NOTES FROM HOMOLOGICAL ALGEBRA

MACIEJ GAŁĄZKA

1. Derived category

Let $\mathscr{C}$ be an abelian category. Let $\mathrm{Kom}(\mathscr{C})$ denote the category of cochain complexes.

Theorem 1.1. *There is category* $\mathrm{D}(\mathscr{C})$ *(the derived category of* $\mathscr{C}$*) and a functor* $Q : \mathrm{Kom}(\mathscr{C}) \to \mathrm{D}(\mathscr{C})$ *such that*

(1) for every quasi-isomorphism $f \in \mathrm{Mor}(\mathrm{Kom}\,\mathscr{C})$ *the morphism* $Q(f)$ *is an isomorphism;*

(2) if any $Q' : \mathrm{Kom}(\mathscr{C}) \to \mathscr{A}$ *(*$\mathscr{A}$ *here is any category) satisfies* (1)*, then there is a unique functor* $P : \mathrm{D}(\mathscr{C}) \to \mathscr{A}$ *such that*

$$\begin{array}{ccc} \mathrm{Kom}(\mathscr{C}) & \xrightarrow{Q} & \mathrm{D}(\mathscr{C}) \\ & \searrow^{Q'} & \downarrow P \\ & & \mathscr{A} \end{array}$$

commutes.

Proof of Theorem 1.1. To construct $\mathrm{D}(\mathscr{C})$, we need to define the localization of a category.

Let $\mathscr{B}$ be a category, S a class of morphisms in $\mathscr{B}$. The localization of $\mathscr{B}$ in S is a category $\mathscr{B}[S^{-1}]$ and a functor $L : \mathscr{B} \to \mathscr{B}[S^{-1}]$ such that L takes any morphism in S to an isomorphism and for any functor $F : \mathscr{B} \to \mathscr{D}$ which takes elements of S to isomorphisms there exists a unique functor $G : \mathscr{B}[S^{-1}] \to \mathscr{D}$ such that

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{L} & \mathscr{B}[S^{-1}] \\ & \searrow_{F} & \downarrow G \\ & & \mathscr{D} \end{array}$$

commutes.

Let us construct $\mathscr{B}[S^{-1}]$. Let Γ be a directed graph with vertices $\mathrm{Ob}\,\mathscr{B}$ and edges $\mathrm{Mor}(\mathscr{B}) \cup \{x_s : s \in S\}$. If $f : X \to Y$, then edge f is directed from X to Y. If $S \ni s : X \to Y$, then edge x_s is directed from $Y \to X$. We define $\mathrm{Hom}_{\mathscr{B}[S^{-1}]}(X, Y)$ to be equivalence classes of paths in Γ from X to Y, where the relations are generated by

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(1) $U \xrightarrow{f} V \xrightarrow{g} W \sim U \xrightarrow{gf} W$,
(2) $U \xrightarrow{s} V \xrightarrow{x_s} U \sim U \xrightarrow{\mathrm{id}_U} U$,
(3) $U \xrightarrow{x_s} V \xrightarrow{s} U \sim U \xrightarrow{\mathrm{id}_U} U$.

(More precisely, we consider for each X, Y an equivalence relation $\equiv_{X,Y}$ on the sets of paths from X to Y. We say that this family of equivalence relations is good if for any two paths $p, q : X \to Y$, any path $u : Y \to Z$, and any path $v : W \to X$ the relation $p \equiv q$ implies both $pv \equiv qv$ and $up \equiv uq$. Then we define our family of relations $\sim_{X,Y}$ to be the intersection of good of families containing the three relations.)

Suppose that we have any $F : \mathscr{B} \to \mathscr{D}$ which takes class S to isomorphisms. Then $G : \mathscr{B}[S^{-1}] \to D$ is uniquely determined on objects. The only way to define G for morphisms is to make it map any path from X to Y into its composition in $\mathscr{D}$ (note that we can compose paths in $\mathscr{D}$ since any $s \in S$ goes under F to an isomorphism). It remains to prove that this agrees with $\sim_{X,Y}$. For any two $X, Y \in \mathrm{Ob}\,\mathscr{B}$ we can define a family of equivalence relations $\equiv_{X,Y}$ on the paths from X to Y by saying that two paths are equivalent if their images under F, after composing in $\mathscr{D}$, are equal. But then $\equiv_{X,Y}$ is good, so $\sim \subseteq \equiv$, in other words, G is well-defined. G is a functor by definition.

We define $\mathrm{D}(\mathscr{C}) = \mathrm{Kom}(\mathscr{C})[(\text{quasi-isomorphisms})^{-1}]$. □

Definition 1.2. A class $S \subseteq \mathrm{Mor}(\mathscr{B})$ is *localizing* if it satisfies:

(a) for all $X \in \mathrm{Ob}(\mathscr{B})$ we have $\mathrm{id}_X \in S$,
(b) if $s, t \in S$, then $st \in S$,
(c) for all $X \xrightarrow{f} Y$ and $Z \xrightarrow{s} Y$ there are $W \xrightarrow{g} Z$ and $W \xrightarrow{t} X$ such that

$$\begin{array}{ccc} W & \overset{g}{\dashrightarrow} & Z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, and for all $W \xrightarrow{g} Z$ and $W \xrightarrow{t} X$ there are $X \xrightarrow{f} Y$ and $Z \xrightarrow{s} Y$ such that

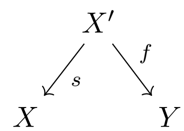
$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \overset{f}{\dashrightarrow} & Y \end{array}$$

commutes (above we require that $s, t \in S$),
(d) for $f, g : X \to Y$ we have

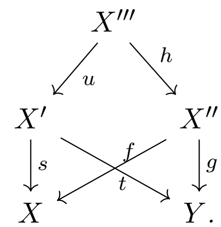
$$\exists_{s \in S}(sf = sg) \iff \exists_{t \in S}(ft = gt).$$

Remark 1.3. The class of quasi-isomorphisms in the category $\mathrm{Kom}(\mathscr{C})$ is *not* localizing.

Lemma 1.4. *If S is localizing in $\mathscr{B}$, then we can present any morphism $X \to Y$ in $\mathscr{B}[S^{-1}]$ as a triangle*



with equivalence



Pairs (s, f) and (t, g) are equivalent if and only if there is a pair (u, h) such that the two squares (or quadrangles) commute. This also applies to left fractions.

Theorem 1.5. *In any of* $\mathrm{K}(\mathscr{C})$, $\mathrm{K}^-(\mathscr{C})$,$\mathrm{K}^+(\mathscr{C})$,$\mathrm{K}^b(\mathscr{C})$ *the class of quasi-isomorphisms is localizing.*

Definition 1.6. Given a cochain complex X, define the complex $X[a]$ by

$$X[a]^i = X^{a+i},$$
$$\mathrm{d}_{X[a]} = (-1)^a \mathrm{d}_X .$$

Definition 1.7. For a morphism $f\colon X \to Y$, where $X, Y \in \mathrm{Kom}(\mathscr{C})$, define the cone $\mathrm{C}(f) \in \mathrm{Kom}(\mathscr{C})$ as

$$\mathrm{C}(f)^i = X[1]^i \oplus Y^i,$$
$$\mathrm{d}_{\mathrm{C}(f)} = (-\mathrm{d}_X \circ \mathrm{pr}_1, f[1] \circ \mathrm{pr}_1 + \mathrm{d}_Y \circ \mathrm{pr}_2).$$

Define the cylinder $\mathrm{Cyl}(f) \in \mathrm{Kom}(\mathscr{C})$ as

$$\mathrm{Cyl}(f)^i = X^i \oplus X[1]^i \oplus Y^i,$$
$$\mathrm{d}_{\mathrm{Cyl}(f)} = (\mathrm{d}_X - id_{X[1]} + 0, 0 - \mathrm{d}_X + 0, 0 + f[1] + \mathrm{d}_Y).$$

Proposition 1.8. *The cone and the cylinder are cochain complexes.*

Proof. For the cone, it is enough to check that

$$\begin{pmatrix} -\mathrm{d}_X & \\ f[1] & \mathrm{d}_Y \end{pmatrix}^2 = 0.$$

But this follows from the fact that $\mathrm{d}_X^2 = 0$ and that f is a morphism of chain complexes.

For the cylinder, it is enough to check that

$$\begin{pmatrix} \mathrm{d}_X & -\mathrm{id}_{X[1]} & \\ & -\mathrm{d}_X & \\ & f[1] & \mathrm{d}_Y \end{pmatrix}^2 = 0.$$

It is true for similar reasons. $\square$

Proposition 1.9. *For any $f\colon X \to Y$ there is the following diagram with exact rows*

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & Y & \xrightarrow{\pi} & \mathrm{C}(f) & \longrightarrow & X[1] & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \alpha_f} & & \Vert & & & & \\
0 & \longrightarrow & X & \xrightarrow{i} & \mathrm{Cyl}(f) & \longrightarrow & \mathrm{C}(f) & \longrightarrow & 0 & & \\
 & & \Vert & & \downarrow{\scriptstyle \beta_f = f+0+\mathrm{id}_Y} & & & & & & \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & & & & & &
\end{array}$$

where the most of the maps are inclusions of direct summands (in particular i is the inclusion of X into the cylinder!) or projections onto them. The maps α_f, β_f are quasi-isomorphisms.

Proof. We have $\beta\alpha = \mathrm{id}_Y$. Enough to prove that $\alpha\beta \sim \mathrm{id}_{\mathrm{Cyl}(f)}$. Note that $\alpha\beta : X \oplus X[1] \oplus Y \to X \oplus X[1] \oplus Y$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ f & 0 & \mathrm{id}_Y \end{pmatrix}$$

We define the homotopy $s : X \oplus X[1] \oplus Y \to X \oplus X[1] \oplus Y$ by $(0, \mathrm{id} + 0 + 0, 0)$, or in other words, the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ \mathrm{id}_{X[1]} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We calculate:

$$\begin{pmatrix} 0 & 0 & 0 \\ \mathrm{id}_{X[1]} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}_X & -\mathrm{id}_{X[1]} & \\ & -\mathrm{d}_X & \\ & f[1] & \mathrm{d}_Y \end{pmatrix} + \begin{pmatrix} \mathrm{d}_X & -\mathrm{id}_X & \\ & -\mathrm{d}_X & \\ & f & \mathrm{d}_Y \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{id}_X & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -\mathrm{id}_X & 0 & 0 \\ 0 & -\mathrm{id}_X & 0 \\ f[-1] & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ f[-1] & 0 & \mathrm{id}_Y \end{pmatrix} - \begin{pmatrix} \mathrm{id}_X & 0 & 0 \\ 0 & \mathrm{id}_X & 0 \\ 0 & 0 & \mathrm{id}_Y \end{pmatrix}.$$

Notice that we wrote d_X and d_Y instead of $\mathrm{d}_{X[-1]}$ and $\mathrm{d}_{Y[-1]}$, because we don't want to change the sign of the coboundary map as in Definition 1.6. $\square$

Let $f, g : X \to Y$. If we take a chain homotopy from f to g, then it gives a map $\mathrm{Cyl}(h) : \mathrm{Cyl}(f) \to \mathrm{Cyl}(g)$ given by the matrix

$$\begin{pmatrix} \mathrm{id}_X & 0 & 0 \\ 0 & \mathrm{id}_{X[1]} & 0 \\ 0 & h & \mathrm{id}_Y \end{pmatrix}$$

and also a map $\mathrm{C}(h) : \mathrm{C}(f) \to \mathrm{C}(g)$ given by

$$\begin{pmatrix} \mathrm{id}_{X[1]} & 0 \\ h & \mathrm{id}_Y \end{pmatrix}.$$

Exercise 1.10. *The maps* $\mathrm{Cyl}(h)$ *and* $\mathrm{C}(h)$ *are chain maps.*

Proposition 1.11. *The maps* $\mathrm{Cyl}(h)$ *and* $\mathrm{C}(h)$ *are isomorphisms in the category* $\mathrm{Kom}(\mathscr{C})$*.*

Proof. The inverses are just given by $\mathrm{Cyl}(-h)$ and $\mathrm{C}(-h)$ (note that $-h$ is a homotopy from g to f). $\square$

Definition 1.12. Let $\mathrm{K}(\mathscr{C})$ be the homotopy category of $\mathrm{Kom}(\mathscr{C})$ (i.e. we mod out the morphisms by the chain homotopy relation).

Definition 1.13. In any category of complexes (like $\mathrm{Kom}(\mathscr{C}, \mathrm{K}(\mathscr{C})$, or $\mathrm{D}(\mathscr{C})$) any sequence of the form

$$X \xrightarrow{\ u\ } Y \xrightarrow{\ v\ } Z \xrightarrow{\ w\ } X[1]$$

is called a triangle. We say that it is distinguished if it is isomorphic (in the corresponding category, or to be more precise, in the category of triangles in the corresponding category) to a triangle

$$X' \to \mathrm{Cyl}(f) \to \mathrm{C}(f) \to X'[1]$$

for some $f : X' \to Y'$.

Proposition 1.14. *Every exact sequence in* $\mathrm{Kom}(\mathscr{C})$ *is quasi-isomorphic to a sequence* $0 \to X \to \mathrm{Cyl}(f) \to \mathrm{C}(f) \to 0$*.*

From the proposition if follows that if $0 \to A \to B \to C \to 0$ is an exact sequence of complexes, then there is a morphism $C \xrightarrow{w} A[1]$ in $\mathrm{D}(\mathscr{C})$ such that $A \to B \to C \xrightarrow{w} A[1]$ is distinguished in $\mathrm{D}(\mathscr{C})$.

Theorem 1.15. *Let* S *be the class of quasi-isomorphisms in* $\mathrm{K}(\mathscr{C})$*. Then* $\mathrm{K}(\mathscr{C})[S^{-1}]$ *is canonically isomorphic to* $\mathrm{D}(\mathscr{C})$*. This applies to any version of* $\mathrm{Kom}(\mathscr{C})$*.*

Proof. We have the following diagram

$$\begin{array}{ccc} \mathrm{Kom}(\mathscr{C}) & \xrightarrow{\ Q\ } & \mathrm{D}(\mathscr{C}) \\ & \searrow & \downarrow{\scriptstyle \bar{Q}} \\ & & \mathrm{K}(\mathscr{C})[S^{-1}] \end{array}$$

Then $\bar{Q}$ is a bijection on objects. Note that $\bar{Q}$ takes paths of morphisms of complexes into paths of corresponding classes (to see this, check that first for paths of length one of the form $X \xrightarrow{f} Y$, and then for paths of the form $X \xrightarrow{x_s} Y$). It follows that $\bar{Q}$ is a surjection on morphisms. We will show the following

Lemma 1.16. *Assume that* $f, g : X \to Y$ *are chain homotopic in* $\mathrm{Kom}(\mathscr{C})$*. Then* $Q(f) = Q(g)$*.*

Let us see how this lemma implies that $\bar{Q}$ is an injection on morphisms. Recall that for each $X, Y \in \mathrm{Ob}\,\mathscr{C}$ we have the equivalence relation $\sim_{X,Y}$ on the sets of paths from X to Y, where two paths are equivalent if and only if they are equal after composing the images in $\mathrm{D}(\mathscr{C})$. We can define another family of relations $\sim'_{X,Y}$, where paths $p \sim'_{X,Y} q$ if and only if they are equal after composing the images in $\mathrm{K}(\mathscr{C})[S^{-1}]$.

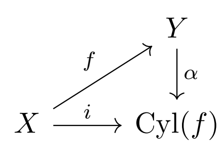
To show that $\bar{Q}$ is injective on morphisms, we need to show that $\sim = \sim'$. But from the definition $\sim'$ is the good family of relations generated by relations (1), (2), (3) and the following

(4) $X \xrightarrow{f} Y \sim' X \xrightarrow{g} Y$ for f, g chain homotopic.

(Here it is possible that we need to use some technical stuff about composing such relations of paths, maybe we need some language of quivers? The point is that we may think of the functor $\mathrm{Kom}(\mathscr{C}) \to \mathrm{K}(\mathscr{C})[S^{-1}]$ as first taking paths of morphisms in $\mathrm{Kom}(\mathscr{C})$, then modding out (1) and (4), then adding edges of the form x_s for $s \in S$, and then modding out (2) and (3).)

But Lemma 1.16 says that (4) is in $\sim$. It follows that $\sim = \sim'$. It remains to prove the Lemma.

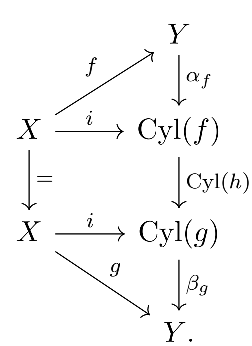
Proof of Lemma 1.16. First note that for $\alpha = \alpha_f$ and $\beta = \beta_f$ defined in Proposition 1.9 we have $Q(\alpha)Q(\beta) = \mathrm{id}_{\mathrm{Cyl}(f)}$ (easy exercise in category theory: if $uv = \mathrm{id}$ and u is an isomorphism, then $vu = \mathrm{id}$; in other words, u and v are mutual inverses). We will prove that the following diagram commutes after applying Q (in the derived category)



(all maps are from Proposition 1.9). Indeed, we have $f = \beta i$, so

$$Q(\alpha_f)Q(f) = Q(\alpha_f)Q(\beta_f)Q(i) = Q(i).$$

Let h be a homotopy from f to g. Consider the following diagram



It commutes in the derived category. But since $\beta_g \operatorname{Cyl}(h)\alpha_f = \mathrm{id}_Y$ (just look at the definition of these maps!), we get that $Q(f) = Q(g)$. □

Remark 1.17. It is a strange proof. We didn't need that $\operatorname{Cyl}(h)$ is an isomorphism, we only needed the fact that it is a chain map, and we needed the calculation $\beta_g \operatorname{Cyl}(h)\alpha_f = \mathrm{id}_Y$.

Remark 1.18. But the idea of the proof is clear: we need some cylinders or cones to translate homotopy equivalences to homotopies (the definition of Q tells us in particular that homotopy equivalences are mapped to isomorphisms; and we need to say something about homotopies). Notice that the lemma and its proof remain valid if we take Q to be any functor from $\mathrm{Kom}(\mathscr{C})$ that maps (chain) homotopy equivalences to isomorphisms.

□

Question 1.19. Can we make this proof work in a more general setting? Let $\mathscr{C}$ be a category with an equivalence class on each Hom (such that everything agrees). Can we define the cone and the cylinder in $\mathscr{C}$ in a way that will make Proposition 1.9 work? Maybe a triangulated structure on $\mathscr{C}$ mod relation is the answer to this question?

Exercise 1.20 (Easy)**.** *Show that $f \in \mathrm{Mor}(X, Y)$ is zero in $D(\mathscr{C})$ is and only if there is a quasi-isomorphism s such that $sf \sim 0$. Similar statement holds for composing from the other side.*

As a result, we get that

$$\begin{aligned} f = 0 \text{ in } \mathrm{Kom}(\mathscr{C}) \implies f = 0 \text{ in } \mathrm{K}(\mathscr{C}) \implies Q(f) = 0 \\ \implies H^i(f) = 0 \text{ for all } i. \end{aligned}$$

None of these implications can be reversed. See [GM03, Exercise III.4.1] for an example why the last one cannot be reversed.

Question 1.21. Does the fact that $Q(f)$ is an isomorphism imply that f is a quasi-isomorphism? If so, we would get a similar chain to the one above.

Answer. Yes, consider the functor $H^\bullet : \mathrm{Kom}(\mathscr{C}) \to \prod_{\mathbb{Z}} \mathscr{C}$. It takes quasi-isomorphisms to isomorphisms. From the universal property of the derived category we get what we want. □

Definition 1.22. For a given i, a complex X is an i-complex (an H^i-complex) if $X_j = 0$ for $j \neq i$ ($H^j(X) = 0$ for $j \neq i$).

Theorem 1.23. *The composition of our localization functor $Q : \mathrm{Kom}(\mathscr{C}) \to \mathrm{D}(\mathscr{C})$ with the embedding $i_0 : \mathscr{C} \to \mathrm{Kom}(\mathscr{C})$ is an equivalence of categories between $\mathscr{C}$ and the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of H^0-complexes.*

Proof. Observe first that $\mathscr{C}$ is isomorphic to the full subcategory of $\mathrm{K}(\mathscr{C})$ consisting of 0-complexes (because any homotopy between 0-complexes is the zero homotopy).

We will show that for $X, Y \in \mathscr{C}$ (so for 0-complexes) the map

$$\alpha : \operatorname{Hom}_{\mathrm{K}(\mathscr{C})}(X, Y) \to \operatorname{Hom}_{\mathrm{D}(\mathscr{C})}(Q(X), Q(Y))$$

given by

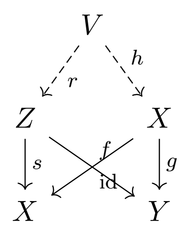
$$\alpha(g : X \to Y) = \begin{array}{ccc} & X & \\ {\scriptstyle \mathrm{id}}\swarrow & & \searrow{\scriptstyle g} \\ X & & Y \end{array}$$

is an isomorphism. The inverse is given by

$$\beta\left(\begin{array}{ccc} & Z & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} \\ X & & Y \end{array}\right) = H^0(f)H^0(s)^{-1} : X \to H^0(Z) \to Y.$$

Here Z is any complex. The fact that $\beta\alpha = \mathrm{id}$ is clear. It remains to prove that $\alpha\beta = \mathrm{id}$. We need to find V, r and h such that the following diagram

(∗)



commutes up to homotopy. Let $V^i = Z^i$ for $i < 0$, $V^0 = \ker \mathrm{d}_Z^0$, $V^i = 0$ for $i > 0$, with the differential d_V induced by d_Z. The map h_i is of course 0 for $i \neq 0$ and equal to $\ker \mathrm{d}_Z^0 \to H^0(Z) \xrightarrow{H^0(s)} X$ for $i = 0$. The map r is the embedding of V into Z. Then r is a quasi-isomorpshism (notice that the only non-trivial cohomology of Z is that of degree zero, this is true because $Z \xrightarrow{s} X$ is a quasi-isomorphism!). Let us prove that the diagrams commute (so something stronger, we need only equalities up to homotopy). After some thought we get that we need to prove that

$$\begin{array}{ccc} \ker \mathrm{d}_Z^0 & \longrightarrow & H^0(Z) \\ \downarrow & & \downarrow{\scriptstyle H^0(s)} \\ Z^0 & \xrightarrow{\quad s \quad} & X \end{array}$$

and

$$\begin{array}{ccc} \ker \mathrm{d}_Z^0 & \longrightarrow & Z^0 \\ \downarrow & & \downarrow{\scriptstyle f} \\ H^0(Z) & \xrightarrow{H^0(f)} & Y \end{array}$$

commute. But these are precisely the definitions of H^0 for morphisms in the special case when the codomains are 0-complexes.

It remains to prove that every H^0-complex is isomorphic in $\mathrm{D}(\mathscr{C})$ to a 0-complex. For this, take the upper part of the diagram $(*)$. $\square$

Remark 1.24. We are not saying this explicitly, but we are using (or are going to use) the fact $W \overset{s}{\leftarrow} V \overset{f}{\rightarrow} Z$ is an isomorphism in $\mathrm{D}(\mathscr{C})$ if and only if f is a quasi-isomorphism.

Definition 1.25. For X, $Y \in \mathscr{C}$ we define

$$\mathrm{Ext}^i_{\mathscr{C}} = \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X[0], Y[i]).$$

Remark 1.26. For any j we have a canonical isomorphism

$$\mathrm{Ext}^i_{\mathscr{C}} = \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X[j], Y[j+i]),$$

which comes from the fact that the point 0 in all these definitions is not distinguished in any way.

Remark 1.27. This definition is valid in any abelian category $\mathscr{C}$ (not necessarily having enough projectives or injectives).

For $i > 0$, we have a standard construction of elements of $\mathrm{Ext}^i_{\mathscr{C}}$. Assume that we have an exact sequence

$$0 \to K^{-i} \to K^{-i+1} \to \ldots \to K^0 \overset{s}{\to} X \to 0,$$

where $K^{-i} = Y$. Then we can produce an element in $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X[0], Y[i])$ by taking

$$\begin{array}{ccccccccccccc}
 & & & & Y & & & & & & & & \\
 & & & & \uparrow{\scriptstyle \mathrm{id}} & & & & & & & & \\
\ldots & \longrightarrow & 0 & \longrightarrow & K^{-i} & \longrightarrow & K^{-i+1} & \longrightarrow \ldots \longrightarrow & K^0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
 & & & & & & & & \downarrow{\scriptstyle s} & & & & \\
 & & & & & & \ldots & \longrightarrow 0 \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}$$

Theorem 1.28. *We have*

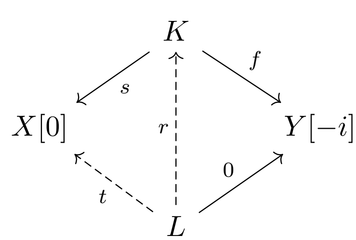
(a) $\mathrm{Ext}^0_{\mathscr{C}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$*,*

(b) $\mathrm{Ext}^i_{\mathscr{C}}(X, Y) = 0$ *for* $i < 0$*,*

(c) for $i > 0$ *every element in* $\mathrm{Ext}^i_{\mathscr{C}}(X, Y)$ *has a presentation as in the previous construction.*

Proof. The point (a) was proved in Theorem 1.23.

For (b), suppose that $i > 0$ and consider a morphism $X[0] \overset{s}{\leftarrow} K \overset{f}{\rightarrow} Y[-i]$. We will construct the following diagram so that it will be commutative



We define

$$L^j = \begin{cases} K^j & \text{for } j < i-1, \\ \ker \mathrm{d}_K^{i-1} & \text{for } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

Let r be the inclusion of L into K, and t be given by $t_0 = s_0$.

For (c), see [GM03, Theorem III.5.5] □

Exercise 1.29. *Prove that if $\mathscr{C}$ has enough injectives, then our definition of* Ext *agrees with the one given by injective resolutions.*

We solve this exercise in a series of small facts:

Fact 1.30. *If X is a complex bounded from below, and Y is a complex of injectives bounded from below, then*

$$\mathrm{Hom}_{\mathrm{K}^+(\mathscr{C})}(X, Y) = \mathrm{Hom}_{\mathrm{D}^+(\mathscr{C})}(X, Y).$$

To prove this, we need a Lemma:

Lemma 1.31. *Let $I \xrightarrow{s} K$ be a quasi-isomorphism, where I is a complex of injectives. Then there is a quasi-isomorphism $K \xrightarrow{t} I$ such that $ts \sim \mathrm{id}_I$.*

To prove the lemma, we will use the following Fact:

Fact 1.32. *Suppose we have the following morphism of complexes:*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & \dots & \longrightarrow & X^i & \longrightarrow & \dots \\ \downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_i} & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots & \longrightarrow & I^i & \longrightarrow & \dots . \end{array}$$

Assume that X is acyclic and that each I^i is injective. Then f is chain homotopic to 0*.*

Proof of Lemma 1.31. Consider the triangle

$$I \xrightarrow{s} K \to \mathrm{C}(s) \xrightarrow{\mathrm{pr}_{I[1]}} I[1].$$

Since s is quasi-iso, we know that $\mathrm{C}(s)$ is acyclic. It follows from Fact 1.32 that there is a homotopy H from $\mathrm{pr}_{I[1]}$ to 0 (which are maps $\mathrm{C}(s) \to I[1]$), let us denote it by

$$I[1] \oplus K \xrightarrow{(h,t)} I.$$

We want to show that h (or, more precisely, $h[-1]$) is a homotopy from ts to id_I which are maps $I \to I$. From the fact that H is a homotopy, we get

$$\begin{bmatrix} h[1] & t[1] \end{bmatrix} \begin{bmatrix} -\mathrm{d}_I & \\ s[1] & \mathrm{d}_K \end{bmatrix} + \mathrm{d}_{I[1]} \begin{bmatrix} h & t \end{bmatrix} = \begin{bmatrix} \mathrm{id}_{I[1]} & 0 \end{bmatrix}.$$

Precomposing with the inclusion of $I[1]$ (the first coordinate), we get that h is a homotopy from $\mathrm{id}_{I[1]}$ to $t[1]s[1]$, so $h[-1]$ is a homotopy from id_I to ts. (Notice that we used that by definition $\mathrm{d}_{X[1]}$ is $-\mathrm{d}_X$ (or, more precisely, $-\mathrm{d}_X[1]$).) □

Proof of Fact 1.30. First we prove that the map

$$\mathrm{Hom}_{\mathrm{K}(\mathscr{C})^+}(X,Y) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{C})^+)}(X,Y)$$

is injective. Suppose $X \xrightarrow{f} Y$ is zero in the derived category, then there is an $Y \xrightarrow{s} Z$ such that $sf \sim 0$. From the Lemma we get a t such that $ts \sim \mathrm{id}$. But then $f \sim tsf \sim 0$.

For surjectivity, we use right fractions. We want to show that $X \xrightarrow{g} L \xleftarrow{s} Y$ is in the image. From the Lemma we know that there is a t such that $ts \sim \mathrm{id}$. We want to show that $X \xrightarrow{tg} Y$ is mapped to $X \xrightarrow{g} L \xleftarrow{s} Y$. For this, notice that the following

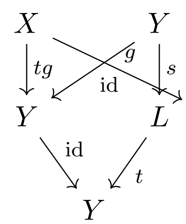


diagram commutes. □

Solution of Exercise 1.29. Let I be an injective resolution of Y. We have

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X[0],Y[i]) \cong \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X[0],I[i]) \cong \mathrm{Hom}_{\mathrm{K}(\mathscr{C})}(X[0],I[i]).$$

The first isomorphism comes from the fact that the inclusion of the 0-complex Y into its injective resolution is a quasi-iso (so an iso in the derived category), and the second follows from Fact 1.30. It is enough to show that the last Hom is the same as the classical definition of Ext^i. For this, note that the fact that $X[0] \xrightarrow{f} I[i]$ is a chain map is the same as $d^i f_i = 0$, which is the same as $f_i \in \ker \mathrm{Hom}(X, d^i)$. Finally, the fact that $f \sim 0$ is equivalent to $f = d^{i-1}h$ for some h, which is the same as $f \in \operatorname{im} \mathrm{Hom}(X, d^{i-1})$.

1.1. **Derived functor.** We will only talk about right derived functors of left exact functors $F : \mathscr{C} \to \mathscr{D}$.

Suppose $F : \mathscr{C} \to \mathscr{D}$ is left exact (and additive). We want to attach to it the right derived functor $RF : \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{D})$. We would like RF to take distiguished triangles into distinguished ones and be such that the composition

$$\mathscr{A} \to \mathrm{D}^+(\mathscr{C}) \xrightarrow{RF} \mathrm{D}^+(\mathscr{D}) \xrightarrow{H^i} \mathscr{D}$$

is the classical i-th right derived functor.

Proposition 1.33. *Suppose $F : \mathscr{C} \to \mathscr{D}$ is exact. Then*

(a) F takes quasi-isomorphisms to quasi-isomorphisms (if we apply F termwise on $\mathrm{Kom}(\mathscr{C})$), so it defines some functor $\mathrm{D}^+ F : \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{D})$.

(b) $\mathrm{D}^+ F$ takes distinguished triangles into distinguished triangles.

□

2. Triangulated categories

Let $\mathscr{C}$ be an additive category with an invertible functor $T : \mathscr{C} \to \mathscr{C}$. We define a triangle in $\mathscr{C}$ to be a sequence of maps

$$X \to Y \to Z \to T(X).$$

Definition 2.1. Such a category $\mathscr{C}$ is called a triangulated category if it is equipped with a class of triangles (called distinguished triangles) such that

TR1. For any morphism $X \xrightarrow{u} Y$ there exists a distiguished triangle

$$X \xrightarrow{u} Y \to Z \to T(X).$$

The triangle

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to T(X)$$

is distinguished. A triangle isomorphic to a distinguished triangle is distinguished.

TR2. The triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is distinguished if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$$

is.

TR3. Assume the diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow f & & \downarrow g & & & & \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

is commutative, and that the rows are distinguished. Then there exists an $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles.

TR4. Assume that

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{\partial} T(X),$$
$$Y \xrightarrow{v} Z \xrightarrow{x} X' \xrightarrow{i} T(Y),$$
$$X \xrightarrow{vu} Z \xrightarrow{y} Y' \xrightarrow{\delta} T(X)$$

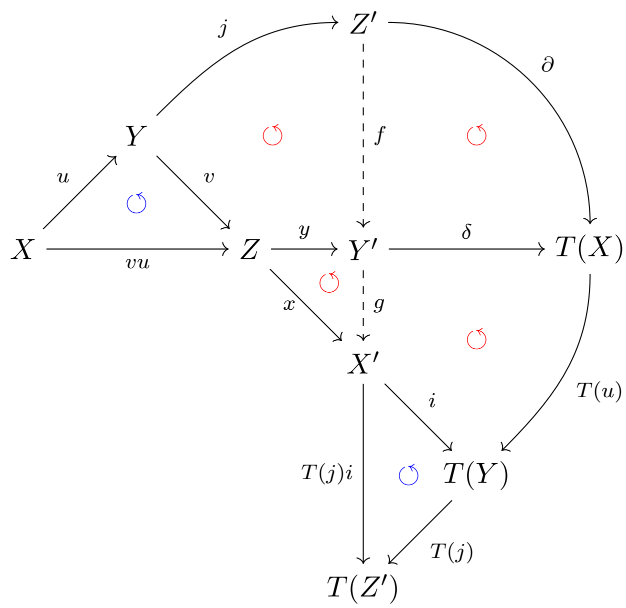
are distinguished. Then there exist $Z' \xrightarrow{f} Y'$ and $Y' \xrightarrow{g} X'$ such that

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{T(j)i} T(Z')$$

is distinguished and

$$\partial = \delta f,$$
$$x = gy,$$
$$yv = fj,$$
$$T(u)\delta = ig.$$

This can be visualized on the following diagram



The blue triangles commute by definition. TR4 says that there exist f and g such that the red polygons commute and the vertical path is distinguished.

Remark 2.2. Every distinguished triangle is determined up to an isomorphism by one of its maps. More concretely, for any commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle \mathrm{id}_Y} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{id}_{X[1]}} \\ X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X[1] \end{array}$$

with distinguished rows we get that h is an isomorphism. It follows that all the data in TR4 is determined by u and v.

We prove this remark in a series of facts (small, but interesting on its own).

Definition 2.3. Let $F : \mathscr{C} \to \mathscr{A}$ be an additive functor from a triangulated category to an abelian category. We say that it is cohomological if for every distinguished triangle

$$X \to Y \to Z \to T(X)$$

in $\mathscr{C}$ the sequence

$$F(X) \to F(Y) \to F(Z)$$

is exact.

Fact 2.4. *If $F : \mathscr{C} \to \mathscr{A}$ is a cohomological functor, and $X \to Y \to Z \to T(X)$ is distinguished, then*

$$\cdots \to F(T^{-1}(Z)) \to F(X) \to F(Y) \to F(Z) \to F(T(X)) \to \dots$$

is exact.

Proof. From TR2 (the "only if" part) applied many times we get that

$$Y \to Z \to T(X) \to T(Y),$$
$$Z \to T(X) \to T(Y) \to T(Z),$$

etc. are distinguished (with some of the signs reversed). From the "if" part we get that

$$T^{-1}(Z) \to X \to Y \to Z,$$
$$T^{-1}(Y) \to T^{-1}(Z) \to X \to Y,$$

etc. are distinguished. Applying F, we get what we wanted (remeber that in an abelian category switching signs of maps does not change exactness). □

Remark 2.5. From the proof we see that we may check exactness for any three consecutive elements and it will be enough.

Example 2.6. For any triangulated category $\mathscr{C}$ and any $U \in \mathscr{C}$ the functor $\mathrm{Hom}(U, -) : \mathscr{C} \to \mathrm{Ab}$ is cohomological.

Proof. We want to prove that for any distinguished triangle $X \to Y \to Z \to T(X)$ the sequence of abelian groups

$$\mathrm{Hom}(U, X) \to \mathrm{Hom}(U, Y) \to \mathrm{Hom}(U, Z)$$

is exact. Consider the commutative diagram

$$\begin{array}{ccccccc} U & \xrightarrow{\mathrm{id}_U} & U & \longrightarrow & 0 & \longrightarrow & T(U) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle uf} & & & & \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X). \end{array}$$

From TR1 the first row is distinguished. From TR3 it can be extented by adding a map from 0 to Z (so that it still commutes). But this means that $vuf = 0$.

Now take any $U \xrightarrow{g} Y$ such that $vg = 0$. We want to find such f that $g = uf$. Take the diagram

$$\begin{array}{ccccccc} U & \longrightarrow & 0 & \longrightarrow & T(U) & \xrightarrow{-\mathrm{id}_{T(U)}} & T(U) \\ \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(g)} \\ Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) & \xrightarrow{-T(u)} & T(Y) \end{array}$$

The rows are distinguished by TR2. From the assumption, we know that the leftmost square is commutative. It follows (from TR3) that there exists an

h such that everything is commutative. Apply T^{-1} to the rightmost square to get $g = uT^{-1}(h)$ (the signs cancel out). $\square$

Fact 2.7. *For any morphism of distinguished triangles*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow T(f) \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X'), \end{array}$$

if f and g are isomorphism, then so is h.

Proof. Applying $\mathrm{Hom}(U, -)$ everywhere, we get a morphism of exact complexes. Every two out of three consecutive arrows are isomorphisms, so from the five lemma the third one also is an isomorphism. So we know that for every $U \in \mathscr{C}$ the map $\mathrm{Hom}(U, Z) \xrightarrow{-\circ h} \mathrm{Hom}(U, Z')$ is an isomorphism. From a version of Yoneda's Lemma it follows that h is an isomorphism. $\square$

This also proves Remark 2.2.

Fact 2.8. *If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is a distinguished triangle, then vu, wv and $T(u)w$ are zero.*

Proof. From TR3 applied to the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 & \longrightarrow & T(X) \\ \downarrow \mathrm{id}_X & & \downarrow u & & & & \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \end{array}$$

we get that vu is zero. To get that the other maps are zero, use the same argument with the help of TR2. $\square$

3. Simplicial objects and stable derived functors

Let $\mathscr{C}$ be an abelian category. In this section we denote by $C_*(\mathscr{C})$ the category of those chain complexes A over $\mathscr{C}$ that $A_i = 0$ for $i < 0$. Let $X \in s\mathscr{C}$ (the category of simplicial objects in $\mathscr{C}$).

Definition 3.1. Let $\mathrm{k}\, X$ be the "standard" chain complex associated with X, i.e. for $n \geq 0$

$$(\mathrm{k}\, X)_n = X_n, \mathrm{d} = \sum_{i=0}^{n} (-1)^i d_i.$$

It is a complex for combinatorial reasons, and k is a functor $s\mathscr{C} \to C_*(\mathscr{C})$.

Definition 3.2. We define $\mathrm{N}\, X$, the normalization of X, by

$$(\mathrm{N}\, X)_n := \bigcap_{i=1}^{n} \ker(d_i : X_n \to X_{n-1}) = \ker\left(X_n \xrightarrow{(d_1, \ldots, d_n)} \prod_{i=1}^{n} X_{n-1} \right),$$
$$\mathrm{d} : (\mathrm{N}\, X)_n \to (\mathrm{N}\, X)_{n-1} \text{ induced by } d_0.$$

To do: check that this definition is ok.

Theorem 3.3. *The natural embedding* $\mathrm{N}\,X \hookrightarrow \mathrm{k}\,X$ *is a chain homotopy equivalence.*

Exercise 3.4. *The map* $(\mathrm{k}\,X)_n \xrightarrow{\mathrm{id}-s_{n-1}d_n} (\mathrm{k}\,X)_{n-1}$ *is a chain map.*

Solution. It suffices to check that

$$\sum_{i=0}^{n}(-1)^i d_i s_{n-1} d_n = \sum_{i=0}^{n}(-1)^i s_{n-2} d_{n-1} d_i.$$

We use the standard formula:

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j, \\ \mathrm{id} & \text{for } i = j \text{ or } i = j+1, \\ s_j d_{i-1} & \text{for } i > j+1. \end{cases}$$

If $i < n-1$, we know that

$$d_i s_{n-1} d_n = s_{n-2} d_i d_n = s_{n-2} d_{n-1} d_i.$$

It is enough to prove that

$$d_{n-1} s_{n-1} d_n - d_n s_{n-1} d_n = s_{n-2} d_{n-1} d_{n-1} - s_{n-2} d_{n-1} d_n.$$

But this follows from the fact that

$$d_{n-1} s_{n-1} = \mathrm{id} = d_n s_{n-1} \text{ and}$$
$$d_{n-1} d_{n-1} = d_{n-1} d_n.$$

□

References

[GM03] Sergei I. Gelfand and Yuri I. Manin, *Methods of homological algebra*, second ed., Springer Monographs in Mathematics, Springer-Verlag, 2003.