Finding a Puiseux expansion of a curve in parametric form
Segovia, YMIS 2010

Maciej Borodzik

Institute of Mathematics, University of Warsaw

11 February 2010
Question

Let us be given a curve in a parametric form

\[
\begin{align*}
  x(t) &= t^4 + 2t^5 + 3t^6 + 4t^7 \\
y(t) &= t^6 + 3t^7 + 11t^8 + 30t^9 + 5t^{10}.
\end{align*}
\]

We ask what is the topological type of the singularity (Puiseux expansion)

\[
y = x^{3/2} + c_1 x^{7/4} + c_2 x^{8/4} + \\
+ c_3 x^{9/4} + c_4 x^{10/4} + c_5 x^{11/4} + \ldots
\]
Let us be given a curve in a parametric form

\[
\begin{align*}
x(t) &= t^4 + 2t^5 + 3t^6 + 4t^7 \\
y(t) &= t^6 + 3t^7 + 11t^8 + 30t^9 + 5t^{10}.
\end{align*}
\]

We ask what is the topological type of the singularity (Puiseux expansion)

\[
y = x^{3/2} + c_1x^{7/4} + c_2x^{8/4} + \\
+ c_3x^{9/4} + c_4x^{10/4} + c_5x^{11/4} + \ldots
\]
Question

Let us be given a curve in a parametric form

\[
\begin{align*}
    x(t) &= t^4 + 2t^5 + 3t^6 + 4t^7 \\
    y(t) &= t^6 + 3t^7 + 11t^8 + 30t^9 + 5t^{10}.
\end{align*}
\]

We ask what is the topological type of the singularity (Puiseux expansion)

\[
y = x^{3/2} + c_1 x^{7/4} + c_2 x^{8/4} + \\
    + c_3 x^{9/4} + c_4 x^{10/4} + c_5 x^{11/4} + \ldots
\]
Let us be given a curve in a parametric form

\[
\begin{align*}
 x(t) &= t^4 + 2t^5 + 3t^6 + 4t^7 \\
y(t) &= t^6 + 3t^7 + 11t^8 + 30t^9 + 5t^{10}.
\end{align*}
\]

We ask what is the topological type of the singularity (Puiseux expansion)

\[
y = c_0 x^{3/2} + c_1 x^{7/4} + c_2 x^{8/4} + \\
+ c_3 x^{9/4} + c_4 x^{10/4} + c_5 x^{11/4} + \ldots
\]
Standard approach

We write

\[ x^{3/2} = \]

Therefore, \( y - x^{3/2} \) is equal to

\[ 5x^5 + 20t^9 + \frac{17}{9} t^{10} + \ldots \]

Hence, \( c_1 = 0 \), \( c_2 = 0 \). Now we look at \( y = x^{3/2} - 5x^2 \). After "simple" computations we get

\[ y - x^{3/2} - 5x^2 = \frac{417}{9} t^9 + \frac{821}{8} t^{10} + \ldots \]

We get that \( c_3 = 0 \), \( c_4 = 0 \) and \( c_5 \neq 0 \) (in the latter we have to compute also \( y = x^{3/2} - 5x^2 + \frac{417}{9} x^{5/2} \)).
I'll be quick, I promise
I am being quick, aren't I?
I've really been trying to be quick, sorry.

Standard approach

We write

\[ x^{3/2} = (t^4 + 2t^5 + 3t^6 + 4t^7)^{3/2} \]

Therefore \( y - x^{3/2} \) is equal to

\[ 5t^8 + 20t^9 + \frac{17}{8} t^{10} + \ldots \]

Hence \( c_1 = 0, c_2 = 5 \). Now we look at \( y - x^{3/2} - 5x^2 \). After "simple" computations we get

\[ y - x^{3/2} - 5x^2 = -\frac{417}{8} t^{10} - \frac{821}{8} t^{11} - \ldots \]

We get that \( c_3 = 0, c_4 \neq 0 \) and \( c_5 \neq 0 \) (in the latter we have to compute also \( y - x^{3/2} - 5x^2 + \frac{417}{8} x^{5/2} \)).
Standard approach

We write

\[ x^{3/2} = t^6 \left(1 + (2t + 3t^2 + 4t^3)\right)^{3/2} \]

Therefore \( y - x^{3/2} \) is equal to

\[ 5t^8 + 20t^9 + \frac{17}{8} t^{10} + \ldots \]

Hence \( c_1 = 0, c_2 = 5 \). Now we look at \( y - x^{3/2} - 5x^2 \). After "simple" computations we get

\[ y - x^{3/2} - 5x^2 = -\frac{417}{8} t^{10} - \frac{821}{8} t^{11} - \ldots \]

We get that \( c_3 = 0, c_4 \neq 0 \) and \( c_5 \neq 0 \) (in the latter we have to compute also \( y - x^{3/2} - 5x^2 + \frac{417}{8} x^{5/2} \)).
Standard approach

We write

\[ x^{3/2} = t^6 \left( 1 + \frac{3}{2}(2t + 3t^2 + 4t^3) + \frac{3}{8}(2t + 3t^2 + 4t^3) + \ldots \right) \]

Therefore \( y - x^{3/2} \) is equal to

\[ 5t^8 + 20t^9 + \frac{17}{8}t^{10} + \ldots \]

Hence \( c_1 = 0, c_2 = 5 \). Now we look at \( y - x^{3/2} - 5x^2 \). After "simple" computations we get

\[ y - x^{3/2} - 5x^2 = -\frac{417}{8}t^{10} - \frac{821}{8}t^{11} - \ldots \]

We get that \( c_3 = 0, c_4 \neq 0 \) and \( c_5 \neq 0 \) (in the latter we have to compute also \( y - x^{3/2} - 5x^2 + \frac{417}{8}x^{5/2} \)).
Standard approach

We write

\[ x^{3/2} = t^6 + 3t^7 + 11t^8 + 30t^9 + \frac{457}{8} t^{10} + \frac{821}{8} t^{11} + \ldots \]

Therefore \( y - x^{3/2} \) is equal to

\[ 5t^8 + 20t^9 + \frac{17}{8} t^{10} + \ldots \]

Hence \( c_1 = 0, c_2 = 5 \). Now we look at \( y - x^{3/2} - 5x^2 \). After "simple" computations we get

\[ y - x^{3/2} - 5x^2 = -\frac{417}{8} t^{10} - \frac{821}{8} t^{11} - \ldots \]

We get that \( c_3 = 0, c_4 \neq 0 \) and \( c_5 \neq 0 \) (in the latter we have to compute also \( y - x^{3/2} - 5x^2 + \frac{417}{8} x^{5/2} \)).
Standard approach

We write

\[ x^{3/2} = t^6 + 3t^7 + 11t^8 + 30t^9 + \frac{457}{8}t^{10} + \frac{821}{8}t^{11} \ldots \]

Therefore \( y - x^{3/2} \) is equal to

\[ 5t^8 + 20t^9 + \frac{17}{8}t^{10} \ldots \]

Hence \( c_1 = 0, c_2 = 5 \). Now we look at \( y - x^{3/2} - 5x^2 \). After "simple" computations we get

\[ y - x^{3/2} - 5x^2 = -\frac{417}{8}t^{10} - \frac{821}{8}t^{11} \ldots \]

We get that \( c_3 = 0, c_4 \neq 0 \) and \( c_5 \neq 0 \) (in the latter we have to compute also \( y - x^{3/2} - 5x^2 + \frac{417}{8}x^{5/2} \)).
Standard approach

We write

\[ x^{3/2} = t^6 + 3t^7 + 11t^8 + 30t^9 + \frac{457}{8}t^{10} + \frac{821}{8}t^{11} + \ldots \]

Therefore \( y - x^{3/2} \) is equal to

\[ 0t^7 + 5t^8 + 20t^9 + \frac{17}{8}t^{10} + \ldots \]

Hence \( c_1 = 0, c_2 = 5 \). Now we look at \( y - x^{3/2} - 5x^2 \). After "simple" computations we get

\[ y - x^{3/2} - 5x^2 = -\frac{417}{8}t^{10} - \frac{821}{8}t^{11} - \ldots \]

We get that \( c_3 = 0, c_4 \neq 0 \) and \( c_5 \neq 0 \) (in the latter we have to compute also \( y - x^{3/2} - 5x^2 + \frac{417}{8}x^{5/2} \)).
Standard approach

We write

\[ x^{3/2} = t^6 + 3t^7 + 11t^8 + 30t^9 + \frac{457}{8} t^{10} + \frac{821}{8} t^{11} + \ldots \]

Therefore \( y - x^{3/2} \) is equal to

\[ 5t^8 + 20t^9 + \frac{17}{8} t^{10} + \ldots \]

Hence \( c_1 = 0, c_2 = 5 \). Now we look at \( y - x^{3/2} - 5x^2 \). After ”simple” computations we get

\[ y - x^{3/2} - 5x^2 = -\frac{417}{8} t^{10} - \frac{821}{8} t^{11} - \ldots . \]

We get that \( c_3 = 0, c_4 \neq 0 \) and \( c_5 \neq 0 \) (in the latter we have to compute also \( y - x^{3/2} - 5x^2 + \frac{417}{8} x^{5/2} \)).
Standard approach

We write

\[ x^{3/2} = t^6 + 3t^7 + 11t^8 + 30t^9 + \frac{457}{8}t^{10} + \frac{821}{8}t^{11} + \ldots \]

Therefore \( y - x^{3/2} \) is equal to

\[ 5t^8 + 20t^9 + \frac{17}{8}t^{10} + \ldots \]

Hence \( c_1 = 0, c_2 = 5 \). Now we look at \( y - x^{3/2} - 5x^2 \). After “simple” computations we get

\[ y - x^{3/2} - 5x^2 = -\frac{417}{8}t^{10} - \frac{821}{8}t^{11} - \ldots \]

We get that \( c_3 = 0, c_4 \neq 0 \) and \( c_5 \neq 0 \) (in the latter we have to compute also \( y - x^{3/2} - 5x^2 + \frac{417}{8}x^{5/2} \)).
Standard approach

We write

\[ x^{3/2} = t^6 + 3t^7 + 11t^8 + 30t^9 + \frac{457}{8} t^{10} + \frac{821}{8} t^{11} + \ldots \]

Therefore \( y - x^{3/2} \) is equal to

\[ 5t^8 + 20t^9 + \frac{17}{8} t^{10} + \ldots \]

Hence \( c_1 = 0, c_2 = 5 \). Now we look at \( y - x^{3/2} - 5x^2 \). After "simple" computations we get

\[ y - x^{3/2} - 5x^2 = -\frac{417}{8} t^{10} - \frac{821}{8} t^{11} - \ldots \]

We get that \( c_3 = 0, c_4 \neq 0 \) and \( c_5 \neq 0 \) (in the latter we have to compute also \( y - x^{3/2} - 5x^2 + \frac{417}{8} x^{5/2} \)).
Look at the order at zero

Write

\[ y = c_0 x^{q/p} + c_1 x^{(q+1)/p} + c_2 x^{(q+2)/p} + \ldots. \]

We divide both sides by \( x^{q/p} \). We get

where

\[ P_1 = \dot{y}x - \frac{q}{p} y\dot{x}. \]

If the order of \( P_1 \) at zero is \( q + (p - 1) + r_1 \), we know that \( c_1 = \ldots = c_{r_1 - 1} = 0 \neq c_{r_1} \). In the above example \( P_1 = 10t^{11} + 65t^{12} + 35t^{13} - 85t^{14} - 165t^{15} - 10t^{16} \), so \( r_1 = 2 \), \( c_1 = 0 \) and \( c_2 = 5 \).
Look at the order at zero

Write

\[ y = c_0 x^{q/p} + c_1 x^{(q+1)/p} + c_2 x^{(q+2)/p} + \ldots. \]

We divide both sides by \( x^{q/p} \). We get

\[ \frac{y}{x^{q/p}} = c_0 + c_1 x^{1/p} + c_2 x^{2/p} + \ldots \]

where

\[ P_1 = \dot{y} x - \frac{q}{p} y \dot{x}. \]

If the order of \( P_1 \) at zero is \( q + (p - 1) + r_1 \), we know that \( c_1 = \ldots = c_{r_1-1} = 0 \neq c_{r_1} \). In the above example \( P_1 = 10t^{11} + 65t^{12} + 35t^{13} - 85t^{14} - 165t^{15} - 10t^{16} \), so \( r_1 = 2 \), \( c_1 = 0 \) and \( c_2 = 5 \).
I’ll be quick, I promise
I’m being quick, aren’t I?
I’ve really been trying to be quick, sorry.

Look at the order at zero

Write

\[ y = c_0 x^{q/p} + c_1 x^{(q+1)/p} + c_2 x^{(q+2)/p} + \ldots. \]

We divide both sides by \( x^{q/p} \), differentiate with respect to \( t \). We get

\[
\frac{y}{x^{q/p}} = c_0 + c_1 x^{1/p} + c_2 x^{2/p} + \ldots
\]

\[
\frac{\dot{y} x - \frac{q}{p} y \dot{x}}{x^{q/p+1}} = c_1 \frac{1}{p} x^{1/p-1} + c_2 \frac{2}{p} x^{2/p-1} + \ldots
\]

where

\[ P_1 = \dot{y} x - \frac{q}{p} y \dot{x}. \]

If the order of \( P_1 \) at zero is \( q + (p-1) + r_1 \), we know that \( c_1 = \ldots = c_{r_1-1} = 0 \neq c_{r_1} \). In the above example...
Look at the order at zero

Write

\[ y = c_0 x^{q/p} + c_1 x^{(q+1)/p} + c_2 x^{(q+2)/p} + \ldots \]

We divide both sides by \( x^{q/p} \), differentiate with respect to \( t \) and multiply back by \( x^{q/p+1} \). We get

\[
\frac{\dot{y} x - \frac{q}{p} y \dot{x}}{x^{q/p+1}} = \frac{1}{p} \ddot{x} x^{1/p-1} + \frac{2}{p} \ddot{x} x^{2/p-1} + \ldots
\]

\[
\frac{\dot{y} x - \frac{q}{p} y \dot{x}}{p} = \frac{c_1}{p} \ddot{x} x^{(q+1)/p} + \frac{2c_2}{p} \ddot{x} x^{(q+2)/p} + \ldots
\]

where

\[ P_1 = \dot{y} x - \frac{q}{p} y \dot{x}. \]

If the order of \( P_1 \) at zero is \( q + (p - 1) + r_1 \), we know that \( c_1 = \cdots = c_{r_1 - 1} = 0 \neq c_r \). In the above example

\[ P_1 = 10 t^{11} + 65 t^{12} + 35 t^{13} - 85 t^{14} - 165 t^{15} - 10 t^{16} \]

so \( r_1 = 2 \).
Look at the order at zero

Write

$$y = c_0 x^{q/p} + c_1 x^{(q+1)/p} + c_2 x^{(q+2)/p} + \ldots.$$  

We divide both sides by $x^{q/p}$, differentiate with respect to $t$ and multiply back by $x^{q/p+1}$. We get

$$P_1(t) = \frac{c_1}{p} \ddot{x} x^{(q+1)/p} + \frac{2c_2}{p} \ddot{x} x^{(q+2)/p} + \ldots,$$

where

$$P_1 = \dot{y}x - \frac{q}{p} y\dot{x}.$$

If the order of $P_1$ at zero is $q + (p - 1) + r_1$, we know that $c_1 = \ldots = c_{r_1-1} = 0 \neq c_{r_1}$. In the above example $P_1 = 10t^{11} + 65t^{12} + 35t^{13} - 85t^{14} - 165t^{15} - 10t^{16}$, so $r_1 = 2$, $c_1 = 0$ and $c_2 = 5$. 

Maciej Borodzik  Finding a Puiseux expansion of a curve in parametric form
Look at the order at zero

Write

\[ y = c_0 x^{q/p} + c_1 x^{(q+1)/p} + c_2 x^{(q+2)/p} + \ldots. \]

We divide both sides by \( x^{q/p} \), differentiate with respect to \( t \) and multiply back by \( x^{q/p+1} \). We get

\[ P_1(t) = \frac{c_1}{p} \frac{\dot{x} x^{(q+1)/p}}{p - 1 + q + 1} + \frac{2c_2}{p} \frac{\dot{x} x^{(q+2)/p}}{p - 1 + q + 2} + \ldots, \]

where

\[ P_1 = \dot{y} x - \frac{q}{p} y \dot{x}. \]

If the order of \( P_1 \) at zero is \( q + (p - 1) + r_1 \), we know that \( c_1 = \cdots = c_{r_1-1} = 0 \neq c_{r_1} \). In the above example, \( P_1 = 10t^{11} + 65t^{12} + 35t^{13} - 85t^{14} - 165t^{15} - 10t^{16} \), so \( r_1 = 2 \), \( c_1 = 0 \) and \( c_2 = 5 \).
I'll be quick, I promise
I am being quick, aren't I?
I've really been trying to be quick, sorry.

Look at the order at zero

Write

\[ y = c_0 x^{q/p} + c_1 x^{(q+1)/p} + c_2 x^{(q+2)/p} + \ldots. \]

We divide both sides by \( x^{q/p} \), differentiate with respect to \( t \) and multiply back by \( x^{q/p+1} \). We get

\[ P_1(t) = \frac{c_1}{p} \dot{x} x^{(q+1)/p} + \frac{2c_2}{p} \ddot{x} x^{(q+2)/p} + \ldots, \]

where

\[ P_1 = \dot{y} x - \frac{q}{p} y \dot{x}. \]

If the order of \( P_1 \) at zero is \( q + (p - 1) + r_1 \), we know that \( c_1 = \cdots = c_{r_1-1} = 0 \neq c_{r_1} \). In the above example \( P_1 = 10t^{11} + 65t^{12} + 35t^{13} - 85t^{14} - 165t^{15} - 10t^{16} \), so \( r_1 = 2 \), \( c_1 = 0 \) and \( c_2 = 5 \).
Look at the order at zero

Write

\[ y = c_0 x^{q/p} + c_1 x^{(q+1)/p} + c_2 x^{(q+2)/p} + \ldots. \]

We divide both sides by \( x^{q/p} \), differentiate with respect to \( t \) and multiply back by \( x^{q/p+1} \). We get

\[ P_1(t) = \frac{c_1}{p} \frac{\dot{x}(q+1)}{p} + \frac{2c_2}{p} \frac{\dot{x}(q+2)}{p} + \ldots, \]

where

\[ P_1 = \dot{y}x - \frac{q}{p} y \dot{x}. \]

If the order of \( P_1 \) at zero is \( q + (p - 1) + r_1 \), we know that \( c_1 = \ldots = c_{r_1-1} = 0 \neq c_{r_1} \). In the above example

\[ P_1 = 10t^{11} + 65t^{12} + 35t^{13} - 85t^{14} - 165t^{15} - 10t^{16}, \]

so \( r_1 = 2 \), \( c_1 = 0 \) and \( c_2 = 5 \).
Looking further at the order at zero

From the equation

$$P_1(t) = \frac{r_1 c_{r_1}}{p} \dot{x} (q + r_1)/p + \frac{(r_1 + 1)c_{r_1+1}}{p} \dot{x} (q + r_1 + 1)/p + \ldots$$

we can go further dividing, differentiating and multiplying. We get

$$P_2 = S_2(r_2) c_{r_2} \dot{x}^3 x (q + r_2)/p + S_2(r_2 + 1) c_{r_2+1} \dot{x}^3 x (q + r_2 + 1)/p + \ldots,$$

where

We see that again $c_{r_1+1} = \cdots = c_{r_2-1} = 0 \neq c_{r_2}$. In our case

$$P_2 = -1680t^{19} - 11520t^{20} - 39060t^{21} - \ldots$$

Hence $c_3 = 0$, $c_4 \neq 0$. 

Maciej Borodzik  Finding a Puiseux expansion of a curve in parametric form
Looking further at the order at zero

From the equation

$$P_1(t) = \frac{r_1c_{r_1}}{p} \dot{x}(q+r_1)/p + \frac{(r_1 + 1)c_{r_1+1}}{p} \ddot{x}(q+r_1+1)/p + \ldots$$

we assume here that ord $P_1 = q + (p - 1) + r_1$ we can go further dividing, differentiating and multiplying. We get

$$P_2 = S_2(r_2)c_{r_2} \dot{x}^3 x(q+r_2)/p +$$

$$S_2(r_2 + 1)c_{r_2+1} \dot{x}^3 x(q+r_2+1)/p + \ldots,$$

where

We see that again $c_{r_1+1} = \cdots = c_{r_2-1} = 0 \neq c_{r_2}$. In our case

$$P_2 = -1680t^{19} - 11520t^{20} - 39060t^{21} - \ldots.$$
Looking further at the order at zero

From the equation

$$P_1(t) = \frac{r_1c_{r_1}}{p} \dot{x}(q+r_1)/p + \frac{(r_1+1)c_{r_1+1}}{p} \ddot{x}(q+r_1+1)/p + \ldots$$

we can go further dividing, differentiating and multiplying. We get

$$P_2 = S_2(r_2) c_{r_2} \dot{x}^3 x(q+r_2)/p +$$

$$S_2(r_2 + 1) c_{r_2+1} \dot{x}^3 x(q+r_2+1)/p + \ldots,$$

where

- $P_2 = xxP_1' - (\frac{q+r_1}{p} x^3 + xx) P_1$.
- $r_2$ is such that $\text{ord}_{t=0} P_2 = q + 3(p-1) + r_2$.

We see that again $c_{r_1+1} = \cdots = c_{r_2-1} = 0 \neq c_{r_2}$. In our case

$$P_2 = -1680t^{19} - 11520t^{20} - 39060t^{21} - \ldots.$$ 

Hence $c_3 = 0$, $c_4 \neq 0$. 
Looking further at the order at zero

From the equation

\[ P_1(t) = \frac{r_1 c_{r_1}}{p} \dot{x} x (q + r_1)/p + \frac{(r_1 + 1) c_{r_1+1}}{p} \dot{x} x (q + r_1 + 1)/p + \ldots \]

we can go further dividing, differentiating and multiplying. We get

\[ P_2 = S_2(r_2) c_{r_2} \dot{x}^3 x (q + r_2)/p + \]

\[ S_2(r_2 + 1) c_{r_2+1} \dot{x}^3 x (q + r_2 + 1)/p + \ldots , \]

where

- \( P_2 = x \dot{x} P'_1 - (\frac{q + r_1}{p} \dot{x}^2 + x \ddot{x}) P_1 \).
- \( r_2 \) is such that \( \text{ord}_{t=0} P_2 = q + 3(p - 1) + r_2 \)
- \( S_2 \) is some coefficient depending on \( q, p \) and \( r_1 \).

We see that again \( c_{r_1+1} = \cdots = c_{r_2-1} = 0 \neq c_{r_2} \). In our case

\[ P_2 = -1680 t^{19} - 11520 t^{20} - 39060 t^{21} - \ldots . \]

Hence \( c_3 = 0, c_4 \neq 0 \).
Looking further at the order at zero

From the equation

\[ P_1(t) = \frac{r_1 c_r_1}{p} \ddot{x}(q+r_1/p) + \frac{(r_1 + 1)c_{r_1+1}}{p} \ddot{x}(q+r_1+1)/p + \ldots \]

we can go further dividing, differentiating and multiplying. We get

\[ P_2 = S_2(r_2) c_{r_2} \ddot{x}^3 x(q+r_2)/p + \]

\[ S_2(r_2 + 1)c_{r_2+1} \ddot{x}^3 x(q+r_2+1)/p + \ldots , \]

where

- \( P_2 = x \dot{x} P'_1 - (\frac{q+r_1}{p} \dddot{x}^2 + x \dddot{x}) P_1. \)
- \( r_2 \) is such that \( \text{ord}_{t=0} P_2 = q + 3(p-1) + r_2 \)
- \( S_2 \) is some coefficient depending on \( q, p \) and \( r_1 \).

We see that again \( c_{r_1+1} = \ldots = c_{r_2-1} = 0 \neq c_{r_2} \). In our case

\[ P_2 = -1680t^{19} - 11520t^{20} - 39060t^{21} - \ldots \]

Hence \( c_3 = 0, c_4 \neq 0. \)
Looking further at the order at zero

From the equation

\[ P_1(t) = \frac{r_1 c_{r_1}}{p} \dot{x}(q + r_1)/p + \frac{(r_1 + 1)c_{r_1+1}}{p} \dot{x}(q + r_1 + 1)/p + \ldots \]

we can go further dividing, differentiating and multiplying. We get

\[ P_2 = S_2(r_2) c_{r_2} \dot{x}^3 x(q + r_2)/p + \]

\[ \quad S_2(r_2 + 1) c_{r_2+1} \dot{x}^3 x(q + r_2 + 1)/p + \ldots, \]

where

- \( P_2 = x \ddot{x} P'_1 - \left( \frac{q + r_1}{p} \dot{x}^2 + x \ddot{x} \right) P_1 \).
- \( r_2 \) is such that \( \text{ord}_{t=0} P_2 = q + 3(p - 1) + r_2 \)
- \( S_2 \) is some coefficient depending on \( q, p \) and \( r_1 \).

We see that again \( c_{r_1+1} = \cdots = c_{r_2-1} = 0 \neq c_{r_2} \). In our case

\[ P_2 = -1680t^{19} - 11520t^{20} - 39060t^{21} - \ldots. \]

Hence \( c_3 = 0, c_4 \neq 0 \).
Looking further at the order at zero

From the equation

\[ P_1(t) = \frac{r_1 c_{r_1}}{p} \dot{x} x (q + r_1)/p + \frac{(r_1 + 1) c_{r_1+1}}{p} \dot{x} x (q + r_1 + 1)/p + \ldots \]

we can go further dividing, differentiating and multiplying. We get

\[ P_2 = S_2(r_2) c_{r_2} \dot{x} x^3 (q + r_2)/p + \]

\[ S_2(r_2 + 1) c_{r_2+1} \dot{x} x^3 (q + r_2 + 1)/p + \ldots , \]

where

- \( P_2 = x \dot{x} P'_1 - \left( \frac{q + r_1}{p} \dot{x}^2 + x \ddot{x} \right) P_1 \).
- \( r_2 \) is such that \( \text{ord}_{t=0} P_2 = q + 3(p - 1) + r_2 \)
- \( S_2 \) is some coefficient depending on \( q, p \) and \( r_1 \).

We see that again \( c_{r_1+1} = \cdots = c_{r_2-1} = 0 \neq c_{r_2} \). In our case

\[ P_2 = -1680t^{19} - 11520t^{20} - 39060t^{21} - \ldots . \]

Hence \( c_3 = 0, \ c_4 \neq 0 \).
Looking further at the order at zero

From the equation

\[ P_1(t) = \frac{r_1 c_{r_1}}{p} \dot{x}(q+r_1)/p + \frac{(r_1 + 1)c_{r_1+1}}{p} \dot{x}(q+r_1+1)/p + \ldots \]

we can go further dividing, differentiating and multiplying. We get

\[ P_2 = S_2(r_2)c_{r_2} \dot{x}^3 x(q+r_2)/p + \]

\[ \quad S_2(r_2 + 1)c_{r_2+1} \dot{x}^3 x(q+r_2+1)/p + \ldots , \]

where

- \( P_2 = x \dot{x} P_1' - (\frac{q+r_1}{p} \dot{x}^2 + x \ddot{x})P_1 \).
- \( r_2 \) is such that \( \text{ord}_{t=0} P_2 = q + 3(p - 1) + r_2 \)
- \( S_2 \) is some coefficient depending on \( q, p \) and \( r_1 \).

We see that again \( c_{r_1+1} = \cdots = c_{r_2-1} = 0 \neq c_{r_2} \). In our case

\[ P_2 = -1680t^{19} - 11520t^{20} - 39060t^{21} - \ldots . \]

Hence \( c_3 = 0, c_4 \neq 0. \)
Looking further at the order at zero

We get

\[ P_2 = S_2(r_2)c_{r_2}\dot{x}^3x^{(q+r_2)}/p + \]

\[ S_2(r_2 + 1)c_{r_2+1}\dot{x}^3x^{(q+r_2+1)}/p + \ldots , \]

where

- \( P_2 = x\dot{x}P_1' - (\frac{q+r_1}{p}\dot{x}^2 + x\ddot{x})P_1 \).
- \( r_2 \) is such that \( \text{ord}_{t=0} P_2 = q + 3(p - 1) + r_2 \)
- \( S_2 \) is some coefficient depending on \( q, p \) and \( r_1 \).

We see that again \( c_{r_1+1} = \cdots = c_{r_2-1} = 0 \neq c_{r_2} \). In our case

\[ P_2 = -1680t^{19} - 11520t^{20} - 39060t^{21} - \ldots . \]

Hence \( c_3 = 0, \ c_4 \neq 0 \).
In general from the expression

\[ P_k = S_k(r_k) \dot{x}^{2k-1} c_{r_k} x^{(q+r_k)/p} + \ldots \]

upon applying the above procedure we get

\[ P_{k+1} = S_{k+1}(r_{k+1}) \dot{x}^{2k+1} c_{r_{k+1}} x^{(q+r_{k+1})/p} + \ldots. \]

Where

\[ P_{k+1} = x \dot{x} P'_k - P_k \left( \frac{q + r_k}{p} \dot{x}^2 + (2k + 1)x \ddot{x} \right). \]

Then, if \( r_{k+1} = \text{ord} P_{k+1} - q - (2k + 1)(p - 1) \) we get that \( c_{r_{k+1}} = \ldots = c_{r_{k+1}-1} = 0 \neq c_{r_{k+1}}. \)
General formula

In general from the expression

\[ P_k = S_k(r_k) \dot{x}^{2k-1} c_{r_k} x^{(q+r_k)/p} + \ldots \]

upon applying the above procedure we get

\[ P_{k+1} = S_{k+1}(r_{k+1}) \dot{x}^{2k+1} c_{r_{k+1}} x^{(q+r_{k+1})/p} + \ldots \]

Where

\[ P_{k+1} = x \dot{x} P'_k - P_k \left( \frac{q + r_k}{p} \dot{x}^2 + (2k + 1)x \ddot{x} \right). \]

Then, if \( r_{k+1} = \text{ord} P_{k+1} - q - (2k + 1)(p - 1) \) we get that \( c_{r_{k+1}} = \cdots = c_{r_{k+1} - 1} = 0 \neq c_{r_{k+1}} \).
General formula

In general from the expression

\[ P_k = S_k(r_k) \dot{x}^{2k-1} c_{r_k} x^{(q+r_k)/p} + \ldots \]

upon applying the above procedure we get

\[ P_{k+1} = S_{k+1}(r_{k+1}) \dot{x}^{2k+1} c_{r_{k+1}} x^{(q+r_{k+1})/p} + \ldots. \]

Where

\[ P_{k+1} = x \dot{x} P'_k - P_k \left( \frac{q + r_k}{p} \dot{x}^2 + (2k + 1) x \ddot{x} \right). \]

Then, if \( r_{k+1} = \text{ord} P_{k+1} - q - (2k + 1)(p - 1) \) we get that \( c_{r_{k+1}} = \ldots = c_{r_{k+1}-1} = 0 \neq c_{r_{k+1}}. \)
General formula

In general from the expression

\[ P_k = S_k(r_k) \dot{x}^{2k-1} c_{r_k} x^{(q+r_k)/p} + \ldots \]

upon applying the above procedure we get

\[ P_{k+1} = S_{k+1}(r_{k+1}) \dot{x}^{2k+1} c_{r_{k+1}} x^{(q+r_{k+1})/p} + \ldots. \]

Where

\[ P_{k+1} = x\dot{x} P'_k - P_k \left( \frac{q + r_k}{p} \dot{x}^2 + (2k + 1)x\ddot{x} \right). \]

Then, if \( r_{k+1} = \text{ord} P_{k+1} - q - (2k + 1)(p - 1) \) we get that

\[ c_{r_{k+1}} = \cdots = c_{r_{k+1}-1} = 0 \neq c_{r_{k+1}}. \]
In general from the expression

\[ P_k = S_k(r_k) \dot{x}^{2k-1} c_r x^{(q+r_k)/p} + \ldots \]

upon applying the above procedure we get

\[ P_{k+1} = S_{k+1}(r_{k+1}) \dot{x}^{2k+1} c_{r_{k+1}} x^{(q+r_{k+1})/p} + \ldots. \]

Where

\[ P_{k+1} = x \dot{x} P'_k - P_k \left( \frac{q + r_k}{p} \dot{x}^2 + (2k + 1)x \ddot{x} \right). \]

Then, if \( r_{k+1} = \text{ord} P_{k+1} - q - (2k + 1)(p - 1) \) we get that
\[ c_{r_{k+1}} = \ldots = c_{r_{k+1}-1} = 0 \neq c_{r_{k+1}}. \]
Advantages

- Quick and elegant way to compute.
- We can also get the Puiseux coefficients, not only whether they are zero or no.
- Hard-core example: \( x(t) = t^{12} + t^{13} + \frac{37}{28} t^{14}, \)
\( y(t) = t^{18} + \frac{3}{2} t^{19} + \frac{33}{14} t^{20} + \frac{13}{14} t^{21} + \frac{675}{1568} t^{22} - \frac{675}{3136} t^{23}. \)
- While Puiseux coefficients do not behave well in deformations, \( P_k \) do carry some information, on passing to the limit.
Advantages

- Quick and elegant way to compute.
- We can also get the Puiseux coefficients, not only whether they are zero or no.

  Hard-core example:  
  
  \[
  x(t) = t^{12} + t^{13} + \frac{37}{28} t^{14},
  \]
  
  \[
  y(t) = t^{18} + \frac{3}{2} t^{19} + \frac{33}{14} t^{20} + \frac{13}{14} t^{21} + \frac{675}{1568} t^{22} - \frac{675}{3136} t^{23}.
  \]

- While Puiseux coefficients do not behave well in deformations, \(P_k\) do carry some information, on passing to the limit.
Advantages

- Quick and elegant way to compute.
- We can also get the Puiseux coefficients, not only whether they are zero or no.
- Hard-core example: $x(t) = t^{12} + t^{13} + \frac{37}{28} t^{14}$,
  $y(t) = t^{18} + \frac{3}{2} t^{19} + \frac{33}{14} t^{20} + \frac{13}{14} t^{21} + \frac{675}{1568} t^{22} - \frac{675}{3136} t^{23}$.
- While Puiseux coefficients do not behave well in deformations, $P_k$ do carry some information, on passing to the limit.
Advantages

- Quick and elegant way to compute.
- We can also get the Puiseux coefficients, not only whether they are zero or no.
- Hard-core example: \( x(t) = t^{12} + t^{13} + \frac{37}{28} t^{14} \),
  \( y(t) = t^{18} + \frac{3}{2} t^{19} + \frac{33}{14} t^{20} + \frac{13}{14} t^{21} + \frac{675}{1568} t^{22} - \frac{675}{3136} t^{23} \).
- While Puiseux coefficients do not behave well in deformations, \( P_k \) do carry some information, on passing to the limit.
Advantages

- Quick and elegant way to compute.
- We can also get the Puiseux coefficients, not only whether they are zero or no.
- Hard-core example: $x(t) = t^{12} + t^{13} + \frac{37}{28} t^{14},$
  $y(t) = t^{18} + \frac{3}{2} t^{19} + \frac{33}{14} t^{20} + \frac{13}{14} t^{21} + \frac{675}{1568} t^{22} - \frac{675}{3136} t^{23}.$
- While Puiseux coefficients do not behave well in deformations, $P_k$ do carry some information, on passing to the limit.
I'll be quick, I promise
I am being quick, aren't I?
I've really been trying to be quick, sorry.

Advantages

- Quick and elegant way to compute.
- We can also get the Puiseux coefficients, not only whether they are zero or no.
- Hard-core example: \( x(t) = t^{12} + t^{13} + \frac{37}{28} t^{14} \),
  \( y(t) = t^{18} + \frac{3}{2} t^{19} + \frac{33}{14} t^{20} + \frac{13}{14} t^{21} + \frac{675}{1568} t^{22} - \frac{675}{3136} t^{23} \).
- While Puiseux coefficients do not behave well in deformations, \( P_k \) do carry some information, on passing to the limit.

Maciej Borodzik  Finding a Puiseux expansion of a curve in parametric form
A deformation is a family \( x_s(t) = a_p(s)t^p + a_{p+1}(s)t^{p+1} + \ldots, \)
\( y_s = b_q(s)t^q + b_{q+1}(s)t^{q+1} + \ldots, \) \( s \in B(0, 1); \)

- If \( a_p(s) \neq 0 \) for \( s \neq 0 \), the Puiseux coefficients \( c_j(s) \) are well-defined.
- If \( a_p(0) = 0 \), the limit \( \lim_{s \to 0} c_j(s) \) might not exist, or loose its topological meaning.
- The polynomials \( P_k(t, s) \) behave well under passing to the limit.
- The orders of \( P_k(t, s) \) at \( t = 0 \) have a topological meaning even for \( s = 0 \).
A deformation is a family $x_s(t) = a_p(s)t^p + a_{p+1}(s)t^{p+1} + \ldots$, $y_s = b_q(s)t^q + b_{q+1}(s)t^{q+1} + \ldots$, $s \in B(0,1)$;

- If $a_p(s) \neq 0$ for $s \neq 0$, the Puiseux coefficients $c_j(s)$ are well-defined.
- If $a_p(0) = 0$, the limit $\lim_{s \to 0} c_j(s)$ might not exist, or lose its topological meaning.
- The polynomials $P_k(t,s)$ behave well under passing to the limit.
- The orders of $P_k(t,s)$ at $t = 0$ have a topological meaning even for $s = 0$. 

Maciej Borodzik  
Finding a Puiseux expansion of a curve in parametric form
Deformations

- A deformation is a family $x_s(t) = a_p(s)t^p + a_{p+1}(s)t^{p+1} + \ldots$, $y_s = b_q(s)t^q + b_{q+1}(s)t^{q+1} + \ldots$, $s \in B(0, 1)$;
- If $a_p(s) \neq 0$ for $s \neq 0$, the Puiseux coefficients $c_j(s)$ are well-defined.
- If $a_p(0) = 0$, the limit $\lim_{s \to 0} c_j(s)$ might not exist, or loose its topological meaning.
- The polynomials $P_k(t, s)$ behave well under passing to the limit.
- The orders of $P_k(t, s)$ at $t = 0$ have a topological meaning even for $s = 0$. 

Maciej Borodzik  
Finding a Puiseux expansion of a curve in parametric form
Deformations

- A deformation is a family $x_s(t) = a_p(s)t^p + a_{p+1}(s)t^{p+1} + \ldots$, $y_s = b_q(s)t^q + b_{q+1}(s)t^{q+1} + \ldots$, $s \in B(0, 1)$;

- If $a_p(s) \neq 0$ for $s \neq 0$, the Puiseux coefficients $c_j(s)$ are well-defined.

- If $a_p(0) = 0$, the limit $\lim_{s \to 0} c_j(s)$ might not exist, or lose its topological meaning.

- The polynomials $P_k(t, s)$ behave well under passing to the limit.

- The orders of $P_k(t, s)$ at $t = 0$ have a topological meaning even for $s = 0$. 
Deformations

- A deformation is a family \( x_s(t) = a_p(s)t^p + a_{p+1}(s)t^{p+1} + \ldots , \)
  \( y_s = b_q(s)t^q + b_{q+1}(s)t^{q+1} + \ldots , s \in B(0, 1); \)
- If \( a_p(s) \neq 0 \) for \( s \neq 0 \), the Puiseux coefficients \( c_j(s) \) are well-defined.
- If \( a_p(0) = 0 \), the limit \( \lim_{s \to 0} c_j(s) \) might not exist, or loose its topological meaning.
- The polynomials \( P_k(t, s) \) behave well under passing to the limit.
- The orders of \( P_k(t, s) \) at \( t = 0 \) have a topological meaning even for \( s = 0 \).