

tutaj $g(u)$ znika dla $u \to \infty$.

wybór

$$g(u) = \frac{1}{1 + e^u}$$

wtedy

$$f_0(x, y) = \int_0^\infty \frac{t^{z-1} e^{ixt^{1-\beta}} dt}{1 + e^{t+yt^\beta}}$$

spełnia $Hf_0 = \lambda f_0$. Związek z $\zeta(s)$ dla $x = y = 0$

$< \psi|\psi >= \infty \;\Rightarrow$ widmo jest ciągłe.

Całka z dodatniej funkcji $= 0 \Rightarrow$
$\Re(s) = \frac{1}{2} \;\Rightarrow$ RH

H=xp AND THE RIEMANN ZEROS

M.V. Berry[1] and J.P. Keating[2]

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[2]School of Mathematics, University Walk,
Bristol BS8 1TW, U.K.,
and Basic Research Institute in the Mathematical Sciences,
Hewlett-Packard Laboratories Bristol, Filton Road,
Stoke Gifford, Bristol BS12 6QZ, U.K.

1. INTRODUCTION

The Riemann hypothesis [1, 2] states that the complex zeros of $\zeta(s)$ lie on the critical line Re s=1/2; that is, the nonimaginary solutions E_n of

$$\zeta\left(\tfrac{1}{2}+\mathrm{i}E_n\right)=0 \qquad (1)$$

are all real. Here we will present some evidence that the E_n are energy levels, that is eigenvalues of a hermitian quantum operator (the 'Riemann operator'), associated with the classical hamiltonian

$$H_{\mathrm{cl}}(x,p)=xp \qquad (2)$$

$$\hat{H} = \frac{1}{2}(xp + px)$$

- $\hat{H}$ ma klasyczny odpowiednik opisujący dynamikę chaotyczną, niestabilną i ograniczoną
- Dynamika Riemanna nie posiada symetrii względem odwrócenia czasu.
- Dynamika Riemanna jest jednowymiarowa

$\hat{H}$ ma liczbę poziomów $< E$:

$$N(E) = \frac{E}{2\pi}\left(\ln\left(\frac{E}{2\pi}\right) - 1\right) + \frac{7}{8} + \dots$$

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + \mathcal{O}(\ln(T))$$

$$\psi_E(x) \sim \frac{const}{|x|^{1/2-iE}}\zeta\left(\frac{1}{2} - iE\right)$$

Hamiltonian for the Zeros of the Riemann Zeta Function

Carl M. Bender,[1] Dorje C. Brody,[2,3] and Markus P. Müller[4,5]
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(Received 23 September 2016; revised manuscript received 17 February 2017; published 30 March 2017)

A Hamiltonian operator $\hat{H}$ is constructed with the property that if the eigenfunctions obey a suitable boundary condition, then the associated eigenvalues correspond to the nontrivial zeros of the Riemann zeta function. The classical limit of $\hat{H}$ is $2xp$, which is consistent with the Berry-Keating conjecture. While $\hat{H}$ is not Hermitian in the conventional sense, $i\hat{H}$ is $\mathcal{PT}$ symmetric with a broken $\mathcal{PT}$ symmetry, thus allowing for the possibility that all eigenvalues of $\hat{H}$ are real. A heuristic analysis is presented for the construction of the metric operator to define an inner-product space, on which the Hamiltonian is Hermitian. If the analysis presented here can be made rigorous to show that $\hat{H}$ is manifestly self-adjoint, then this implies that the

ABSTRACTIONS BLOG

Physicists Attack Math's $1,000,000 Question

Physicists are attempting to map the distribution of the prime numbers to the energy levels of a particular quantum system.

Riemann hypothesis: is Bender-Brody-Müller Hamiltonian a new line of attack?

There is a beautiful paper in Physical Review Letters [PRL 118, 130201 (2017), DOI:10.1103/PhysRevLett.118.130201] by Carl Bender, Dorje Brody, and Markus Müller (BBM) on a Hamiltonian approach to the Riemann Hypothesis. The paper is surprisingly easy to follow for a physicist.

BBM define a Hamiltonian

$$\hat{H} = (1 - e^{-i\hat{p}})^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}})$$

where $p = -i\partial_x$ is the momentum operator in $\hbar = 1$ units.

The authors show that eigenfunctions of $\hat{H}$ vanishing at infinity must be be in the form of Hurwitz

$$\hat{H}_{\mathrm{BBM}} = \frac{\mathbb{1}}{\mathbb{1} - e^{-\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(\mathbb{1} - e^{-\hat{p}})$$

funkcje własne: dla $\Re(s) > 1$ oraz $q \neq -1, -2, -3, \ldots$

$$\zeta(s, q) = \sum_{n=1}^{\infty} \frac{1}{(q+n)^s}$$

$$\zeta(s) = \zeta(s, 0)$$

$\psi_s(x) = -\zeta(s, x+1)$ spełnia

$$\hat{H}_{\mathrm{BBM}} \psi_s(x) = i(2s-1)\psi_s(x)$$

warunek brzegowy:
$\psi_s(0) = 0 = \zeta(s, 0) = \zeta(s)$
wtedy s są zerami funkcji $\zeta(s)$

Jeżeli HR $\Rightarrow i(2s-1) = -2\gamma_n$.

e-mail z 15.X 2019 od Dorje Brody:

The fact that once a Hilbert space condition is imposed then one is restricting analysis on the critical line, however, is a generic feature, applicable to other operators, and hence makes the Hilbert-Pólya programme harder than what people might have thought of it previously.

Hipotetyczny układ kwantowy (abstrakcyjny „pierwiastek”, którego jądro posiada poziomy energetyczne pokrywające się z γ_n) został nazwany przez O. Bohigasa „Riemannium”, analogicznie do mionium, czyli atomu zbudowanego z antymionu i elektronu albo pozytonium: układu złożonego z elektronu e^- i pozytonu e^+, czyli antyelektronu.

American Scientist, Volume 91 2003 July–August

The Spectrum of Riemannium

Brian Hayes

The year: 1972. The scene: Afternoon tea in Fuld Hall at the Institute for Advanced Study. The camera pans around the Common Room, passing by several Princetonians in tweeds and corduroys, then zooms in on Hugh Montgomery, boyish Midwestern number theorist with sideburns. He has just been introduced to Freeman Dyson, dapper British physicist.

Dyson: So tell me, Montgomery, what have you been up to?

Montgomery: Well, lately I've been looking into the distribution of the zeros of the Riemann zeta function.

Dyson: Yes? And?

Montgomery: It seems the two-point correlations go as.... (*turning to write on a nearby blackboard*):

$$1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$

Dyson: Extraordinary! Do you realize that's the games of solitaire, one-dimensional gases and chaotic quantum systems. Is it all just a cosmic coincidence, or is there something going on behind the scenes?

The Spectrum of Interstatium

How things distribute themselves in space or time or along some more abstract dimension is a question that comes up in all the sciences. An astronomer wants to know how galaxies are scattered around the universe; a biologist might study the distribution of genes along a strand of chromatin; a seismologist records the temporal pattern of earthquakes; a mathematician ponders the sprinkling of prime numbers among the integers. Here I shall consider only discrete, one-dimensional distributions, where the positions of items can be plotted along a line.

Figure 1 shows samples of several such distributions, some of them mathematically defined

Uwaga: dla trywialnych zer $-2n$ funkcji $\zeta(s)$:
kwantowy oscylator harmoniczny

$$\frac{1}{2}\left(-\frac{d^2}{dx^2}+x^2\right)\psi_n(x)=(n+\tfrac{1}{2})\psi_n(x)$$

$$\psi_n(x)=e^{-\frac{x^2}{2}}\cdot H_n(x), \qquad n=0,1,2,\ldots$$

$H_n(x)$ to wielomiany Hermite'a

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

$$\left(\frac{d^2}{dx^2}-x^2+1\right)\psi_n(x)=-2n\psi_n(x).$$

$$\hat{H}_{tryw}=\frac{d^2}{dx^2}-x^2+1$$

Sierpień 1996, konferencja w Seattle poświęcona 100-leciu PNT: Peter Sarnak oferuje butelkę wina dla fizyków, którzy z prawa Montgomery'ego–Odlyzki potrafią otrzymać informacje nie znane wcześniej matematykom.

Wrzesień 1998, konferencja w Wiedniu: Jon Keating podał rozwiązanie (ale nie dowód) tzw. problemu momentów dzety.

G.H. Hardy i J.E. Littlewood (1918):

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+\imath t)|^2 dt \sim \ln(T), \quad \text{gdy } T \to \infty.$$

A.E. Ingham w 1926 roku:

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+\imath t)|^4 dt \sim \tfrac{1}{2\pi}\ln^4(T), \quad T \to \infty.$$

przypuszczalna postać dla $k = 3$ i dla $k = 4$ (B. Conrey i A. Ghosh, 1998):

$$\int_1^T |\zeta(\tfrac{1}{2} + it)|^6 \, dt \sim$$

$$\frac{42}{9!} \prod_p \left\{ \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \right\} T \ln^9 T$$

dla dużych T

$$\int_0^T |\zeta(1/2+it)|^8 \, dt \sim$$

$$\frac{24024}{16!} \prod_p \left(\left(1 - \frac{1}{p}\right)^9 \left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right) \right)$$

$$\times T \log^{16} T \quad \text{dla dużych } T$$

Keating i Snaith (2000):

$$\frac{1}{T}\int_0^T \left|\zeta(\tfrac{1}{2}+\imath t)\right|^{2k}\ dt \sim f_k a(k)(\ln T)^{k^2},$$

gdzie

$$a(k) = \prod_{p \ \text{pierwsza}} \left(1-\frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2 p^{-}$$

a liczby f_k są dane wzorem:

$$f_k = \frac{G^2(k+1)}{G(2k+1)}.$$

$G(\cdot)$ jest funkcją Barnes'a: $G(z+1) = \Gamma(z)G(z)$

Ciepło właściwe ciał stałych i obszar wolny od zer $\zeta(s)$:

$$c_V(T) = \int_0^\infty \left(\frac{\hbar\omega}{k_B T}\right)^2 \frac{e^{\hbar\omega/k_B T}}{(e^{\hbar\omega/k_B T} - 1)^2} g(\omega) d\omega$$

(Cheng 1990, Ming *et al* 2003):

$$g(\omega) = \frac{1}{2\pi\omega} \int_{-\infty}^{\infty} \frac{\omega^{ik+s} Q(k)}{\Gamma(ik+s+2)\zeta(ik+s+1)} dk$$

$Q(k) = \int_0^\infty u^{ik+s-1} c_V(1/u) du$

s — parametr regularyzacyjny, $s_1 < s < s_2$, gdzie s_1 i s_2 są wykładnikami T odpowiednio w wysokiej i niskiej temperaturze

$$c_V(T) \sim T^{s_1} \qquad (T \to \infty)$$

$$c_V(T) \sim T^{s_2} \qquad (T \to 0)$$

Prawo Dulonga–Petita daje $s_1 = 0$ i $s_2 = d$ (d–wymiar przestrzeni, $c_V \sim T^d$).
Obszar wolny od zer $[1, 1 + d]$.

Physica 138A (1986) 310–319
North-Holland, Amsterdam

ON THE RIEMANN HYPOTHESIS: A FRACTAL RANDOM WALK APPROACH

Michael F. SHLESINGER

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In his investigation of the distribution of prime numbers Riemann, in 1859, introduced the zeta function with a complex argument. His analysis led him to hypothesize that all the complex zeros of the zeta function lie on a vertical line in the complex plane. The proof or disproof of this hypothesis has been a famous outstanding problem in mathematics. We are able to recast Riemann's Hypothesis into a probabilistic framework connected to the fractal behavior of a lattice random walk. Fractal random walks were introduced by P. Levy, and in the continuum are called Levy flights. For one particular lattice version of a Levy flight we show the connection to Weierstrass' continuous but nowhere differentiable function. For a different lattice version, using a Mellin transform analysis, we show how the zeroes of the zeta function become

Shlesinger rozważa ruch jednowymiarowy z prawdopodobieństwem wykonania skoku $\pm l$ danym przez wzór:

$$p(\pm l) = \frac{1}{2} C \left(\frac{1}{l^{\,1+\beta}} \pm \frac{\mu(l)}{l^{\,1+\beta-\epsilon}} \right), \qquad \beta > 0,$$

$$C = \frac{1}{\zeta(1+\beta) + \frac{1}{\zeta(1+\beta)}}$$

Ruch losowy Riemanna–Mobiusa

$$\lambda(k) = \sum_{l=-\infty}^{\infty} e^{ikl} p(l)$$

$\lambda(k)$ osiąga maksimum przy $k = 0$ dla każdego RW. Jeżeli RH to

$$\lambda(k) \sim 1 - k^{\beta} \pm k^{\beta+1/2+\epsilon}$$

Gdy są zera dowolnie bliskie 0 lub 1:

$$\lambda(k) \sim 1 \pm k^{\beta-\epsilon} + k^{\beta} \qquad k \to 0$$

Nawet jedno zero poza $\Re(s) = \frac{1}{2}$ będzie dominować w zachowaniu $\lambda(k)$.

Open Circular Billiards and the Riemann Hypothesis

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(Received 5 August 2004; published 18 March 2005)

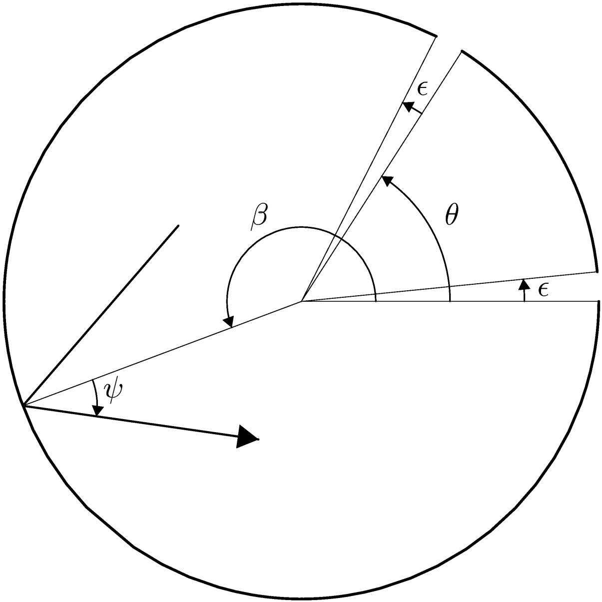
A comparison of escape rates from one and from two holes in an experimental container (e.g., a laser trap) can be used to obtain information about the dynamics inside the container. If this dynamics is simple enough one can hope to obtain exact formulas. Here we obtain exact formulas for escape from a circular billiard with one and with two holes. The corresponding quantities are expressed as sums over zeros of the Riemann zeta function. Thus we demonstrate a direct connection between recent experiments and a major unsolved problem in mathematics, the Riemann hypothesis.

DOI: 10.1103/PhysRevLett.94.100201 PACS numbers: 02.10.De, 02.30.Ik, 05.60.Cd, 45.50.Dd

Billiard systems, in which a point particle moves freely except for specular reflections from rigid walls, permit close connections between rigorous mathematics and experimental physics. Very general physical situations, in which particles or waves are confined to cavities or other homogeneous regions, are related to well understood billiard dynamical systems, directly for particles and via semiclassical (short wavelength) theories for waves. Precise billiard experiments have used microwaves in metal [1,2] and superconducting [3] cavities and with wave guides [4], visible light reflected from mirrors [5], phonons in quartz blocks [6], electrons in semiconductors

At long times, the probability of a particle remaining in an integrable billiard with a hole is well-known to exhibit power law decay, in contrast to exponential decay from strongly chaotic billiards [19]; however the coefficient of the power ("escape rate") in the integrable case has not been computed exactly to our knowledge. Numerical simulations can be misleading; for example, a power law decay at long times can be masked by an exponential term at short times.

Here we consider the circle billiard, which is integrable due to angular momentum conservation. Some three dimensional cases, namely, the cylinder and sphere, can be


ϵ
β
θ
ϵ
ψ

Dynamika kuli bilardowej

$$T(\beta, \psi) = (\beta + \pi - 2\psi, \psi)$$

Prawdopodobieństwo, że kulka nie ucieknie do chwili t:

$$tP(t) = \frac{1}{8\pi} \sum_{n=1}^{\infty} n(\phi(n) - \mu(n)) \times \left[g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g\left(\theta' - \epsilon\right) \right]$$

$\phi(n)$ funkcja totient Euler: liczba dodatnich $m \leq n$ takich, że $gcd(m, n) = 1$

$$g(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\sin(\alpha) = \frac{(e^{i\alpha} - e^{-i\alpha})}{2}$$

Tożsamość Ramanujana:

$$\sum_{\substack{m=0 \\ \gcd(m,n)=1}}^{n-1} e^{2\pi i m/n} = \mu(n)$$

Dla jednego otworu RH jest równoważna temu, że:

$$\lim_{\epsilon \to 0} \lim_{t \to \infty} \epsilon^{\delta - 1/2}(tP(t) - 2/\epsilon) = 0$$

dla każdego $\delta > 0$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \;\; s \in \mathbb{C}\setminus\{0,1\}$$

Niesymetryczna postać równania funkcyjnego

$$2\Gamma(s)\cos\left(\frac{\pi}{2}s\right)\zeta(s) = (2\pi)^s\zeta(1-s)$$

Relacje Kramersa–Wanniera. 2-wymiarowy model Isinga

$$Z(J) = 2^N(cosh(J))^{2N}(tanh(J))^N Z(\widetilde{J})$$

N liczba spinów, $e^{-2\widetilde{J}} = tanh(J)$. Model spinowy z funkcją stanów $Z(\beta)$ wyrażoną przez $\zeta(s)$. Knauf: $Z(\beta) = \zeta(\beta-1)/\zeta(\beta)$.

Twierdzenie Lee–Yanga o zerach sumy stanów: sumy stanów mają zera zespolone leżące na okręgu. W granicy termodynamicznej zespolone zera mają punkt skupienia przy rzeczywistej wartości temperatury.

Twierdzenia "kołowe".

Twierdzenie Lee-Yanga $\rightarrow$ RH.

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Mathematical
Physics

The Number-Theoretical Spin Chain and the Riemann Zeroes

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Abstract: It is an empirical observation that the Riemann zeta function can be well approximated in its critical strip using the Number-Theoretical Spin Chain. A proof of this would imply the Riemann Hypothesis. Here we relate that question to the one of spectral radii of a family of Markov chains. This in turn leads to the question whether certain graphs are Ramanujan.

The general idea is to explain the pseudorandom features of certain number-theoretical functions by considering them as observables of a spin chain of statistical mechanics. In an appendix we relate the free energy of that chain to the Lewis Equation of modular theory.

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Riemann hypothesis and quantum mechanics

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Riemann zeros from a periodically-driven trapped ion

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(Dated: February 16, 2021)

The non-trivial zeros of the Riemann zeta function are central objects in number theory. In particular, they enable one to reproduce the prime numbers. They have also attracted the attention of physicists working in Random Matrix Theory and Quantum Chaos for decades. Here we present an experimental observation of the lowest non-trivial Riemann zeros by using a trapped ion qubit in a Paul trap, periodically driven with microwave fields. The waveform of the driving is engineered such that the dynamics of the ion is frozen when the driving parameters coincide with a zero of the real component of the zeta function. Scanning over the driving amplitude thus enables the locations of the Riemann zeros to be measured experimentally to a high degree of accuracy, providing a physical embodiment of these fascinating mathematical objects in the quantum realm.

13 Feb 2021

I. MAIN

GUE. These were explained later by Berry and

No.	Exact	$\Omega = 5$	$\Omega = 8$	$\Omega = 12$	$\Omega = 16$
1	14.135	14.07(1)	14.06(2)	13.99(4)	14.03(3)
2	21.022	21.04(2)	21.00(2)	20.93(5)	20.82(3)
3	25.011	24.70(3)	24.87(2)	24.87(7)	24.99(4)
4	30.425	30.59(2)	30.31(2)	30.29(3)	30.27(4)
5	32.935	32.76(3)	32.72(3)	32.57(8)	32.29(23)
6	37.586	37.64(2)	37.62(2)	37.39(2)	37.59(4)
7	40.919	40.95(2)	40.89(3)	40.78(3)	40.70(4)
8	43.327	42.85(9)	43.12(4)	43.23(4)	42.74(40)
9	48.005	48.23(4)	47.87(6)	47.94(6)	47.75(9)
10	49.774	49.26(19)	49.67(3)	49.36(23)	49.23(22)
11	52.970	52.93(2)	52.83(4)	52.88(5)	52.78(5)
12	56.446	56.56(3)	56.58(3)	56.28(3)	56.49(5)
13	59.347	59.44(5)	59.33(9)	59.35(6)	59.08(28)

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