Differential geometry I, problem set
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Vector fields and differential forms

Problem 1. Let $v$ be a vector field on $M$ which is a gradient–like vector field for some Morse function $f$. Prove that there exists a Riemannian metric on $M$ such that $v$ is actually a gradient vector field for $M$.

Problem 2. Let $M$ be a compact manifold presented as a union of $M_+ \cup M_-$ along their common boundary $M_0$. Suppose that there are two smooth functions $f_\pm : M_\pm \to \mathbb{R}_\pm$ such that $f_-^{-1}(0) = M_0$ and have non-vanishing gradient on $M_0$. Prove that there exists a smooth function $f : M \to \mathbb{R}$, which is equal to $f_\pm$ away from an arbitrary small neighbourhood of $M_0$ and such that the set of critical points of $f$ is a union of the set of critical points of $f_+$ and $f_-$. Hint: construct $f$ as an integral of a suitably defined vector field.

Problem 3. Let $M$ be a compact manifold with boundary $-M_0 \sqcup M_1$. Suppose $v$ is a vector field with finitely many critical points, all of them in the interior of $M$. Assume that near each critical point the vector field can be written in local coordinates as $(\pm x_1, \ldots, \pm x_n)$. Suppose that for any $x \in M$ a trajectory of $v$ starts either on $M_0$ or at a critical point of $v$ and ends either on $M_1$ or at a critical point of $v$. Prove that if there are no 'broken circular trajectories' (unions of trajectories that start at one point and end at the same), then the vector field is a gradient–like vector field for some function $f$.

Problem 4. Prove that $S^n$ has a nowhere vanishing vector field if and only if $n$ is odd.

Problem 5. Prove that $*$ is an idempotent. The eigenvalues of $*$ are $\pm 1$ and the eigenspaces have the same dimension. We denote them by $V_{tot}^+$ and $V_{tot}^-$. 

Problem 6. Suppose dim $V = 4$. Find explicitly the basis of $V_+^2$ and $V_-^2$. * Using this prove that $SO(4) = SO(3) \times SO(3)$.

Problem 7. Compute the star operator for the induced metric on a torus in $\mathbb{R}^2$.

Problem 8. Use the star operator to compute Laplace operator in polar coordinates in $\mathbb{R}^4$.

Problem 9. Prove that for a vector field $v$ on a manifold $M$ the divergence is given by $di_v \omega$, where $\omega$ is the volume form, $i_v$ is a contraction and $d$ the exterior derivative.

Problem 10. Argue that the Liouville theorem $L_X \omega = i_X d\omega + di_X \omega$ is morally the same statement as trace is the derivative of the determinant.

Vector bundles

Problem 11. Prove that over a compact manifold $M$ every vector bundle is a subbundle of a trivial bundle.
Problem 12. Suppose $E$ is a rank $k$ vector subbundle of trivial bundle $F \cong \mathbb{F}^m$ (here $F$ is the base field) over a compact space $X$. There is an induced map $\rho_{E,m} : M \to G_{\mathbb{F}}(k,m)$.

(a) Prove that there is a $1-1$ correspondence between homotopy classes of maps from $X$ to $G_{\mathbb{F}}(k,m)$ and homotopy classes of rank $k$ vector subbundles of $\mathbb{F}^m$ (the notion of a homotopy class of a bundle is simple: it is the homotopy class of transition functions).

(b) There is a map $p_m : G_{\mathbb{F}}(k,m) \to G_{\mathbb{F}}(k,m+1)$ given by the inclusion. Prove that $\rho_{E,m+1}$ is homotopy equivalent to $p_m \circ \rho_{E,m}$.

(c) Deduce that all homotopy types of rank $k$ vector bundles are in a $1-1$ correspondence with homotopy types of maps from $M$ to $G_{\mathbb{F}}(k,\infty) = \lim G_{\mathbb{F}}(k,m)$.

Problem 13. Determine the tangent bundle to the projective plane and to the grassmanian in terms of the tautological bundle.

Problem 14. Let $\pi : E \to F$ be a linear map between vector bundles over the same manifold $M$. Suppose the rank of the image $\pi_x : E_x \to F_x$ (the subscript $x$ means that we take the fibre over a point $x \in M$) is independent of $x$. Does this mean that $\ker \pi$ is a vector bundle?

Problem 15. A Riemannian metric on the vector bundle $E$ is a choice of a scalar product in each fibre. Prove that a bundle over a paracompact manifold admits a Riemannian metric. Show that if $E$ has a Riemannian metric, then the transition functions can be chosen to sit in the orthogonal group.

Problem 16. Let $E$ be a real vector bundle over $M$ of rank $2k$. Suppose that the transition functions of $E$ sit in $U(k)$. Prove that $E$ can be given a structure of a complex bundle of rank $k$.

Problem 17. Let $E$ be a real vector bundle over $M$ of rank $2k$. Explain that it has a complex structure if and only if a corresponding map $\rho_E : M \to G_{\mathbb{R}}(2k,\infty)$ lifts to a map $G_{\mathbb{C}}(k,\infty)$.

Problem 18. Prove or find in the literature, that the tangent bundle to $S^{2k}$ for $k \neq 1, 3$ does not admit a complex structure (the default proof is completely different that the maps into grassmanian). The case $k = 2$ is ‘easiest’, the case $k$ is big is also not too hard.

Morse homology.

For some problems an elementary knowledge of homological algebra might be helpful. You can consult Weibel’s book for instance.

Problem 19. Let $M$ be a closed smooth manifold of dimension $n$ admitting a Morse function with exactly two critical points. Prove that $M$ is homeomorphic to $S^n$.

Problem 20. Let $D$ be a two–dimensional disk and $f : D \to \mathbb{R}$ be a smooth function with $f|_{\partial D} \equiv 0$ and $f$ takes both positive and negative values in the interior. Prove that $f$ must have at least three critital points (note that $f$ doesn’t have to be Morse).
Problem 21. Prove, using Morse homology, that if $M$ and $N$ are two closed manifolds, then $H_*(M \times N; \mathbb{F}) \cong H_*(M; \mathbb{F}) \times H_*(N; \mathbb{F})$ for any field $\mathbb{F}$. Hint: show that there is a quasiisomorphism $C_*(M \times N) \cong C_*(M) \otimes C_*(N)$. This is easy.

Problem 22. Using Morse homology (over a field) prove the Poincaré duality for a closed manifold $M$ of dimension $n$: $H_k(M) \cong H_{n-k}(M)$.

Problem 23. Suppose $M$ is a compact manifold with boundary. Consider Morse functions $f: M \rightarrow [0, 1]$ that are identically equal to zero (respectively one) on $\partial M$. For such a function consider the Morse complex $C_*(M)$. Prove that the homology of this complex is either $H_*(M)$ or $H_*(M; \partial M)$ (singular homology of a space or singular homology of the space). Which is which?

Problem 24. Establish a variant of the Mayer-Vietoris exact sequence in Morse homology. That is, we assume that a closed manifold $M$ is presented as a union of $M_1$ and $M_2$ along their common boundary $N$. The Mayer-Vietoris sequence computes $H_*(M)$ in terms of $H_*(M_i)$ and $H_*(N)$.

Problem 25. Show that the Euler characteristic of a closed manifold is equal to the alternating sum of numbers of critical points of a Morse function on it. Notice that the Morse–Smale condition is not necessary.

Transversality theorems.

Problem 26. Let $A, B$ be two closed submanifolds of a compact manifold $M$ intersecting transversally. Prove that $A \cap B$ is a smooth manifold of dimension $\dim A + \dim B - \dim M$.

Problem 27. Let $A, B$ be as above. Suppose $x_n$ is a sequence of points in $A \cap B$ converging to $x_0$, such that $x_n \neq x_0$. Prove that $T_{x_0}A \cap T_{x_0}B$ has positive dimension. Do not use the fact that $A \cap B$ is a manifold.

Problem 28. Show that given a generic closed surface in $\mathbb{R}^3$ there is no line tangent to it with tangency order 5.

Problem 29. Find out all possible singularities that can occur in a generic one-parameter family of smooth functions on a closed manifold $M$. What are singularities in two-parameter families? Hint: see Arnold–Varchenko–Gussein-Zade’s book or Cerf’s paper. At best both.

Riemannian geometry