Labelled Well Quasi Ordered Classes of Bounded Linear Clique-Width

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University of Warsaw

MFCS'2025
Warsaw, Poland

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Presented by Maël Dumas

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ZYGMUNT ZALESKI STICHTING

Graphs and Induced Subgraphs

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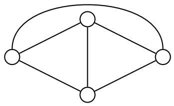
▷ Graphs are undirected, without self-loops



H

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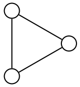

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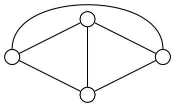

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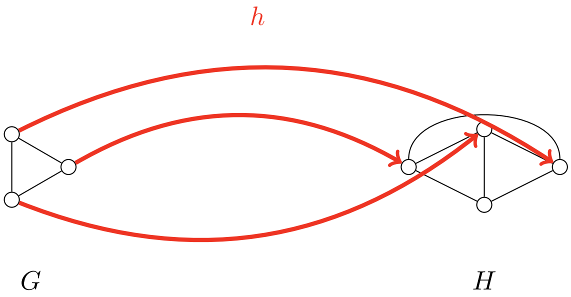

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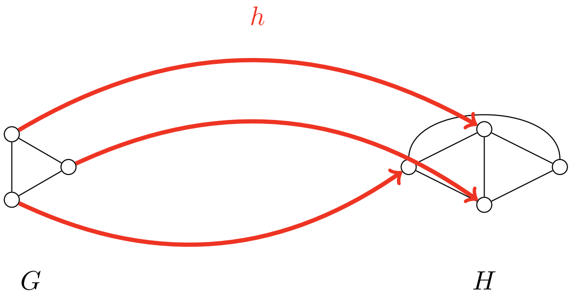

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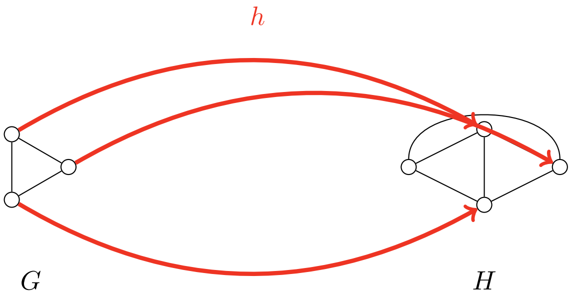

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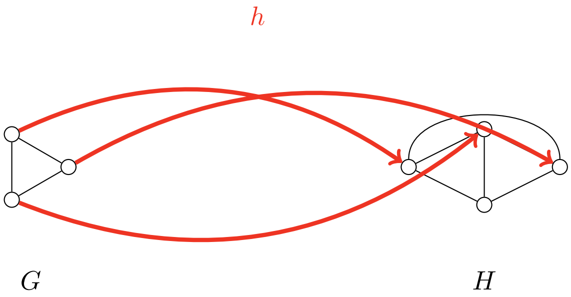

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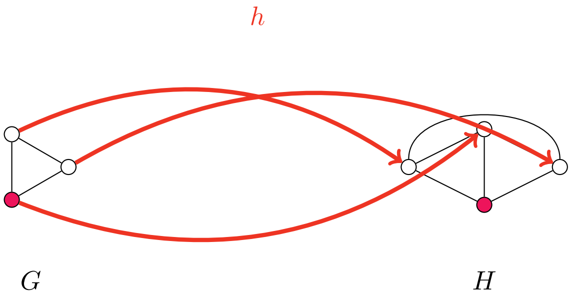

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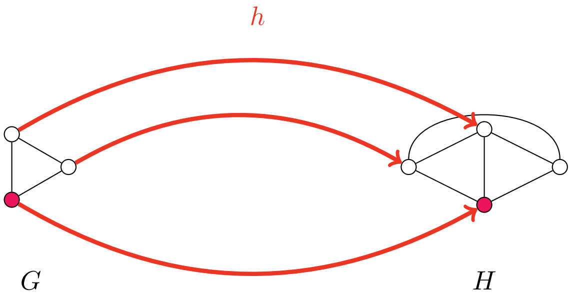

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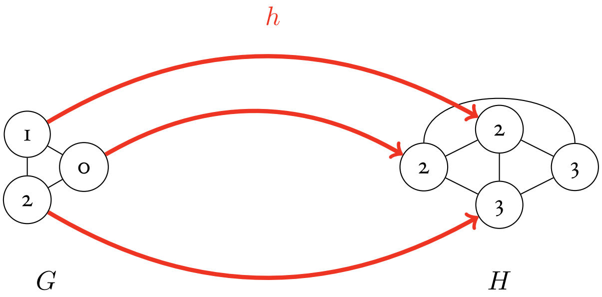

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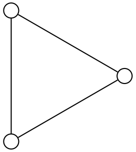

Freely Labeled Graphs

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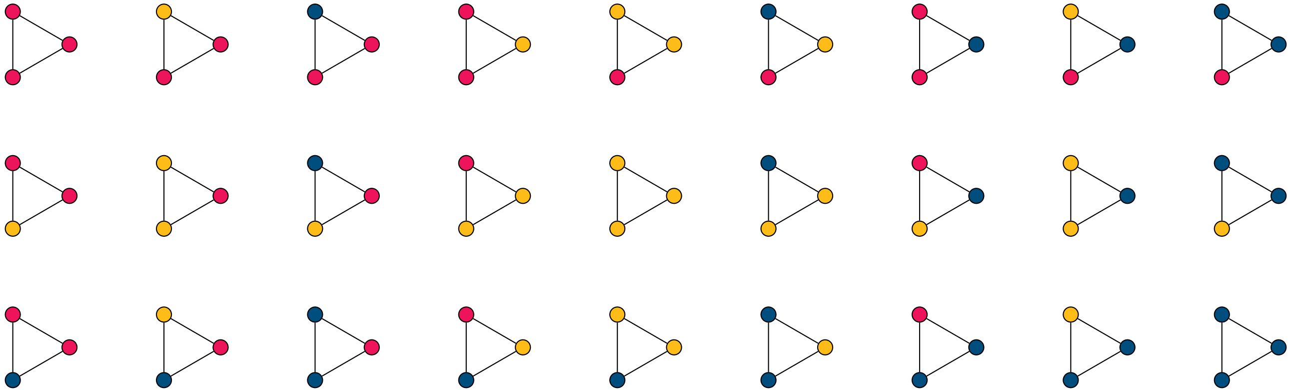
$\mathcal{C} = \{C_3\}$ and $X = \{\bullet, \bullet, \bullet\}$



Well Quasi Ordered Clases of Graphs

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▷ A sequence of graphs $G_1, G_2, \ldots$ is a **good sequence** if there exists a pair $i < j$ such that G_i *embeds* in G_j.

$$G_0 \qquad G_1 \qquad G_2 \qquad \cdots \qquad G_i \qquad \cdots \qquad G_j \qquad \cdots$$

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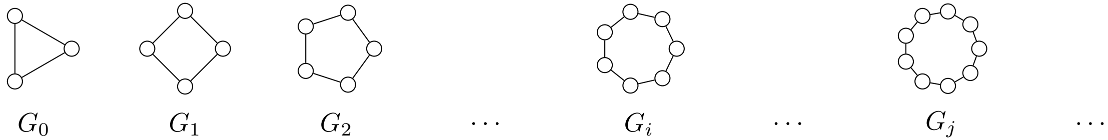

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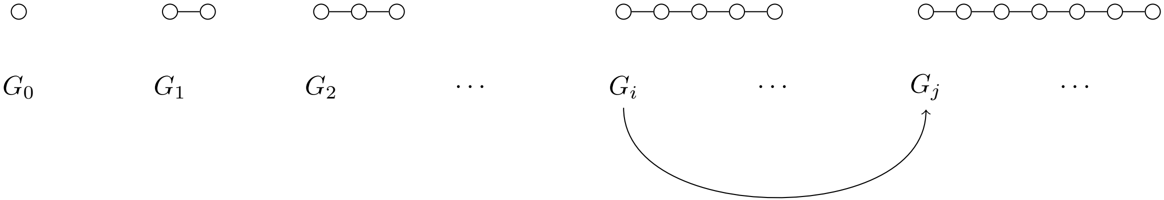

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Bad Sequence

Examples.
Cycles (NO), Paths (YES), Colored paths (NO)

Examples.
Cliques : $\forall X$ wqo, X-wqo,
Paths : wqo, but not 2-wqo, Cycles : not wqo

Why care about Well-Quasi-Ordered Classes?

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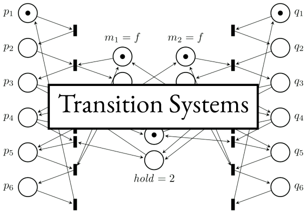

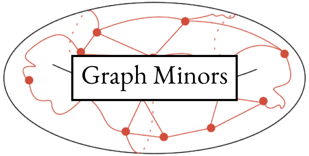

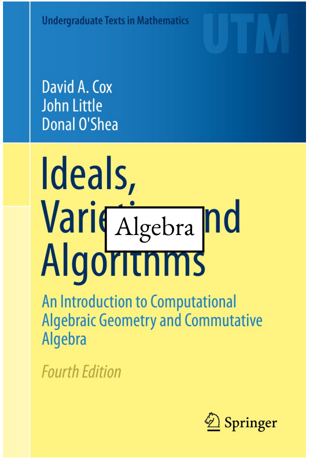

Related Work

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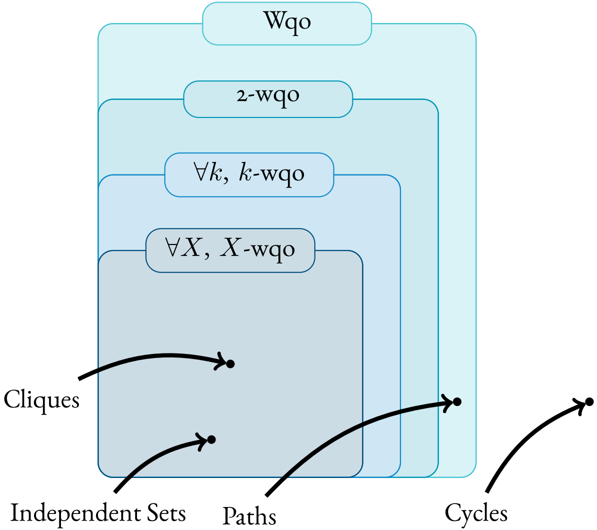
Inclusions of Properties



Related Work

Conjecture 1 [Pouzet'72].
(2-wqo) $\implies$ ($\forall k \in \mathbb{N}$, k-wqo)

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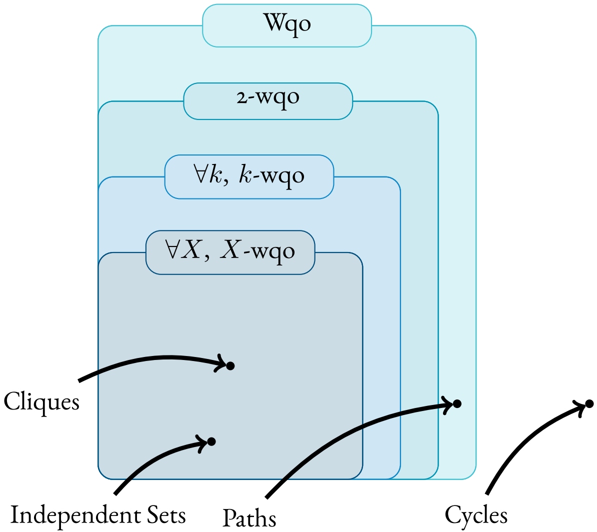

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Bounded tree-depth implies $\forall X$, X-wqo

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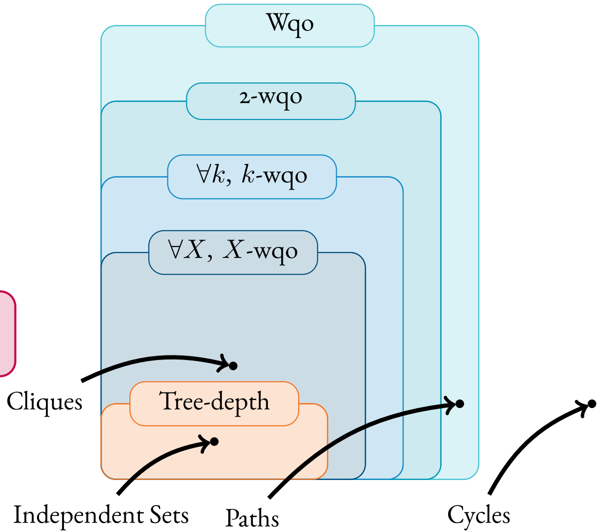

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For **some classes** of bounded clique-width, both conjectures hold

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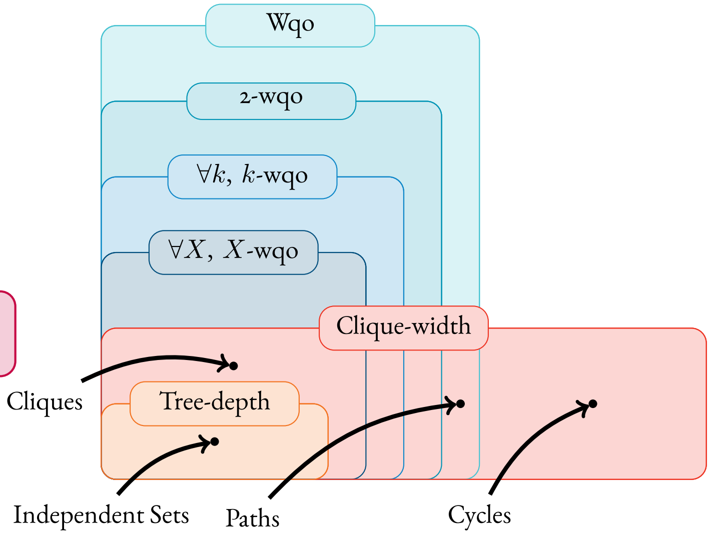

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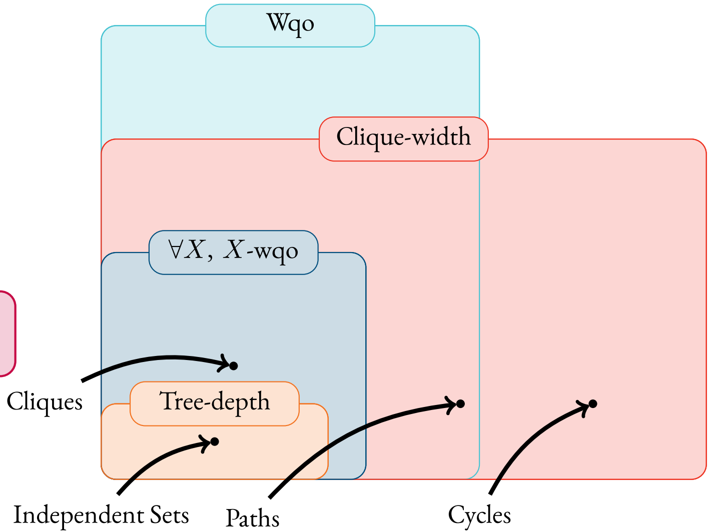
Conjectured Inclusions of Properties



$\mathsf{NLC}_Q^{\mathcal{F}}$ Expressions and Bounded Clique-Width

Relabel Expressions.

Select a finite set Q of colors, and a finite set $\mathcal{F}$ of functions from Q to Q.

- $\mathsf{vertex}(q)$ for $q \in Q$,
- $\mathsf{relabel}_f(g)$ for $f \in \mathcal{F}$, g an expression,
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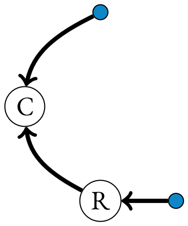

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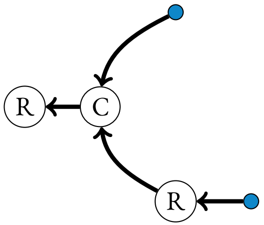

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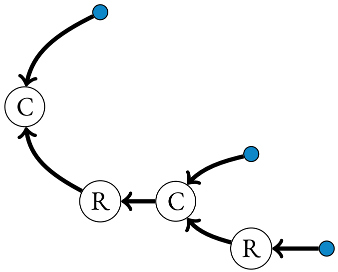

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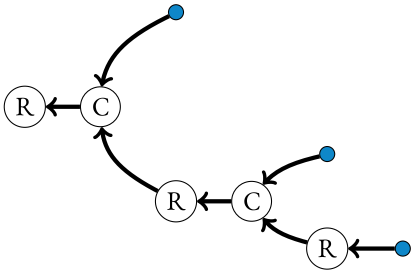

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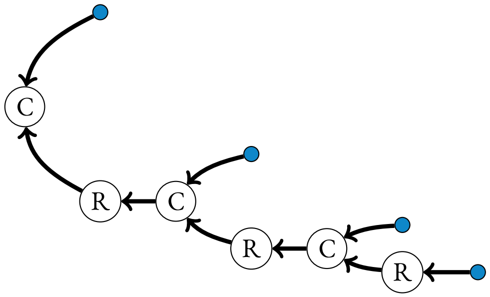

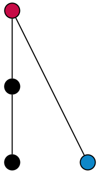

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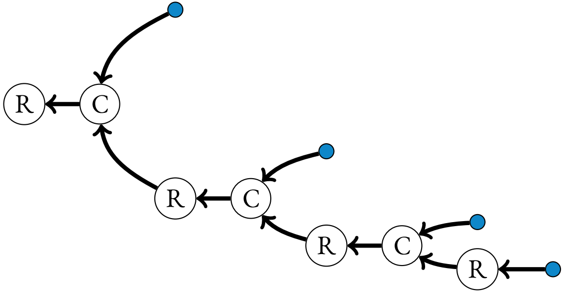

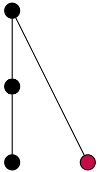

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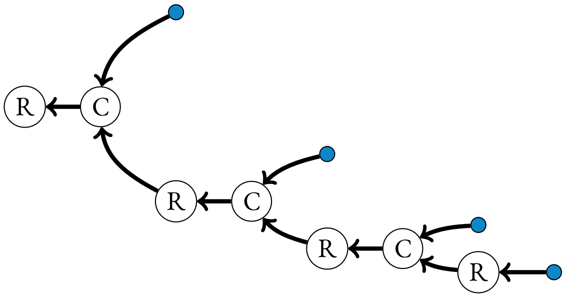
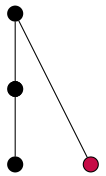
Theorem [Daligault, Rao, Thomassé'10].
For every $Q, \mathcal{F}$, one can decide whether $\mathsf{NLC}_Q^{\mathcal{F}}$ is 2-wqo.
Furthermore, the following are equivalent :

- $\mathsf{NLC}_Q^{\mathcal{F}}$ is 2-wqo,
- $\mathsf{NLC}_Q^{\mathcal{F}}$ is X-wqo, $\forall X$,
- $\mathsf{NLC}_Q^{\mathcal{F}}$ does not contain arbitrarily large paths

Therefore, conjectures 1 and 2 are true for classes $\mathsf{NLC}_Q^{\mathcal{F}}$





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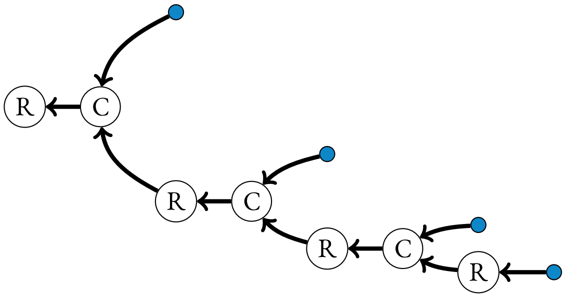

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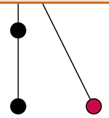
Theorem [Courcelle].
A class has *bounded clique-width* if and only if it is **contained** in some $\mathsf{NLC}_Q^{\mathcal{F}}$.



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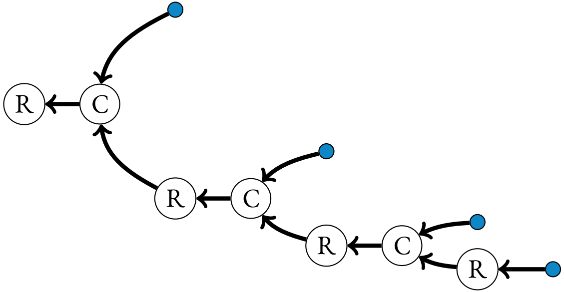

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Problem.
This does not prove conjectures 1 and 2 for classes of bounded clique-width : subsets could still be WQO!

Results

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Setting.
We restrict ourselves to the linear NLC expressions, and we fix how we add edges once and for all :

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linear shaped expression trees!

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Theorem.
Given $Q, \mathcal{F}, P$, the following properties are equivalent and **decidable** :

- $\mathsf{linNLC}_Q^{\mathcal{F},P}$ is X-wqo, for all X,
- $\mathsf{linNLC}_Q^{\mathcal{F},P}$ is k-wqo, for all k,
- $\mathsf{linNLC}_Q^{\mathcal{F},P}$ is $(|\mathcal{F}|^3 \times 2)$-wqo.

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The theorem extends to **subsets**, unlike the result of [Daligault, Rao, Thomassé'10].

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Corollary.
For all classes $\mathcal{C}$ of bounded **linear** clique-width, $\mathcal{C}$ is X-wqo (for all X) if and only if it is k-wqo (for all k).

Proof Sketch

Let $\mathcal{C} = \mathsf{linNLC}_{\mathcal{Q}}^{\mathcal{F},\mathcal{P}}$, be $(|\mathcal{F}|^3 \times 2)$-wqo.

Prove that it is X-wqo for all X wqo.

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$$\forall \quad G_0 \quad G_1 \quad G_2 \quad \cdots \quad G_i \quad \cdots \quad G_j \quad \cdots$$

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$\exists$	w_0	w_1	w_2		w_i		w_j	
$\forall$	G_0	G_1	G_2	$\cdots$	G_i	$\cdots$	G_j	$\cdots$

By definition.
$\forall G_i, \exists w_i$

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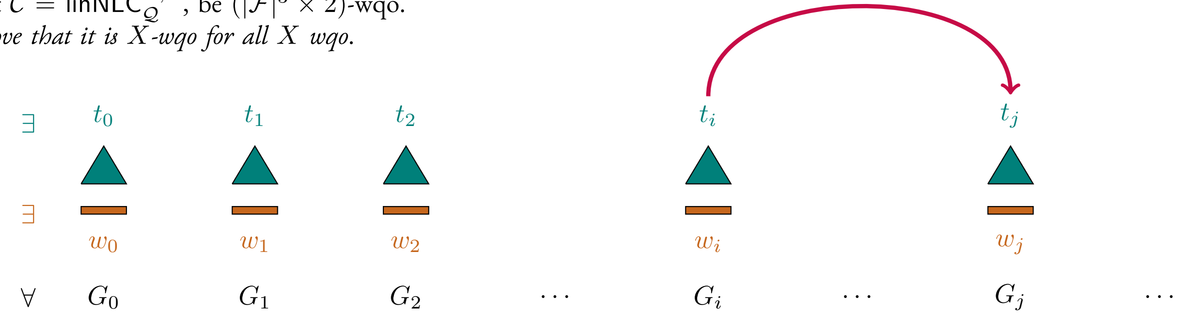

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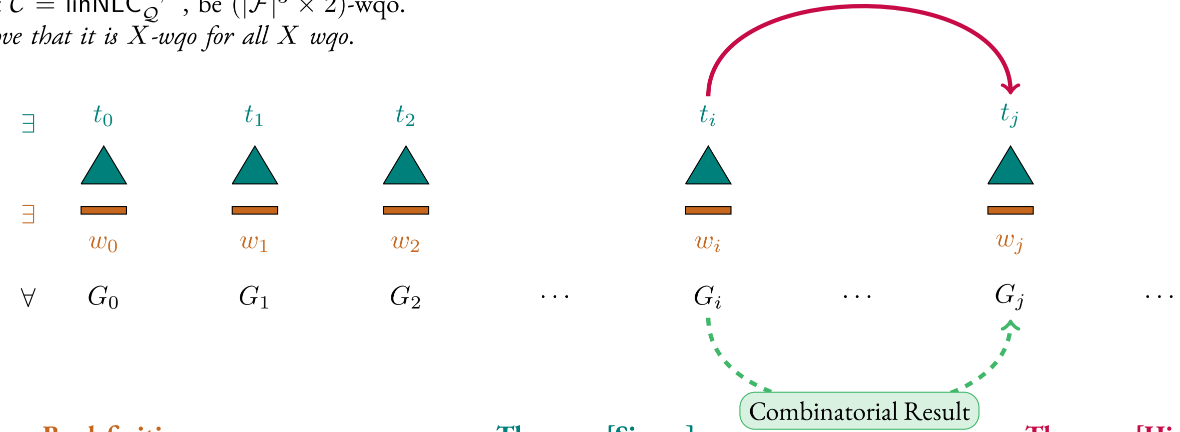

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Conclusion(s)

Theorem.
Half of Pouzet's conjecture holds for classes of bounded linear clique-width :
$\forall k \implies \forall X$

Remark.
In their paper, Daligault, Rao, and Thomassé reproved part of Simon's factorisation theorem.

Motto.
Automata / Semigroup theory applied to well-quasi-orders and structural properties of graph classes

Future Work

Towards 2-wqo.
Better analysis of the proof should lead to 2-wqo

Towards trees.
Higman'lemma $\longrightarrow$ Kruskal's tree theorem
Simon's word factorisation $\longrightarrow$ Colcombet's tree factorization

Interpretations.
It is conjectured that for all hereditary classes $\mathcal{C}$:
$\mathcal{C}$ is 2-wqo
$\iff$
one cannot interpret all **paths** using existential first order formulas