Let $F = x^3yu^3v$. Then $F^\perp = (a^4, b^2, c^4, d^2)$. We show that multihomogeneous rank of F is 12. Let $I \subseteq F^\perp$ be a B-saturated radical ideal of at most 11 points. But then $\dim(T/I)_{(3,3)} \leq 11$, so $\dim I_{(3,3)} \geq 5$. We know that

$$(F^\perp)_{(3,3)} = a^2d^2\langle a,b\rangle\langle c,d\rangle + b^2c^2\langle a,b\rangle\langle c,d\rangle + b^2d^2\langle a,b\rangle\langle c,d\rangle.$$

1. Since $\dim I_{(3,3)} \geq 5$, and $\dim(b^2c^2\langle a,b\rangle\langle c,d\rangle + b^2d^2\langle a,b\rangle\langle c,d\rangle) = 8$, we get that there is a non-zero $\hat{g} \in I_{(3,3)} \cap (b^2c^2\langle a,b\rangle\langle c,d\rangle + b^2d^2\langle a,b\rangle\langle c,d\rangle)$. However, I is radical, so also $g = \hat{g}/b \in I$. But $I \subseteq F^\perp$, hence

   $$g \in \langle b^2c^3, b^2c^2d, abcd^2, abd^3, b^2cd^2, b^2d^3\rangle.$$

   By exchanging the roles of a, b with c, d, we obtain that there exists a non-zero $f \in I$ such that

   $$f \in \langle a^3d^2, a^2bd^2, ab^2cd, b^3cd, ab^2d^2, b^3d^2\rangle.$$

2. Let

   $$\begin{aligned} g &= p_0b^2c^3 + p_1b^2c^2d + p_2abcd^2 + p_3abd^3 + p_4b^2cd^2 + p_5b^2d^3,\\ f &= q_0a^3d^2 + q_1a^2bd^2 + q_2ab^2cd + q_3b^3cd + q_4ab^2d^2 + q_5b^3d^2. \end{aligned}$$

   Assume that $q_0 \neq 0$, and consider the polynomial $R = a^2g - b(Bc + Cd)f$, where $B, C \in \mathbb{C}$. We choose B, C so that R does not contain monomials a^3bcd^2, a^3bd^3 (this can be done because of the assumption $q_0 \neq 0$). Then

   $$\begin{aligned} R &= p_0a^2b^2c^3 + p_1a^2b^2c^2d + p_4a^2b^2cd^2 + p_5a^2b^2d^3\\ &\quad - B(q_1a^2b^2cd^2 + q_2ab^3c^2d + q_3b^4c^2d + q_4ab^3cd^2 + q_5b^4cd^2)\\ &\quad - C(q_1a^2b^2d^3 + q_2ab^3cd^2 + q_3b^4cd^2 + q_4ab^3d^3 + q_5b^4d^3)\\ &= p_0a^2b^2c^3 + p_1a^2b^2c^2d + (p_4 - Bq_1)a^2b^2cd^2 + (p_5 - Cq_1)a^2b^2d^3\\ &\quad - Bq_2ab^3c^2d - Bq_3b^4c^2d - (Bq_4 + Cq_2)ab^3cd^2 - (Bq_5 + Cq_3)b^4cd^2\\ &\quad - Cq_4ab^3d^3 - Cq_5b^4d^3. \end{aligned}$$

   Thus we can divide R by b, and still get something from I. We get that $p_0 = p_1 = 0$.

3. In this step, we assume that there is a non-zero element

   $$g = p_1b^2c^2d + p_2abcd^2 + p_3abd^3 + p_4b^2cd^2 + p_5b^2d^3 \in I,$$

   Let $h \in I_{(3,3)}$, and let m_0, m_1 be the coeffcients of h corresponding to monomials a^3cd^2, a^3d^3, respectively. We claim that either $p_1 = 0$, or $m_0 = m_1 = 0$. We know that

   $$(g/b) \cdot h - g \cdot (h - m_0a^3cd^2 - m_1a^3d^3)/b \in I,$$

   but this polynomial is equal to

   $$g/b \cdot (m_0a^3cd^2 + m_1a^3d^3) = \\ p_1m_0a^3bc^3d^3 + p_2m_0a^4c^2d^4 + (p_4m_0 + p_1m_1)a^3bc^2d^4 + d^5Q$$

for some polynomial Q. We divide by d^2, and conclude that $p_1 m_0 = 0$. Then we can divide by d again and obtain $p_4 m_0 + p_1 m_1 = 0$. Thus if $p_1 \neq 0$, then $m_0 = 0$, which implies $m_1 = 0$ from the second equation.

4. In this step, we assume that there is a non-zero element

$$g' = bd(k_0 ad + k_1 bc + k_2 bd) \in I,$$

Let $h \in I_{(3,3)}$, and let m_0, m_1 be the coeffcients of h corresponding to monomials a^3cd^2, a^3d^3, respectively. We claim that $m_0 = m_1 = 0$. We know that

$$(g'/b) \cdot h - g' \cdot (h - m_0 a^3cd^2 - m_1 a^3d^3)/b \in I,$$

but this polynomial is equal to

$$g'/b \cdot (m_0 a^3cd^2 + m_1 a^3d^3) \\ = k_1 m_0 a^3bc^2d^3 + k_0 m_0 a^4cd^4 + (k_2 m_0 + k_1 m_1) a^3bcd^4 + d^5 Q'$$

for some polynomial Q'. We divide by d^2, and conclude that $k_1 m_0 = 0$. Then we can divide by d again and obtain $k_4 m_0 + k_1 m_1 = 0$. Thus if $k_1 \neq 0$, then $m_0 = 0$, which implies $m_1 = 0$ from the second equation. If $k_1 = 0$, we divide g' by d and get a contradiction.

5. We claim that from Steps 3 and 4 it follows that for any $h \in I_{(3,3)}$, the coefficients of h corresponding to monomials a^3cd^2, a^3d^3 are zero. Indeed, from Step 3 we get that if those coeffcients are not zero, then $p_1 = 0$. But if $p_1 = 0$, we can divide g by d, and get that $p_3 abd^2 + p_4 b^2cd + p_5 b^2d^2 \in I$. Then we can use Step 4 to conclude.

From the fact that these two coefficients are zero, using that $\dim I_{(3,3)} \geq 5$, from Step 1 we obtain that

$$\dim(I \cap \langle b^2c^3, b^2c^2d, abcd^2, abd^3, b^2cd^2, b^2d^3 \rangle) \geq 3.$$

By Step 2, we have

$$\dim(I \cap \langle b^2c^2d, abcd^2, abd^3, b^2cd^2, b^2d^3 \rangle) \geq 3.$$

Therefore

$$\dim(I \cap \langle abcd^2, abd^3, b^2cd^2, b^2d^3 \rangle) \geq 2.$$

We can divide this two-dimensional space by d, and we get that

$$\dim I_{(2,2)} \geq 2,$$

which is impossible by [Gał20, Theorem 1.5(iii)].

References

[Gał20] Maciej Gałązka. Multigraded apolarity. arXiv:1601.06211 [math.AG], 2020.