Affine algebraic curves with zero Euler characteristics

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Problem

• Curve $C \subset \mathbb{C}^2$. 
Problem

• Curve $\mathcal{C} \subset \mathbb{C}^2$.
• $\overline{\mathcal{C}} \subset \mathbb{C}P^2$ its closure.
Problem

• Curve $\mathcal{C} \subset \mathbb{C}^2$.
• $\bar{\mathcal{C}} \subset \mathbb{C}P^2$ its closure.
• $\bar{\mathcal{C}}$ is rational.
Problem

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- $\overline{C} \subset \mathbb{C}P^2$ its closure.
- $\overline{C}$ is rational.
- $\chi(C) = 0$. 
Problem

- Curve $\mathcal{C} \subset \mathbb{C}^2$.
- $\bar{\mathcal{C}} \subset \mathbb{C}P^2$ its closure.
- $\bar{\mathcal{C}}$ is rational.
- $\chi(\mathcal{C}) = 0$.

It follows, that either
Problem

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• $\overline{C} \subset \mathbb{C}P^2$ its closure.
• $\overline{C}$ is rational.
• $\chi(C) = 0$.

It follows, that either

• $C \cong \mathbb{C}^*$ and $C$ has no finite self–intersections.
Problem

- Curve $C \subset \mathbb{C}^2$.
- $\bar{C} \subset \mathbb{C}P^2$ its closure.
- $\bar{C}$ is rational.
- $\chi(C) = 0$.

It follows, that either

- $C \simeq \mathbb{C}^*$ and $C$ has no finite self–intersections.
- $C$ has one place at infinity and one finite self–intersection.
Known results
Known results

If $\chi(C) = 1$, then $C$ is homeomorphic to a line.
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Zajdenberg–Lin theorem: $C \simeq \{x^p = y^q\}$ with $p, q$ coprime.
Known results

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Zajdenberg–Lin theorem: $C \simeq \{x^p = y^q\}$ with $p, q$ coprime.

Koras, Russell case $C \simeq \mathbb{C}^*$ and $C$ smooth.
Our result

It is restricted to regular curves.
Our result

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Our result

It is restricted to regular curves.

• Conjecture: All curves are regular.
Our result

It is restricted to regular curves.

• Conjecture: All curves are regular.
• Lots of evidence.
Our result

It is restricted to regular curves.

- **Conjecture:** All curves are regular.
- Lots of evidence.
- A gap in the proof.
Two points of view

• Equation
Two points of view

- **Equation**
- \( C = \{ (x, y) \in \mathbb{C}^2 : f(x, y) = 0 \} \)
Two points of view

- **Equation**
  \[ C = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\} \]
- **Genus is difficult.**
Two points of view

- Equation
  
  \[ C = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\} \]

- Genus is difficult.

- Singular points \( f'_x(x_0, y_0) = f'_y(x_0, y_0) = 0 \)
Two points of view

• Equation

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• Singular points \( f'_x(x_0, y_0) = f'_y(x_0, y_0) = 0 \)

• Self–intersections are singular points.
Two points of view

- Equation

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- Parametrisation
Two points of view

- Equation
  \[ C = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\} \]
- Genus is difficult.
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- Self–intersections are singular points.

- Parametrisation
  \[ C = \{(x(t), y(t)) \in \mathbb{C}^2, t \in \mathbb{C}P^1\} \]
Two points of view

- **Equation**
  
  \[ C = \{ (x, y) \in \mathbb{C}^2 : f(x, y) = 0 \} \]

- Genus is difficult.

- **Singular points**
  
  \[ f'_x(x_0, y_0) = f'_y(x_0, y_0) = 0 \]

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---

- **Parametrisation**

  \[ C = \{ (x(t), y(t)) \in \mathbb{C}^2, t \in \mathbb{C}P^1 \} \]

- genus is zero.
Two points of view

- **Equation**
  
  \[ C = \left\{ (x, y) \in \mathbb{C}^2 : f(x, y) = 0 \right\} \]

- **Genus is difficult.**

- **Singular points**
  \[ f'_x(x_0, y_0) = f'_y(x_0, y_0) = 0 \]

- **Self-intersections are singular points.**

---

**Parametrisation**

- \[ C = \left\{ (x(t), y(t)) \in \mathbb{C}^2, t \in \mathbb{C}P^1 \right\} \]

- **genus is zero.**
Two points of view

- **Equation**
  
  \[ C = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\} \]

- **Genus is difficult.**

- **Singular points**
  
  \[ f_x'(x_0, y_0) = f_y'(x_0, y_0) = 0 \]

- **Self–intersections are singular points.**

---

- **Parametrisation**

  - **Equation**
    
    \[ C = \{(x(t), y(t)) \in \mathbb{C}^2, t \in \mathbb{C}P^1\} \]

  - **Genus is zero.**

  - **Singular points:**
    
    \[ x'(t_0) = y'(t_0) = 0 \]
Two points of view

- **Equation**
  - \( C = \{ (x, y) \in \mathbb{C}^2 : f(x, y) = 0 \} \)
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  - Singular points \( f'_x(x_0, y_0) = f'_y(x_0, y_0) = 0 \)
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- **Parametrisation**
  - \( C = \{ (x(t), y(t)) \in \mathbb{C}^2, t \in \mathbb{C}P^1 \} \)
  - genus is zero.
  - Singular points: \( x'(t_0) = y'(t_0) = 0 \)
  - Self–intersections are difficult.
Parametric curves

Rational curve $C$ with one place at infinity is given by a polynomial
Parametric curves

Rational curve $C$ with one place at infinity is given by a polynomial

\[
\begin{align*}
x(t) &= t^a + \alpha_1 t^{a-1} + \cdots + \alpha_a \\
y(t) &= t^c + \beta_1 t^{c-1} + \cdots + \beta_c.
\end{align*}
\]
Parametric curves

Rational curve $C$ with one place at infinity is given by a polynomial

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Any rational $C$ with two branches at infinity is given by a polynomial in $t$ and $t^{-1}$. 
Parametric curves

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Any rational $C$ with two branches at infinity is given by a polynomial in $t$ and $t^{-1}$.

so

\[
\begin{align*}
x(t) &= t^a + \alpha_1 t^{a-1} + \cdots + \alpha_{a+b} t^{-b} \\
y(t) &= t^c + \beta_1 t^{c-1} + \cdots + \beta_{c+d} t^{-d}.
\end{align*}
\]
δ invariant

For singular point with Milnor number $\mu$ and $r$ branches set
$\delta$ invariant

For singular point with Milnor number $\mu$ and $r$ branches set

$$2\delta = \mu + r - 1.$$
δ \textit{invariant}

For singular point with Milnor number \( \mu \) and \( r \) branches set

\[
2\delta = \mu + r - 1.
\]

If \((x_0, y_0)\) is a singular point of \((x(t), y(t))\),
\[\delta \text{ invariant}\]

For singular point with Milnor number \(\mu\) and \(r\) branches set

\[2\delta = \mu + r - 1.\]

If \((x_0, y_0)\) is a singular point of \((x(t), y(t))\), \(2\delta\) is the number of solutions to
δ invariant

For singular point with Milnor number $\mu$ and $r$ branches set

$$2\delta = \mu + r - 1.$$ 

If $(x_0, y_0)$ is a singular point of $(x(t), y(t))$, $2\delta$ is the number of solutions to

$$\begin{cases} 
\frac{x(s_1) - x(s_2)}{s_1 - s_2} = 0 \\
\frac{y(s_1) - y(s_2)}{s_1 - s_2} = 0 
\end{cases}$$

such that $x(s_1) = x_0$ i $y(s_1) = y_0$. 
δ invariant

For singular point with Milnor number $\mu$ and $r$ branches set

$$2\delta = \mu + r - 1.$$ 

If $(x_0, y_0)$ is a singular point of $(x(t), y(t))$, $2\delta$ is the number of solutions to

**double point equation**

$$\begin{align*}
\frac{x(s_1) - x(s_2)}{s_1 - s_2} &= 0 \\
\frac{y(s_1) - y(s_2)}{s_1 - s_2} &= 0
\end{align*}$$

such that $x(s_1) = x_0$ i $y(s_1) = y_0$. 

\textbf{\(\delta\) invariant}

For singular point with Milnor number \(\mu\) and \(r\) branches set

\[
2\delta = \mu + r - 1.
\]

If \((x_0, y_0)\) is a singular point of \((x(t), y(t))\), \(2\delta\) is the number of solutions to

\[
\begin{cases}
\frac{x(s_1) - x(s_2)}{s_1 - s_2} = 0 \\
\frac{y(s_1) - y(s_2)}{s_1 - s_2} = 0
\end{cases}
\]

such that \(x(s_1) = x_0\) i \(y(s_1) = y_0\).

For an ordinary double point we have \(2\delta = 2\).
Example
Example

\[ y^2 = x^3 + \lambda x^2, \]
Example

\[ y^2 = x^3 + \lambda x^2, \quad \lambda = 2. \]
Example

\[ y^2 = x^3 + \lambda x^2, \quad \lambda = 1. \]
Example

$$y^2 = x^3 + \lambda x^2, \quad \lambda = \frac{1}{2}.$$
Example

$$y^2 = x^3 + \lambda x^2, \quad \lambda = 0.$$ 

One double point „hides” in a singular point. $$2\delta = 2.$$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]
Another example

Curves depend on $\lambda$.

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\begin{align*}
x_\lambda(t) &= t^3 - 15\lambda^2t \\
y_\lambda(t) &= t^5 - 30\lambda^2t^3 + 10\lambda^3t^2 + 201\lambda^4t,
\end{align*}
\]

$\lambda = 1$
Another example

Curves depend on $\lambda$.

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\begin{align*}
  x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.99$
Another example

Curves depend on $\lambda$.

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\begin{align*}
x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.98$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
    x_\lambda(t) &= t^3 - 15\lambda^2 t \\
    y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.97$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
  x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.96$
Another example

Curves depend on $\lambda$.

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\begin{align*}
x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.95$
Another example

Curves depend on $\lambda$.

\[
\begin{cases}
x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{cases}
\]

$\lambda = 0.94$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
  x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.93$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
  x_\lambda(t) &= t^3 - 15\lambda^2 t \\
  y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.92$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
    x_\lambda(t) & = t^3 - 15\lambda^2 t \\
    y_\lambda(t) & = t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.91$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
    x_\lambda(t) &= t^3 - 15\lambda^2 t \\
    y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.89$
Another example

Curves depend on $\lambda$.

\[
\begin{aligned}
  x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{aligned}
\]

$\lambda = 0.87$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
    x_\lambda(t) &= t^3 - 15\lambda^2 t \\
    y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.85$
Another example

Curves depend on $\lambda$.

\[
\begin{cases}
    x_\lambda(t) &= t^3 - 15\lambda^2 t \\
    y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{cases}
\]

$\lambda = 0.83$
Another example

Curves depend on $\lambda$.

\[
\begin{cases}
  x_\lambda(t) = t^3 - 15\lambda^2 t \\
  y_\lambda(t) = t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{cases}
\]

$\lambda = 0.81$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
  x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.79$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
    x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.75$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
    x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.7$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
  x_\lambda(t) &= t^3 - 15\lambda^2 t \\
  y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.65$
Another example

Curves depend on $\lambda$.

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\begin{align*}
    x_\lambda(t) &= t^3 - 15\lambda^2 t \\
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\end{align*}
\]

$\lambda = 0.6$
Another example
Curves depend on $\lambda$.

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\begin{align*}
  x_\lambda(t) &= t^3 - 15\lambda^2 t \\
  y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.55$
Another example

Curves depend on $\lambda$.

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\begin{align*}
x_\lambda(t) &= t^3 - 15\lambda^2 t \\
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\end{align*}
\]

$\lambda = 0.5$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
  x_\lambda(t) &= t^3 - 15\lambda^2 t \\
  y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.45$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
x_\lambda(t) &= t^3 - 15\lambda^2 t \\
y_\lambda(t) &= t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t,
\end{align*}
\]

$\lambda = 0.4$
Another example

Curves depend on $\lambda$.

\[
\begin{align*}
    x_\lambda(t) &= t^3 - 15\lambda^2t \\
y_\lambda(t) &= t^5 - 30\lambda^2t^3 + 10\lambda^3t^2 + 201\lambda^4t,
\end{align*}
\]

$\lambda = 0$

Four double points hide in a singular point $(t^3, t^5)$. Thus $2\delta = 8$. 
Serre formula

For a curve $C$ of degree $d$ we have
Serre formula

For a curve $C$ of degree $d$ we have

$$g = \frac{(d - 1)(d - 2)}{2} - \sum \delta_i$$
Serre formula

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$g$ — genus.
Serre formula

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$g$ — genus.

$d$ — degree.
Serre formula

For a curve $C$ of degree $d$ we have

\[ g = \frac{(d - 1)(d - 2)}{2} - \sum \delta_i \]

- $g$ — genus.
- $d$ — degree.
- $\delta_i$ — $\delta$ invariant of a singular point.
Serre formula

For a curve $C$ of degree $d$ we have

$$g = \frac{(d - 1)(d - 2)}{2} - \sum \delta_i$$

$g$ — genus.

$d$ — degree.

$\delta_i$ — $\delta$ invariant of a singular point.

$\sum$ — sum over all singular points and double points.
Serre formula II

We require $g = 0$. Thus

$$
\sum 2\delta_i = (d - 1)(d - 2).
$$
Serre formula II

We require $g = 0$. Thus

$$\sum 2\delta_i = (d - 1)(d - 2).$$

- For a typical curve $\delta_i$ correspond to ordinary double points.
Serre formula II

We require $g = 0$. Thus

$$\sum 2\delta_i = (d - 1)(d - 2).$$

- For a typical curve $\delta_i$ correspond to ordinary double points.
- If $C$ has no finite double points (or only one), all other points must be hidden in singular points.
Serre formula II

We require $g = 0$. Thus

$$\sum 2\delta_i = (d - 1)(d - 2).$$

- For a typical curve $\delta_i$ correspond to ordinary double points.
- If $C$ has no finite double points (or only one), all other points must be hidden in singular points.
- Maybe at infinity.
Codimension of a singular point.

To control the deformations of a parametric curves we introduce
Codimension of a singular point.

To control the deformations of a parametric curves we introduce the codimension.
Codimension of a singular point.

To control the deformations of a parametric curves we introduce the codimension. *Strongly resembles $\bar{M}$ number of Orevkov.*
Codimension of a singular point.

To control the deformations of a parametric curves we introduce the codimension. Parametrise locally $x(t) \sim t^p$, $y(t) \sim t^q + \ldots$. Write

$$y = c_1 x^{1/p} + c_2 x^{2/p} + \cdots + c_i x^{i/p} + \ldots.$$
Codimension of a singular point.

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$c_1, c_2, \ldots, c_i, \ldots$ — Puiseux coefficients
Codimension of a singular point.

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$c_1, c_2, \ldots, c_i, \ldots$ — Puiseux coefficients

- The local *codimension* $\nu$ is the number of vanishing *essential* Puiseux coefficients.
Codimension of a singular point.

To control the deformations of a parametric curves we introduce the codimension.

Parametrise locally $x(t) \sim t^p$, $y(t) \sim t^q + \ldots$. Write

$$y = c_1 x^{1/p} + c_2 x^{2/p} + \cdots + c_i x^{i/p} + \ldots.$$ 

$c_1, c_2, \ldots, c_i, \ldots$ — Puiseux coefficients

- The local *codimension* $\nu$ is the number of vanishing *essential* Puiseux coefficients.
- $\nu$ is determined by the characteristic sequence and the order $p$. 
Codimension inequality

If $x(t) \sim t^p$, $y(t) \sim t^q$ we have:

\[ \mu \leq p\nu. \]
Codimension inequality

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$\mu \leq \rho \nu$. 
Codimension inequality

If $x(t) \sim t^p$, $y(t) \sim t^q$ we have:

$$\mu \leq p\nu.$$  

where $\mu$ Milnor number ($= 2\delta$),
Codimension inequality

If $x(t) \sim t^p$, $y(t) \sim t^q$ we have:

$$\mu \leq p \nu.$$

where $\mu$ Milnor number ($= 2\delta$), $\nu$ the local codimension
Codimension inequality

If $x(t) \sim t^p$, $y(t) \sim t^q$ we have:

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where $\mu$ Milnor number ($= 2\delta$), $\nu$ the local codimension
Codimension inequality

If $x(t) \sim t^p$, $y(t) \sim t^q$ we have:

$$\mu \leq p \nu.$$ 

where $\mu$ Milnor number ($= 2\delta$), $\nu$ the local codimension

- One can find all cases with an equality.
Codimension inequality

If \( x(t) \sim t^p, y(t) \sim t^q \) we have:

\[
\mu \leq p\nu.
\]

where \( \mu \) Milnor number \( (= 2\delta) \), \( \nu \) the local codimension

- One can find all cases with an equality.
- Direct calculations.
Codimension inequality

If $x(t) \sim t^p$, $y(t) \sim t^q$ we have:

$$
\mu \leq p\nu.
$$

where $\mu$ Milnor number ($= 2\delta$), $\nu$ the local codimension

- One can find all cases with an equality.
- Direct calculations.
- Resembles Zajdenberg–Orevkov inequality.
Example

\[ \begin{align*}
    x &= t^4, \\
    y &= 2t^4 + t^6 + 2t^8 + t^9
\end{align*} \]
Example

\[
\begin{aligned}
\begin{cases}
x &= t^4, \\
y &= 2t^4 + t^6 + 2t^8 + t^9
\end{cases}
\end{aligned}
\]

\[
y = 2x^{4/4} + x^{6/4} + 2x^{8/4} + x^{9/4}.
\]
Example

\[
\begin{aligned}
    x &= t^4, \\
    y &= 2t^4 + t^6 + 2t^8 + t^9 \\
    y &= 2x + x^{3/2} + 2x^2 + x^{9/4}.
\end{aligned}
\]

\[c_1 = c_2 = c_3 = c_5 = c_7 = 0.\]
Example

\[
\begin{aligned}
x &= t^4, \\
y &= 2t^4 + t^6 + 2t^8 + t^9
\end{aligned}
\]

\[y = 2x + x^{3/2} + 2x^2 + x^{9/4}.
\]

\[c_1 = c_2 = c_3 = c_5 = c_7 = 0.\]

Hence \(\nu = 5\). Also

\[\mu = 15 + 3 = 18 \leq 4 \cdot 5\]
Example

\[
\begin{align*}
  x &= t^4, \\
  y &= 2t^4 + t^6 + 2t^8 + t^9
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\]

\[y = 2x + x^{3/2} + 2x^2 + x^{9/4}.\]

\[c_1 = c_2 = c_3 = c_5 = c_7 = 0.\]

Hence \(\nu = 5\). Also

\[\mu = 15 + 3 = 18 \leq 4 \cdot 5\]

Change 9 to 13.
Example

\[
\begin{aligned}
    x &= t^4, \\
    y &= 2t^4 + t^6 + 2t^8 + t^{13}
\end{aligned}
\]

\[
y = 2x + x^{3/2} + 2x^2 + x^{13/4}.
\]

\[
c_1 = c_2 = c_3 = c_5 = c_7 = 0.
\]

Hence $\nu = 5$. Also

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Example

\[
\begin{align*}
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\end{align*}
\]

\[
y = 2x + x^{3/2} + 2x^2 + x^{13/4}.
\]

\[
c_1 = c_2 = c_3 = c_5 = c_7 = c_9 = c_{11} = 0.
\]

Now \( \nu = 7 \). And

\[
\mu = 15 + 7 = 22 \leq 4 \cdot 7
\]
Example

\[
\begin{align*}
  x &= t^4, \\
  y &= 2t^4 + t^6 + 2t^8 + t^{13} \\

  y &= 2x + x^{3/2} + 2x^2 + x^{13/4}.
\end{align*}
\]

\[
\begin{array}{cccc}
  c_1 &= c_2 &= c_3 &= c_5 = c_7 = c_9 = c_{11} = 0.
\end{array}
\]

Now \( \nu = 7 \). And

\[
\begin{align*}
  \mu &= 15 + 7 = 22 \leq 4 \cdot 7
\end{align*}
\]

The more complicated singularity, the less sharp is the inequality.
Tangent codimension

- Two branches at a singular point

\[ y = c_1 x^{1/p_1} + c_2 x^{2/p_1} + \cdots + c_k x^{k/p_1} + \ldots \]

\[ y = d_1 x^{1/p_2} + d_2 x^{2/p_2} + \cdots + d_l x^{l/p_2} + \ldots \]
Tangent codimension

• Two branches at a singular point

\[ y = c_1 x^{1/p_1} + c_2 x^{2/p_1} + \cdots + c_k x^{k/p_1} + \cdots \]

\[ y = d_1 x^{1/p_2} + d_2 x^{2/p_2} + \cdots + d_l x^{l/p_2} + \cdots . \]

• The singularity is described by
Tangent codimension

- Two branches at a singular point

\[ y = c_1 x^{1/p_1} + c_2 x^{2/p_1} + \cdots + c_k x^{k/p_1} + \ldots \]
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- The singularity is described by
  — vanishing of \( \nu_1 \) \( c \)'s, \( \nu_2 \), \( d \)'s
Tangent codimension

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\[ y = c_1 x^{1/p_1} + c_2 x^{2/p_1} + \cdots + c_k x^{k/p_1} + \ldots \]

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- The singularity is described by
  — vanishing of \( \nu_1 \) \( c \)'s, \( \nu_2 \) \( d \)'s
  — and possibly some equality relations between non–vanishing \( c \)'s and \( d \)'s.
Tangent codimension

- Two branches at a singular point

\[ y = c_1 x^{1/p_1} + c_2 x^{2/p_1} + \cdots + c_k x^{k/p_1} + \ldots \]
\[ y = d_1 x^{1/p_2} + d_2 x^{2/p_2} + \cdots + d_l x^{l/p_2} + \ldots . \]

- The singularity is described by
  — vanishing of \( \nu_1 \) c’s, \( \nu_2 \) d’s
  — and possibly some equality relations between non-vanishing c’s and d’s.

- number of these relation \( \nu_{tan} \): the tangent codimension.
Example

Branch I \[
\begin{align*}
x &= t^4 \\
y &= t^8 + 3t^{10} + 2t^{14} + 5t^{15}
\end{align*}
\]

Branch II \[
\begin{align*}
x &= u^6 \\
y &= u^{12} - 3u^{15} + 2u^{21} + 3t^{22}
\end{align*}
\]
Example

Branch I \( \begin{align*}
  x &= t^4 \\
  y &= t^8 + 3t^{10} + 2t^{14} + 5t^{15}
\end{align*} \)

Branch II \( \begin{align*}
  x &= u^6 \\
  y &= u^{12} - 3u^{15} + 2u^{21} + 3t^{22}
\end{align*} \)

Consider Puiseux expansion
Example

Branch I \[ \begin{align*}
x & = t^4 \\
y & = t^8 + 3t^{10} + 2t^{14} + 5t^{15}
\end{align*} \]

Branch II \[ \begin{align*}
x & = u^6 \\
y & = u^{12} - 3u^{15} + 2u^{21} + 3t^{22}
\end{align*} \]

\[ y = x^2 + 3x^{5/2} + 2x^{7/2} + 5x^{15/4} \]

\[ y = x^2 + 3x^{5/2} - 2x^{7/2} + 4x^{22/6}. \]
Example

Branch I \[
\begin{align*}
  x &= t^4 \\
  y &= t^8 + 3t^{10} + 2t^{14} + 5t^{15}
\end{align*}
\]

Branch II \[
\begin{align*}
  x &= u^6 \\
  y &= u^{12} - 3u^{15} + 2u^{21} + 3t^{22}
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\[
y = x^2 + 3x^{5/2} + 2x^{7/2} + 5x^{15/4}
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Example

Branch I \[
\begin{align*}
  x &= t^4 \\
  y &= t^8 + 3t^{10} + 2t^{14} + 5t^{15}
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\]

Branch II \[
\begin{align*}
  x &= u^6 \\
  y &= u^{12} - 3u^{15} + 2u^{21} + 3t^{22}
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\]

\[
\begin{align*}
  y &= x^2 + 3x^{5/2} + 2x^{7/2} + 5x^{15/4} \\
  y &= x^2 + 3x^{5/2} - 2x^{7/2} + 4x^{22/6}.
\end{align*}
\]

- The sign change results from choosing different root of unity of order 6.
Example

Branch I \[ \begin{cases} x &= t^4 \\ y &= t^8 + 3t^{10} + 2t^{14} + 5t^{15} \end{cases} \]

Branch II \[ \begin{cases} x &= u^6 \\ y &= u^{12} - 3u^{15} + 2u^{21} + 3t^{22} \end{cases} \]

\[ y = x^2 + 3x^{5/2} + 2x^{7/2} + 5x^{15/4} \]

\[ y = x^2 + 3x^{5/2} - 2x^{7/2} + 4x^{22/6}. \]

Here terms at \( x, \ x^2, \ x^{5/2}, \ x^3 \) agree.
Example

Branch I \[
\begin{align*}
x &= t^4 \\
y &= t^8 + 3t^{10} + 2t^{14} + 5t^{15}
\end{align*}
\]

Branch II \[
\begin{align*}
x &= u^6 \\
y &= u^{12} - 3u^{15} + 2u^{21} + 3t^{22}
\end{align*}
\]

\[
y = x^2 + 3x^{5/2} + 2x^{7/2} + 5x^{15/4}
\]

\[
y = x^2 + 3x^{5/2} - 2x^{7/2} + 4x^{22/6}.
\]

In other words, \( c_4 = d_6, c_8 = d_{12}, c_{10} = d_{15} \) and \( c_{12} = d_{18} \).
Example

Branch I
\[
\begin{align*}
x &= t^4 \\
y &= t^8 + 3t^{10} + 2t^{14} + 5t^{15}
\end{align*}
\]

Branch II
\[
\begin{align*}
x &= u^6 \\
y &= u^{12} - 3u^{15} + 2u^{21} + 3t^{22} \\
y &= x^2 + 3x^{5/2} + 2x^{7/2} + 5x^{15/4} \\
y &= x^2 + 3x^{5/2} - 2x^{7/2} + 4x^{22/6}.
\end{align*}
\]

\[\nu_{tan} = 4\]
Example

Branch I \[
\begin{align*}
x &= t^4 \\
y &= t^8 + 3t^{10} + 2t^{14} + 5t^{15}
\end{align*}
\]

Branch II \[
\begin{align*}
x &= u^6 \\
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\end{align*}
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\[
y = x^2 + 3x^{5/2} + 2x^{7/2} + 5x^{15/4}
\]

\[
y = x^2 + 3x^{5/2} - 2x^{7/2} + 4x^{22/6}.
\]

Note, that \( c_{14} \neq d_{21} \)
Example

Branch I \[
\begin{align*}
x &= t^4 \\
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Branch II \[
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x &= u^6 \\
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\]

\[
y = x^2 + 3x^{5/2} - 2x^{7/2} + 4x^{22/6}.
\]

Note, that \(c_{14} \neq d_{21}\)
Codimension inequality II.

- Two branches.
Codimension inequality II.

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- $p_1, p_2$ — orders of $x$. 
Codimension inequality II.

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- $\nu_1, \nu_2$ — local codimensions.
Codimension inequality II.

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- $\nu_{tan}$ — tangent codimension.
Codimension inequality II.

- Two branches.
- $p_1, p_2$ — orders of $x$.
- $\nu_1, \nu_2$ — local codimensions.
- $\nu_{tan}$ — tangent codimension.

$$2\delta \leq (p_1 + p_2)(\nu_1 + \nu_2 + \nu_{tan} + 1).$$
Codimension inequality II.

• Two branches.
• $p_1, p_2$ — orders of $x$.
• $\nu_1, \nu_2$ — local codimensions.
• $\nu_{tan}$ — tangent codimension.

$$2\delta \leq (p_1 + p_2)(\nu_1 + \nu_2 + \nu_{tan} + 1).$$

*This +1 is very inconvenient. We can get rid of it almost all cases.*
Codimension inequality II.

- Two branches.
- $p_1, p_2$ — orders of $x$.
- $\nu_1, \nu_2$ — local codimensions.
- $\nu_{tan}$ — tangent codimension.

$$2\delta \leq (p_1 + p_2)(\nu_1 + \nu_2 + \nu_{tan} + 1).$$

- Assume $q_1$ and $q_2$ are orders of $y$. 
Codimension inequality II.

- Two branches.
- \( p_1, p_2 \) — orders of \( x \).
- \( \nu_1, \nu_2 \) — local codimensions.
- \( \nu_{\text{tan}} \) — tangent codimension.

\[
2\delta \leq (p_1 + p_2)(\nu_1 + \nu_2 + \nu_{\text{tan}} + 1).
\]

- Assume \( q_1 \) and \( q_2 \) are orders of \( y \).
- For \( q_2 p_1 \neq q_1 p_2 \), the intersection index of branches is fixed:
Codimension inequality II.

- Two branches.
- $p_1, p_2$ — orders of $x$.
- $\nu_1, \nu_2$ — local codimensions.
- $\nu_{tan}$ — tangent codimension.

$$2\delta \leq (p_1 + p_2)(\nu_1 + \nu_2 + \nu_{tan} + 1).$$

- Assume $q_1$ and $q_2$ are orders of $y$.
- For $q_2p_1 \neq q_1p_2$, the intersection index of branches is fixed:
  it equals $\min(q_1p_2, q_2p_1)$ — leads to better estimate.
External codimension
For the singularity with one branch
External codimension

For the singularity with one branch

\[ \text{ext } \nu = \nu + p - 2. \]
External codimension

For the singularity with one branch

$$\text{ext } \nu = \nu + p - 2.$$  

The subspace of curves with such singularity in the space curves $x = t^p + \cdots + a_0$, $y = t^q + b_1 t^{q-1} + \ldots$ for $p, q$ sufficiently large has codimension $\text{ext } \nu$. 
External codimension

For the singularity with one branch

\[ \text{ext } \nu = \nu + p - 2. \]

- We have \( p - 1 \) condition on \( x, \nu \) on \( y \) and can move parameter \( t \).
External codimension

For the singularity with one branch

\[ \text{ext } \nu = \nu + p - 2. \]

- We have \( p - 1 \) condition on \( x \), \( \nu \) on \( y \) and can move parameter \( t \).

*If we swap \( x \) with \( y \), the codimension may change.*
External codimension

For the singularity with one branch

\[ \text{ext } \nu = \nu + p - 2. \]

- We have \( p - 1 \) condition on \( x \), \( \nu \) on \( y \) and can move parameter \( t \).

For \( x = t^4, \ y = t^8 + t^9 \), we have \( \text{ext } \nu = 8 \).

For \( x = t^8 + t^9, \ y = t^4 \) we have \( \text{ext } \nu = 9 \).
External codimension

For the singularity with one branch

\[ \text{ext } \nu = \nu + p - 2. \]

- We have \( p - 1 \) condition on \( x \), \( \nu \) on \( y \) and can move parameter \( t \).

The codimension is minimal if \( \text{ord } x = \text{multiplicity} \).
External codimension

For the singularity with one branch

\[ \text{ext } \nu = \nu + p - 2. \]

• We have \( p - 1 \) condition on \( x, \nu \) on \( y \) and can move parameter \( t \).

• If we have 2 branches with \( \text{ext } \nu_1, \text{ext } \nu_2 \),
  \[ \text{ext } \nu = \text{ext } \nu_1 + \text{ext } \nu_2 + \nu_{\text{tan}}^{(12)} + 2. \]
External codimension

For the singularity with one branch

\[ \text{ext } \nu = \nu + p - 2. \]

- We have \( p - 1 \) condition on \( x, \nu \) on \( y \) and can move parameter \( t \).
- If we have 2 branches with \( \text{ext } \nu_1, \text{ext } \nu_2 \),
  \[ \text{ext } \nu = \text{ext } \nu_1 + \text{ext } \nu_2 + \nu^{(12)}_{\text{tan}} + 2. \]

*Additional 2 comes from the condition*
\[ x(t_0) = x(t_1), \ y(t_0) = y(t_1). \]
External codimension

For the singularity with one branch

\[ \text{ext } \nu = \nu + p - 2. \]

- We have \( p - 1 \) condition on \( x, \nu \) on \( y \) and can move parameter \( t \).
- If we have 2 branches with \( \text{ext } \nu_1, \text{ext } \nu_2 \),
  \[ \text{ext } \nu = \text{ext } \nu_1 + \text{ext } \nu_2 + \nu_{tan}^{(12)} + 2. \]
External codimension

For the singularity with one branch

$$\text{ext } \nu = \nu + p - 2.$$  

- We have $p - 1$ condition on $x$, $\nu$ on $y$ and can move parameter $t$.
- If we have 2 branches with $\text{ext } \nu_1$, $\text{ext } \nu_2$,  
  $$\text{ext } \nu = \text{ext } \nu_1 + \text{ext } \nu_2 + \nu^{(12)}_{tan} + 2.$$  
- Definition for more branches is similar.
External codimension II

Setup
External codimension II

Setup

- $(\mathcal{C}, x_0)$ is a curve on a surface $X$.
- $\tilde{\mathcal{C}} \subset \tilde{X}$ resolution of singular point $x_0$. 
External codimension II

Setup

- $(C, x_0)$ is a curve on a surface $X$.
- $\tilde{C} \subset \tilde{X}$ resolution of singular point $x_0$.

$E$ the reduced exceptional divisor.
External codimension II

Setup

- \((C, x_0)\) is a curve on a surface \(X\).
- \(\tilde{C} \subset \tilde{X}\) resolution of singular point \(x_0\).

\(K\) is the projection of the canonical divisor onto the subgroup of \(\text{Pic}(\tilde{X}) \otimes \mathbb{Q}\) spanned by components of \(E\).
External codimension II

Setup

- $(\tilde{C}, x_0)$ is a curve on a surface $X$.
- $\tilde{C} \subset \tilde{X}$ resolution of singular point $x_0$.
- $D = \tilde{C} + E$. 
External codimension II

Setup

• \((C, x_0)\) is a curve on a surface \(X\).
• \(\tilde{C} \subset \tilde{X}\) resolution of singular point \(x_0\).
• \(D = \tilde{C} + E\).
• Let \(\tilde{M} = K(K + D)\): modified Orevkov \(\tilde{M}\) number.
External codimension II

Setup

• \((C, x_0)\) is a curve on a surface \(X\).
• \(\tilde{C} \subset \tilde{X}\) resolution of singular point \(x_0\).
• \(D = \tilde{C} + E\).
• Let \(\tilde{M} = K(K + D)\): modified Orevkov \(\tilde{M}\) number.

\[\text{In fact } \tilde{M} = K(K + D) + \# \text{branches} - 1.\]
External codimension II

Setup

- $(C, x_0)$ is a curve on a surface $X$.
- $\tilde{C} \subset \tilde{X}$ resolution of singular point $x_0$.
- $D = \tilde{C} + E$.
- Let $\bar{M} = K(K + D)$: modified Orevkov $\bar{M}$ number.

**Proposition.** For a given singular curve $C \subset \mathbb{C}^2$, if orders of $x$ at $C$ all branches are multiplicities, then

$$\text{ext } \nu = K(K + D).$$
External codimension II

Setup

• \((C, x_0)\) is a curve on a surface \(X\).
• \(\tilde{C} \subset \tilde{X}\) resolution of singular point \(x_0\).
• \(D = \tilde{C} + E\).
• Let \(\tilde{M} = K(K + D)\): modified Orevkov \(\tilde{M}\) number.

Proposition. For a given singular curve \(C \subset \mathbb{C}^2\), if orders of \(x\) at \(C\) all branches are multiplicities, then

\[
\text{ext } \nu = K(K + D).
\]

The proof follows from calculating both quantities in terms of Eisenbud–Neumann diagrams.
Regularity I

• Space $Cur_{a,c}$ of curves with one place at infinitity
Regularity I

- Space $\text{Cur}_{a,c}$ of curves with one place at infinity

\[
\begin{align*}
    x &= t^a + \alpha_1 t^{a-1} + \alpha_2 t^{a-2} + \cdots + \alpha_a \\
y &= t^c + \beta_1 t^{c-1} + \cdots + \beta_c
\end{align*}
\]
Regularity I

• Space $\text{Cur}_{a,c}$ of curves with one place at infinity has dimension $a + c$. 
Regularity I

- Space $Cur_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $Curv_{a,b,c,d}$ of curves with two branches at infinity.
Regularity I

- Space $\text{Curv}_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $\text{Curv}_{a,b,c,d}$ of curves with two branches at infinity

\[
\begin{aligned}
  x &= t^a + \alpha_1 t^{a-1} + \alpha_2 t^{a-2} + \cdots + \alpha_{a+b} t^{-b} \\
  y &= t^c + \beta_1 t^{c-1} + \cdots + \beta_{c+d} t^{-d}
\end{aligned}
\]
Regularity I

- Space $\text{Cur}_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $\text{Curv}_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$. 
Regularity I

• Space $Curv_{a,c}$ of curves with one place at infinity has dimension $a + c$.

• Space $Curv_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.

$a, b, c$ and $d$ need not be positive. We will discuss it later.
Regularity I

- Space $Cur_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $Curv_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.
- Let $g$ be the dimension of the automorphism group.
Regularity I

- Space $Cur_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $Curv_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.
- Let $g$ be the dimension of the automorphism group.
- This group consists, in first case, of changes
Regularity I

• Space $Cur_{a,c}$ of curves with one place at infinity has dimension $a + c$.

• Space $Curv_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.

• Let $g$ be the dimension of the automorphism group.

• This group consists, in first case, of changes
  - $y \rightarrow y + \alpha_k x^k$ for $c \geq ka$, 
Regularity I

- Space $Cur_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $Curv_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.
- Let $g$ be the dimension of the automorphism group.
- This group consists, in first case, of changes
  - $y \rightarrow y + \alpha_k x^k$ for $c \geq ka$,
  - $y \rightarrow y + \text{const}$, $x \rightarrow x + \text{const}$,
Regularity I

• Space $\text{Cur}_{a,c}$ of curves with one place at infinity has dimension $a + c$.
• Space $\text{Curv}_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.
• Let $g$ be the dimension of the automorphism group.
• This group consists, in first case, of changes
  • $y \rightarrow y + \alpha_k x^k$ for $c \geq ka$,
  • $y \rightarrow y + \text{const}$, $x \rightarrow x + \text{const}$,
  • $t \rightarrow \lambda t$ followed by $x \rightarrow \lambda^{-a} x$, $y \rightarrow \lambda^{-c} y$,
Regularity I

- Space $\text{Cur}_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $\text{Curv}_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.
- Let $g$ be the dimension of the automorphism group.
- This group consists, in first case, of changes
  - $y \to y + \alpha_k x^k$ for $c \geq ka$,
  - $y \to y + \text{const}$, $x \to x + \text{const}$,
  - $t \to \lambda t$ followed by $x \to \lambda^{-a} x$, $y \to \lambda^{-c} y$,
  - $t \to t + a$. 
Regularity I

- Space $Cur_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $Curv_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.
- Let $g$ be the dimension of the automorphism group.

*In case of two branches the actual structure of the group depends heavily on $a$, $b$, $c$ and $d.*
Regularity I

- Space $Cur_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $Curv_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.
- Let $g$ be the dimension of the automorphism group.

*For example if $b < 0$, the change $x \rightarrow x + \text{const}$ is not allowed.*
Regularity I

- Space $\text{Cur}_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $\text{Curv}_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.
- Let $g$ be the dimension of the automorphism group.
- The *regularity* means, that the space of curves with singularities of codimension $\text{ext } \nu_1, \ldots, \text{ext } \nu_k$ forms a subspace of the codimension at most $\text{ext } \nu_1 + \cdots + \text{ext } \nu_k$. Thus
Regularity I

- Space $Cur_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $Curv_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.
- Let $g$ be the dimension of the automorphism group.
- The *regularity* means, that the space of curves with singularities of codimension $\text{ext } \nu_1, \ldots, \text{ext } \nu_k$ forms a subspace of the codimension at most $\text{ext } \nu_1 + \cdots + \text{ext } \nu_k$. Thus

$$\sum \text{ext } \nu_i \leq a + c - g$$
Regularity I

- Space $\text{Cur}_{a,c}$ of curves with one place at infinity has dimension $a + c$.
- Space $\text{Curv}_{a,b,c,d}$ of curves with two branches at infinity has dimension $a + b + c + d$.
- Let $g$ be the dimension of the automorphism group.
- The *regularity* means, that the space of curves with singularities of codimension $\text{ext} \ \nu_1, \ldots, \text{ext} \ \nu_k$ forms a subspace of the codimension at most $\text{ext} \ \nu_1 + \cdots + \text{ext} \ \nu_k$. Thus
  \[ \sum \text{ext} \ \nu_i \leq a + c - g \]
  Sum all singular points together with infinity.
Regularity I

- Space \( \text{Cur} \, a,c \) of curves with one place at infinity has dimension \( a + c \).
- Space \( \text{Cur} \, v \, a,b,c,d \) of curves with two branches at infinity has dimension \( a + b + c + d \).
- Let \( g \) be the dimension of the automorphism group.
- The **regularity** means, that the space of curves with singularities of codimension \( \text{ext} \, \nu_1, \ldots, \text{ext} \, \nu_k \) forms a subspace of the codimension at most \( \text{ext} \, \nu_1 + \cdots + \text{ext} \, \nu_k \). Thus

\[
\sum \text{ext} \, \nu_i \leq a + c - g
\]

\( \text{ext} \, \nu_i \) external codimensions.
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- Let $g$ be the dimension of the automorphism group.
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$$\sum \text{ext } \nu_i \leq a + b + c + d - g$$

Codimension is really a codimension.
Regularity II

• Regularity is stronger than the inequality
  \[ \sum \bar{M}_i \leq 3d - 4. \]
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- Evidence: all cases found by Koras and Russell turn out to be regular.
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• Evidence: all cases found by Koras and Russell turn out to be regular.

• All our examples calculated by hand are regular.
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• The most difficult part is that of one singular point. If we know that, we can apply induction.

• Evidence: all cases found by Koras and Russell turn out to be regular.

• All our examples calculated by hand are regular.

• Slightly more general regularity conjecture fail.
Genus formula revisited

Take curve $C$

\[
\begin{align*}
    x(t) &= t^a + \alpha_1 t^{a-1} + \cdots + \alpha_a \\
    y(t) &= t^c + \beta_1 t^{c-1} + \cdots + \beta_c.
\end{align*}
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Suppose $a < c$. 
Genus formula revisited

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Suppose $a < c$.

Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c - 1)$. 
Genus formula revisited

Take curve $C$ Suppose $a < c$.

Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c - 1)$. But $2\delta_\infty = (c - 1)(c - a - 1) + 2\delta'_\infty$. 
Genus formula revisited

Take curve $C$. Suppose $a < c$.

Then $\text{deg } C = c$ and $\sum 2\delta_i + 2\delta_{\infty} = c(c - 1)$. But $2\delta_{\infty} = (c - 1)(c - a - 1) + 2\delta'_{\infty}$. Hence

$$\sum 2\delta_i + 2\delta_{\infty} = c(c - 1).$$
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$$\sum 2\delta_i + 2\delta_\infty = c(c - 1).$$

Now we plug the $2\delta'_\infty$. 
Genus formula revisited

Take curve $C$. Suppose $a < c$.

Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c - 1)$. But $2\delta_\infty = (c - 1)(c - a - 1) + 2\delta'_\infty$. Hence

$$\sum 2\delta_i + 2\delta'_\infty = (a - 1)(c - 1).$$
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Then $\text{deg } C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c - 1)$. But $2\delta_\infty = (c - 1)(c - a - 1) + 2\delta'_\infty$. Hence

$$\sum 2\delta_i + 2\delta'_\infty = (a - 1)(c - 1).$$

From this sum we exclude the only finite double point.
Genus formula revisited

Take curve $C$. Suppose $a < c$.

Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_{\infty} = c(c - 1)$. But $2\delta_{\infty} = (c - 1)(c - a - 1) + 2\delta'_{\infty}$. Hence

$$\sum' 2\delta_i + 2\delta_{dbl} + 2\delta'_{\infty} = (a - 1)(c - 1).$$
Genus formula revisited

Take curve $C$. Suppose $a < c$.

Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c - 1)$. But $2\delta_\infty = (c - 1)(c - a - 1) + 2\delta'_\infty$. Hence

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$$\sum' 2\delta_i + 2\delta_{dbl} + 2\delta'_\infty = (a - 1)(c - 1).$$

Use the inequality $2\delta_i \leq p_i\nu_i$ for singular point with one branch.
Genus formula revisited

Take curve $C$ Suppose $a < c$.

Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c - 1)$. But $2\delta_\infty = (c - 1)(c - a - 1) + 2\delta'_\infty$. Hence

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Genus formula revisited

Take curve $C'$. Suppose $a < c$.

Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c - 1)$. But $2\delta_\infty = (c - 1)(c - a - 1) + 2\delta'_\infty$. Hence

$$\sum p_i \nu_i + 2\delta_{dbl} + 2\delta'_\infty \leq (a - 1)(c - 1).$$

The inequality for $2\delta'_\infty$ is similar.

$2\delta'_\infty \leq a'\nu'_\infty + a' - 1$, where $a' = \gcd(a, c)$. 
Genus formula revisited

Take curve $C$. Suppose $a < c$.

Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c - 1)$. But $2\delta_\infty = (c - 1)(c - a - 1) + 2\delta'_\infty$. Hence

$$\sum p_i \nu_i + 2\delta_{dbl} + a'\nu'_\infty + a' - 1 \leq (a - 1)(c - 1).$$
Genus formula revisited

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Genus formula revisited

Take curve $C$. Suppose $a < c$.

Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c - 1)$. But $2\delta_\infty = (c - 1)(c - a - 1) + 2\delta'_\infty$. Hence

$$\sum p_i \nu_i + 2\delta_{dbl} + a' \nu'_\infty + a' - 1 \leq (a - 1)(c - 1).$$

Now we estimate $2\delta_{dbl}$. 
Genus formula revisited

Take curve $C$. Suppose $a < c$.

Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c - 1)$. But

$$2\delta_\infty = (c - 1)(c - a - 1) + 2\delta'_\infty.$$  Hence

$$\sum p_i\nu_i + (p_{01} + p_{02})(\nu_{01} + \nu_{02} + \nu_{tan} + 1) + a' - 1 \leq (a - 1)(c - 1).$$

$p_{01}$ and $p_{02}$ are orders of $x$ at two branches of the double locus.
Genus formula for annuli.

For curve
Genus formula for annuli.

For curve

\[
\begin{aligned}
x &= t^a + \alpha_1 t^{a-1} + \alpha_2 t^{a-2} + \cdots + \alpha_{a+b} t^{-b} \\
y &= t^c + \beta_1 t^{c-1} + \cdots + \beta_{c+d} t^{-d}
\end{aligned}
\]
Genus formula for annuli.

For curve

$$\begin{cases} x = t^a + \alpha_1 t^{a-1} + \alpha_2 t^{a-2} + \cdots + \alpha_{a+b} t^{-b} \\ y = t^c + \beta_1 t^{c-1} + \cdots + \beta_{c+d} t^{-d} \end{cases}$$

if $ad - bc \neq 0$ and $a + b \leq c + d$ we get
Genus formula for annuli.

For curve

\[
\begin{align*}
  x &= t^a + \alpha_1 t^{a-1} + \alpha_2 t^{a-2} + \cdots + \alpha_{a+b} t^{-b} \\
  y &= t^c + \beta_1 t^{c-1} + \cdots + \beta_{c+d} t^{-d}
\end{align*}
\]

if \( ad - bc \neq 0 \) and \( a + b \leq c + d \) we get

\[
\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + |ad - bc| - a' - b' + 1 - a' \nu'_\infty - b' \nu'_0,
\]
Genus formula for annuli.

For curve

\[
\begin{align*}
  x &= t^a + \alpha_1 t^{a-1} + \alpha_2 t^{a-2} + \cdots + \alpha_{a+b} t^{-b} \\
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\end{align*}
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if \( ad - bc \neq 0 \) and \( a + b \leq c + d \) we get

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\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + \\
+ |ad - bc| - a' - b' + 1 - a' \nu'_\infty - b' \nu'_0,
\]

where \( a' = \gcd(a, c) \), \( b' = \gcd(b, d) \), \( \nu'_0 \), \( \nu'_\infty \) are codimensions at zero and infinity.
Genus formula for annuli.

For curve

\[
\begin{align*}
  x &= t^a + \alpha_1 t^{a-1} + \alpha_2 t^{a-2} + \cdots + \alpha_{a+b} t^{-b} \\
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if \( ad - bc \neq 0 \) and \( a + b \leq c + d \) we get

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\]

If \( ad = bc \) we need to take into account the tangency of branches at infinity.
Genus formula for annuli.

For curve

\[
\begin{align*}
  x &= t^a + \alpha_1 t^{a-1} + \alpha_2 t^{a-2} + \cdots + \alpha_{a+b} t^{-b} \\
  y &= t^c + \beta_1 t^{c-1} + \cdots + \beta_{c+d} t^{-d}
\end{align*}
\]

if \(ad - bc \neq 0\) and \(a + b \leq c + d\) we get

\[
\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + \left(- a' - b' + 1 - (a' + b')(\nu_{\inf}' + 1)\right),
\]

If \(ad = bc\) we need to take into account the tangency of branches at infinity. The formula is suitably changed.
Estimates for annuli.

$C$ annulus with $ad \neq bc$. 
Estimates for annuli.

$C$ annulus with $ad \neq bc$. Then

$$\det' = |ad - bc| - a' - b' + 1 \geq 0.$$
Estimates for annuli.

$C$ annulus with $ad \neq bc$. Then
\[
\det' = |ad - bc| - a' - b' + 1 \geq 0.
\]

\[
\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) - a' \nu'_\infty - b' \nu'_0 + \det'.
\]
Estimates for annuli.

A annulus with $ad \neq bc$. Then
\[ \det' = |ad - bc| - a' - b' + 1 \geq 0. \]

- \[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) \]
  \[ - a' \nu'_\infty - b' \nu'_0 + \det'. \]
- \[ \sum (\nu_i + p_i - 2) + \nu'_0 + \nu'_\infty \leq \]
  \[ a + b + c + d - 1 - K - D \]
Estimates for annuli.

$C$ annulus with $ad \neq bc$. Then
\[
\det' = |ad - bc| - a' - b' + 1 \geq 0.
\]

\begin{itemize}
  \item \[\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) - a' \nu'_\infty - b' \nu'_0 + \det'.\]
  \item \[\sum (\nu_i + p_i - 2) + \nu'_0 + \nu'_\infty \leq a + b + c + d - 1 - K - D\]
  \item \[\sum (p_i - 1) \leq \begin{cases} a + b - 1, & b \leq 0 \\ a + b, & b > 0 \end{cases}\]
\end{itemize}
Estimates for annuli.

$C$ annulus with $ad \neq bc$. Then

$$\det' = |ad - bc| - a' - b' + 1 \geq 0.$$ 

- $$\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) - a' \nu'_{\infty} - b' \nu'_{0} + \det'.$$
- $$\sum (\nu_i + p_i - 2) + \nu'_{0} + \nu'_{\infty} \leq a + b + c + d - 1 - K - D$$
- $$\sum (p_i - 1) \leq \begin{cases} a + b - 1, & b \leq 0 \\ a + b, & b > 0 \end{cases}$$

- The genus formula.
Estimates for annuli.

C annulus with $ad \neq bc$. Then
\[
\det' = |ad - bc| - a' - b' + 1 \geq 0.
\]

- \[
\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) - a' \nu_\infty' - b' \nu_0' + \det'.
\]

- \[
\sum (\nu_i + p_i - 2) + \nu_0' + \nu_\infty' \leq a + b + c + d - 1 - K - D
\]

- \[
\sum (p_i - 1) \leq \begin{cases} 
a + b - 1, & b \leq 0 
a + b, & b > 0
\end{cases}
\]

- The regularity condition.
Estimates for annuli.

$C$ annulus with $ad \neq bc$. Then

\[
\det' = |ad - bc| - a' - b' + 1 \geq 0.
\]

- \[
\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) - a' \nu_\infty' - b' \nu_0' + \det'.
\]

- \[
\sum (\nu_i + p_i - 2) + \nu_0' + \nu_\infty' \leq a + b + c + d - 1 - K - D
\]

- \[
\sum (p_i - 1) \leq \begin{cases} 
    a + b - 1, & b \leq 0 \\
    a + b, & b > 0
\end{cases}
\]

- Counting zeros of \( \frac{d}{dt} x(t) \).
Estimates for annuli.

$C$ annulus with $ad \neq bc$. Then

$$\det' = |ad - bc| - a' - b' + 1 \geq 0.$$  

- $\sum p_i \nu_i \leq (a + b - 1)(c + d - 1)$
  $$-a' \nu'_{\infty} - b' \nu'_0 + \det'.$$

- $\sum (\nu_i + p_i - 2) + \nu'_0 + \nu'_\infty \leq a + b + c + d - 1 - K - D$

- $\sum (p_i - 1) \leq \begin{cases} a + b - 1, & b \leq 0 \\ a + b, & b > 0 \end{cases}$

- $K$ is maximal non-negative integer such that $Ka \leq c$ and $Kb \leq d$. 
Estimates for annuli.

$C$ annulus with $ad \neq bc$. Then

$$det' = |ad - bc| - a' - b' + 1 \geq 0.$$  

- $\sum p_i \nu_i \leq (a + b - 1)(c + d - 1)$
- $-a' \nu'_\infty - b' \nu'_0 + det'$.  
- $\sum (\nu_i + p_i - 2) + \nu'_0 + \nu'_\infty \leq a + b + c + d - 1 - K - D$

- $\sum (p_i - 1) \leq \begin{cases} a + b - 1, & b \leq 0 \\ a + b, & b > 0 \end{cases}$

- $D \in \{0, 1, 2\}$ is the number of constants: if we can add a constant to $x$ or $y$. 
Estimates for annuli.

$C$ annulus with $ad \neq bc$. Then

$$\det' = |ad - bc| - a' - b' + 1 \geq 0.$$  

- $$\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) - a' \nu'_\infty - b' \nu'_0 + \det'.$$

- $$\sum (\nu_i + p_i - 2) + \nu'_0 + \nu'_\infty \leq a + b + c + d - 1 - K - D$$

- $$\sum (p_i - 1) \leq \begin{cases} a + b - 1, & b \leq 0 \\ a + b, & b > 0 \end{cases}$$
Different types.

- Appearance of $D$ and $K$. Suggests different types.
Different types.

- Appearance of $D$ and $K$. Suggests different types.
- Type $\left(\frac{+}{+}\right)$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. $K$ is here very important.
Different types.

• Appearance of $D$ and $K$. Suggests different types.

• Type $(\dagger)$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. $K$ is here very important.
Different types.

- Appearance of $D$ and $K$. Suggests different types.
- Type $(\uparrow \downarrow)$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. $K$ is here very important.

*In type $(\uparrow \downarrow)$ distinguish $ad \neq bc$ and $ad = bc$. 
Different types.

• Appearance of $D$ and $K$. Suggests different types.

• Type $(\dagger)$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. $K$ is here very important.

If $ad \neq bc$, many singular cases.
Different types.

• Appearance of $D$ and $K$. Suggests different types.

• Type $(\dagger\dagger)$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. $K$ is here very important.

\[ ad = bc: \text{strong condition on } a, b, c \text{ and } d. \]
Different types.

• Appearance of $D$ and $K$. Suggests different types.

• Type $(\text{++})$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. $K$ is here very important.

• Type $(\text{+-})$: $0 < a < c$, $0 < d \leq b$, $a + b \leq c + d$. $D = 2$ and $K = 0$. 
Different types.

- Appearance of $D$ and $K$. Suggests different types.
- Type $(\begin{array}{c}+ \\ + \end{array})$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. $K$ is here very important.
- Type $(\begin{array}{c}- \\ + \end{array})$: $0 < a < c$, $0 < d \leq b$, $a + b \leq c + d$. $D = 2$ and $K = 0$. $|ad - bc|$ is here large. Case does not even require regularity.
Different types.

- Appearance of $D$ and $K$. Suggests different types.

- Type $(\oplus)$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. $K$ is here very important.

- Type $(\ominus \oplus)$: $0 < a < c$, $0 < d \leq b$, $a + b \leq c + d$. $D = 2$ and $K = 0$.

- Type $(\ominus)$: $a, d > 0$, $bc < 0$, $a + b \leq c + d$. $D = 1$, $K$ varies.
Different types.

• Appearance of $D$ and $K$. Suggests different types.

• Type $(\dagger)$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. $K$ is here very important.

• Type $(\ddagger)$: $0 < a < c$, $0 < d \leq b$, $a + b \leq c + d$. $D = 2$ and $K = 0$.

• Type $(-\dagger)$: $a, d > 0$, $bc < 0$, $a + b \leq c + d$. $D = 1$, $K$ varies.

• Annoying type. Many subcases, i.e. $a > c$, $a < c$ etc.
Different types.

• Appearance of $D$ and $K$. Suggests different types.

• Type $(\dagger)$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. $K$ is here very important.

• Type $(\dagger\dagger)$: $0 < a < c$, $0 < d \leq b$, $a + b \leq c + d$. $D = 2$ and $K = 0$.

• Type $(\ddagger)$: $a, d > 0$, $bc < 0$, $a + b \leq c + d$. $D = 1$, $K$ varies.

• Type $(\dagger\dagger)$: $a, d > 0$, $b, c < 0$. $D = K = 0$.

Here we do not need regularity neither.
Different types.

- Appearance of $D$ and $K$. Suggests different types.
- Type $(\begin{array}{c}+ \\ + \end{array})$: $0 < a < c, 0 < b < d$. Then $D = 2, K \geq 1$. $K$ is here very important.
- Type $(\begin{array}{c}- + \\ - - \end{array})$: $0 < a < c, 0 < d \leq b, a + b \leq c + d$. $D = 2$ and $K = 0$.
- Type $(\begin{array}{c}- \\ + \end{array})$: $a, d > 0, bc < 0, a + b \leq c + d$. $D = 1, K$ varies.
- Type $(\begin{array}{c}- \\ - \end{array})$: $a, d > 0, b, c < 0$. $D = K = 0$.
- Most important. Contains all smooth cases.
Choice of presentation.
Recall
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + |ad - bc| - a' - b' + 1 - a'\nu'_\infty - b'\nu'_0. \]
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + \\
\hspace{1cm} + |ad - bc| - a' - b' + 1 - a'\nu'_\infty - b'\nu'_0. \]

- Assume \( a | c \).
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + |ad - bc| - a' - b' + 1 - a'\nu'_\infty - b'\nu'_0. \]

- Assume \( a \mid c \). Then \( a' = a \) is large.
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + \]
\[ + |ad - bc| - a' - b' + 1 - a' \nu'_{\infty} - b' \nu'_0. \]

- **Assume** \( a \mid c \). Then \( a' = a \) is large.
- **The term** \( a' \nu'_{\infty} \) **may dominate.**
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + \]
\[ + |ad - bc| - a' - b' + 1 - a' \nu'_\infty - b' \nu'_0. \]

- Assume \( a | c \). Then \( a' = a \) is large.
- The term \( a' \nu'_\infty \) may dominate. *Especially if \( a + b \) is small.*
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + \]
\[ + |ad - bc| - a' - b' + 1 - a'\nu'_\infty - b'\nu'_0. \]

- Assume \(a|c\). Then \(a' = a\) is large.
- The term \(a'\nu'_\infty\) may dominate. \textit{It cannot happen, that } \(a|c\) \textit{and } b|d \textit{ at the same time.}
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + 
+ |ad - bc| - a' - b' + 1 - a' \nu'_{\infty} - b' \nu'_0. \]

- Assume \( a|c \). Then \( a' = a \) is large.
- The term \( a'\nu'_{\infty} \) may dominate. \textit{It cannot happen, that} \( a|c \) \textit{and} \( b|d \) \textit{at the same time. We can reduce the case by the suitable change} \( y \rightarrow y - \text{const} \cdot x^{\min(c/a, d/b)} \).
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + |ad - bc| - a' - b' + 1 - a' \nu'_{\infty} - b' \nu'_{0}. \]

- Assume \( a | c \). Then \( a' = a \) is large.
- The term \( a' \nu'_{\infty} \) may dominate.

There are conditions on \( a, b, c \) and \( d \) solely such that
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + \]
\[ + |ad - bc| - a' - b' + 1 - a' \nu'_{\infty} - b' \nu'_{0}. \]

- Assume \( a \mid c \). Then \( a' = a \) is large.
- The term \( a' \nu'_{\infty} \) may dominate.

There are conditions on \( a, b, c \) and \( d \) solely such that — one deals easily with case \( a \mid c \).
Choice of presentation.

Recall

$$\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) +$$
$$+ |ad - bc| - a' - b' + 1 - a' \nu_{\infty} - b' \nu_0.$$

- Assume $a|c$. Then $a' = a$ is large.
- The term $a' \nu_{\infty}$ may dominate.

There are conditions on $a$, $b$, $c$ and $d$ solely such that
— one deals easily with case $a|c$.
— we can make each curve satisfy this condition.
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + \]
\[ + |ad - bc| - a' - b' + 1 - a' \nu'_\infty - b' \nu'_0. \]

- Assume $a \mid c$. Then $a' = a$ is large.
- The term $a' \nu'_\infty$ may dominate.

There are conditions on $a$, $b$, $c$ and $d$ solely such that
- one deals easily with case $a \mid c$.
- we can make each curve satisfy this condition.
- we call curve satisfying it handsome.
Choice of presentation.

Recall

$$\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) +$$

$$+ |ad - bc| - a' - b' + 1 - a' \nu'_\infty - b' \nu'_0.$$ 

- Assume $a | c$. Then $a' = a$ is large.
- The term $a' \nu'_\infty$ may dominate.

There are conditions on $a, b, c$ and $d$ solely such that
- one deals easily with case $a | c$.
- we can make each curve satisfy this condition.
- each curve is isomorphic to a handsome curve.
Choice of presentation.

Recall

$$\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + |ad - bc| - a' - b' + 1 - a' \nu'_{\infty} - b' \nu'_0.$$  

- Assume $a|c$. Then $a' = a$ is large.
- The term $a' \nu'_{\infty}$ may dominate.

There are conditions on $a, b, c$ and $d$ solely such that
- one deals easily with case $a|c$.
- we can make each curve satisfy this condition.
- this provides a choice of coordinates on $\mathbb{C}^2$. 
Choice of presentation.

Recall

\[ \sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + \]
\[ + |ad - bc| - a' - b' + 1 - a' \nu'_{\infty} - b' \nu'_0. \]

- Assume \( a \mid c \). Then \( a' = a \) is large.
- The term \( a' \nu'_\infty \) may dominate.

There are conditions on \( a, b, c \) and \( d \) solely such that
- one deals easily with case \( a \mid c \).
- we can make each curve satisfy this condition.
- this provides a choice of coordinates on \( \mathbb{C}^2 \).

The most difficult is then case \( (\_\_\_) \).
Example

Instead of the definition of handsomeness.
Example

Consider a curve
Example

Consider a curve

\[
\begin{align*}
x &= t^4 + t^{-2} \\
y &= t + 2t^{-2} - t^{-4} + 3t^{-6}.
\end{align*}
\]
Example

Consider a curve

\[
\begin{align*}
x &= t^4 + t^{-2} \\
y &= t + 2t^{-2} - t^{-4} + 3t^{-6}.
\end{align*}
\]

We can apply different changes of type \( y \rightarrow y - x^k \), \( x \rightarrow x - y^l \) to that curve.
Example

Consider a curve

\[
\begin{align*}
    x &= t^4 + t^{-2} \\
    y &= t + 2t^{-2} - t^{-4} + 3t^{-6}.
\end{align*}
\]

We can apply different changes of type \( y \rightarrow y - x^k \), \( x \rightarrow x - y^l \) to that curve.

\[
y \rightarrow y^{(1)} = -y/3 + x^3
\]
Example

Consider a curve

\[
\begin{aligned}
x &= t^4 + t^{-2} \\
y^{(1)} &= t^{12} + 3t^6 - \frac{1}{3}t + 3 + \frac{2}{3}t^{-2} - \frac{1}{3}t^{-4}.
\end{aligned}
\]

We can apply different changes of type \( y \rightarrow y - x^k \), \( x \rightarrow x - y^l \) to that curve.

\[
y \rightarrow y^{(1)} = -y/3 + x^3
\]
Example

Consider a curve

\[
\begin{aligned}
    x & = t^4 + t^{-2} \\
y^{(1)} & = t^{12} + 3t^6 - \frac{1}{3}t + 3 + \frac{2}{3}t^{-2} - \frac{1}{3}t^{-4}.
\end{aligned}
\]

We can apply different changes of type \( y \rightarrow y - x^k, \)
\( x \rightarrow x - y^l \) to that curve.

\[
y^{(1)} \rightarrow y^{(2)} = y^{(1)} + \frac{1}{3}x^2
\]
Example

Consider a curve

\[
\begin{aligned}
x &= t^4 + t^{-2} \\
y^{(2)} &= t^{12} + \frac{1}{3}t^8 + 3t^6 + \frac{2}{3}t^2 - \frac{1}{3}t + 3 + \frac{2}{3}t^{-2}.
\end{aligned}
\]

We can apply different changes of type \( y \to y - x^k \), \( x \to x - y^l \) to that curve.

\[
y^{(1)} \to y^{(2)} = y^{(1)} + \frac{1}{3}x^2
\]
Example

Consider a curve

\[
\begin{cases}
  x = t^4 + t^{-2} \\
  y^{(2)} = t^{12} + \frac{1}{3} t^8 + 3t^6 + \frac{2}{3} t^2 - \frac{1}{3} t + 3 + \frac{2}{3} t^{-2}.
\end{cases}
\]

We can apply different changes of type \( y \rightarrow y - x^k \), \( x \rightarrow x - y^l \) to that curve.

\[
y^{(2)} \rightarrow y^{(3)} = y^{(2)} - \frac{2}{3} x + 3
\]
**Example**

Consider a curve

\[
\begin{align*}
    x &= t^4 + t^{-2} \\
    y^{(3)} &= t^{12} + \frac{1}{3}t^8 + 3t^6 - \frac{2}{3}t^4 + \frac{2}{3}t^2 - \frac{1}{3}t.
\end{align*}
\]

We can apply different changes of type \( y \rightarrow y - x^k \), \( x \rightarrow x - y^l \) to that curve.

\[
y^{(2)} \rightarrow y^{(3)} = y^{(2)} - \frac{2}{3}x + 3
\]
Example

Consider a curve

\[
\begin{align*}
x &= t^4 + t^{-2} \\
y^{(3)} &= t^{12} + \frac{1}{3}t^8 + 3t^6 - \frac{2}{3}t^4 + \frac{2}{3}t^2 - \frac{1}{3}t.
\end{align*}
\]

We can apply different changes of type \( y \rightarrow y - x^k \), \( x \rightarrow x - y^l \) to that curve.

The same is with \( x \). Return to previous \( y \).
Example

Consider a curve

\[
\begin{aligned}
x &= t^4 + t^{-2} \\
y &= t + 2t^{-2} - t^{-4} + 3t^{-6}.
\end{aligned}
\]

We can apply different changes of type \( y \to y - x^k \), \( x \to x - y^l \) to that curve.

\[
x \to x^{(1)} = \frac{1}{8}(y^4 - x).
\]
**Example**

Consider a curve

\[
\begin{cases}
  x^{(1)} = t + t^{-1} - \frac{1}{8} t^{-2} + \cdots + \frac{81}{8} t^{-24} \\
y = t + 2t^{-2} - t^{-4} + 3t^{-6}.
\end{cases}
\]

We can apply different changes of type \( y \to y - x^k \), \( x \to x - y^l \) to that curve.

\[
x \to x^{(1)} = \frac{1}{8} (x - y^4).
\]
Example

Consider a curve

\[ \begin{align*}
  x^{(1)} &= t + t^{-1} - \frac{1}{8}t^{-2} + \cdots + \frac{81}{8}t^{-24} \\
y &= t + 2t^{-2} - t^{-4} + 3t^{-6}.
\end{align*} \]

We can apply different changes of type \( y \to y - x^k \), \( x \to x - y^l \) to that curve.

\[ x^{(1)} \to x^{(2)} = x^{(1)} - y. \]
Example

Consider a curve

\[
\begin{align*}
    x^{(2)} &= t^{-1} - \frac{17}{8} t^{-2} + \cdots + \frac{81}{8} t^{-24} \\
    y &= t + 2t^{-2} - t^{-4} + 3t^{-6}.
\end{align*}
\]

We can apply different changes of type \( y \rightarrow y - x^k \), \( x \rightarrow x - y^l \) to that curve.

\[
x^{(1)} \rightarrow x^{(2)} = x^{(1)} - y.
\]
Example

Consider a curve

\[
\begin{align*}
  x^{(2)} &= t^{-1} - \frac{17}{8} t^{-2} + \cdots + \frac{81}{8} t^{-24} \\
  y &= t + 2t^{-2} - t^{-4} + 3t^{-6}.
\end{align*}
\]

We can apply different changes of type \( y \to y - x^k \), \( x \to x - y^l \) to that curve.
Example

Consider a curve

\[
\begin{align*}
x &= t^4 + t^{-2} \\
y &= t + 2t^{-2} - t^{-4} + 3t^{-6}.
\end{align*}
\]

We can apply different changes of type \( y \rightarrow y - x^k \), \( x \rightarrow x - y^l \) to that curve.

Which parametrisation is the best?
Example

Consider a curve

\[
\begin{aligned}
x &= t^4 + t^{-2} \\
y &= t + 2t^{-2} - t^{-4} + 3t^{-6}.
\end{aligned}
\]

We can apply different changes of type \( y \rightarrow y - x^k \), \( x \rightarrow x - y^l \) to that curve.

Handsomeness. This one!
Dealing with inequalities

Essentially three methods
Dealing with inequalities

Essentially three methods

calculations,
Dealing with inequalities

Essentially three methods

calculations,
Dealing with inequalities

Essentially three methods:
calculations,
Dealing with inequalities

More seriously. In all cases but $\leq$. 
Dealing with inequalities

More seriously. In all cases but (−).

• exclude smooth curves.
Dealing with inequalities

More seriously. In all cases but \((\_\_\_\_\_)\).

- exclude smooth curves.
- order multiplicities of \(x\): \(p_1 \geq p_2 \geq p_3 \cdots \geq p_N\).
Dealing with inequalities

More seriously. In all cases but (\text{- -}).

- exclude smooth curves.
- order multiplicities of $x$: $p_1 \geq p_2 \geq p_3 \cdots \geq p_N$.

\textit{N is the number of finite singular points.}
Dealing with inequalities

More seriously. In all cases but (=).

- exclude smooth curves.
- order multiplicities of $x$: $p_1 \geq p_2 \geq p_3 \cdots \geq p_N$.
- exclude cases with $N \geq 2, 3, 4$ (depending on type).
Dealing with inequalities

More seriously. In all cases but (\(\Box\)).

- exclude smooth curves.
- order multiplicities of \(x\): \(p_1 \geq p_2 \geq p_3 \cdots \geq p_N\).
- exclude cases with \(N \geq 2, 3, 4\) (depending on type).
- deal with cases with \(N \geq 2\).
Dealing with inequalities

More seriously. In all cases but $(-)$. 

- exclude smooth curves.
- order multiplicities of $x$: $p_1 \geq p_2 \geq p_3 \cdots \geq p_N$.
- exclude cases with $N \geq 2, 3, 4$ (depending on type).
- deal with cases with $N \geq 2$.

We are left with case $N = 1$. 
Dealing with inequalities

More seriously. In all cases but (―).

• exclude smooth curves.
• order multiplicities of $x$: $p_1 \geq p_2 \geq p_3 \cdots \geq p_N$.
• exclude cases with $N \geq 2, 3, 4$ (depending on type).
• deal with cases with $N \geq 2$.

Reject cases with $\nu'_0 + \nu'_\infty \geq 2$ (if $ad \neq bc$).
Dealing with inequalities

More seriously. In all cases but (\(_{-}\)).

- exclude smooth curves.
- order multiplicities of \(x\): \(p_1 \geq p_2 \geq p_3 \cdots \geq p_N\).
- exclude cases with \(N \geq 2, 3, 4\) (depending on type).
- deal with cases with \(N \geq 2\).

*Left with something like*

\[ p_1(a + b + c + d - K - D - p_1 + 2) \leq (a + b - 1)(c + d - 1) + \det'. \]
Dealing with inequalities

More seriously. In all cases but (Ⅱ).

- exclude smooth curves.
- order multiplicities of $x$: $p_1 \geq p_2 \geq p_3 \cdots \geq p_N$.
- exclude cases with $N \geq 2, 3, 4$ (depending on type).
- deal with cases with $N \geq 2$.

*Left with something like*

$$p_1(a + b + c + d - K - D - p_1 + 2) \leq (a + b - 1)(c + d - 1) + \text{det}' .$$

- Now reject $p_1 \leq a + b - 2$ and consider other cases.
Result

In case of polynomial curves with one double locus there are
Result

In case of polynomial curves with one double locus there are $16$ series
Result

In case of polynomial curves with one double locus there are 16 series and
Result

In case of polynomial curves with one double locus there are 16 series and 5 special cases.
Result

In case of polynomial curves with one double locus there are 16 series and 5 special cases.

For annuli we find
Result

In case of polynomial curves with one double locus there are 16 series and 5 special cases.

For annuli we find 19 series.
**Result**

In case of polynomial curves with one double locus there are 16 series and 5 special cases.

For annuli we find 19 series and
Result

In case of polynomial curves with one double locus there are 16 series and 5 special cases.

For annuli we find 19 series and 4 special cases,
Result

In case of polynomial curves with one double locus there are **16** series and **5** special cases.

For annuli we find **19** series and **4** special cases, including one series with continuous parameters.
Result

In case of polynomial curves with one double locus there are 16 series and 5 special cases.

For annuli we find 19 series and 4 special cases, including one series with continuous parameters.

*namely*

\[
\begin{align*}
  x &= t^a \\
  y &= \lambda_1 t^{-a} + \lambda_2 t^{-2a} + \cdots + \lambda_k t^{-ka} + t^{-c},
\end{align*}
\]

*with* \( a \neq c \).
Result

In case of polynomial curves with one double locus there are 16 series and 5 special cases.

For annuli we find 19 series and 4 special cases, including one series with continuous parameters.

Moreover these 23 cases contain
Result

In case of polynomial curves with one double locus there are 16 series and 5 special cases.

For annuli we find 19 series and 4 special cases, including one series with continuous parameters.

Moreover these 23 cases contain 7 series
Result

In case of polynomial curves with one double locus there are 16 series and 5 special cases.

For annuli we find 19 series and 4 special cases, including one series with continuous parameters.

Moreover these 23 cases contain 7 series and
**Result**

In case of polynomial curves with one double locus there are \(16\) series and \(5\) special cases.

For annuli we find \(19\) series and \(4\) special cases, including one series with continuous parameters.

Moreover these 23 cases contain \(7\) series and \(2\) special cases
Result

In case of polynomial curves with one double locus there are 16 series and 5 special cases.

For annuli we find 19 series and 4 special cases, including one series with continuous parameters.

Moreover these 23 cases contain 7 series and 2 special cases of smooth embeddings $\mathbb{C}^* \rightarrow \mathbb{C}^2$. 
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(a) $x = t^2, \ y = (t^2 - 1)^k t^{2l+1}, \ k = 1, 2, \ldots, \ l = 0, 1, \ldots$;
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(a) $x = t^2, \ y = (t^2 - 1)^k t^{2l+1}, \ k = 1, 2, \ldots, \ l = 0, 1, \ldots$;
(b) $x = t^3, \ y = t^{3k+2} - t^{3k+1}, \ k = 1, 2, \ldots$;
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(a) $x = t^2, \ y = (t^2 - 1)^k t^{2l+1}, \ k = 1, 2, \ldots, l = 0, 1, \ldots$;
(b) $x = t^3, \ y = t^{3k+2} - t^{3k+1}, \ k = 1, 2, \ldots$;
(c) $x = t^4, \ y = t^{4k+2} - t^{4k+1}, \ k = 1, 2, \ldots$;
Maps $\mathbb{C} \to \mathbb{C}^2$

(a) $x = t^2, \ y = (t^2 - 1)^k t^{2l+1}, \ k = 1, 2, \ldots, \ l = 0, 1, \ldots$;

(b) $x = t^3, \ y = t^{3k+2} - t^{3k+1}, \ k = 1, 2, \ldots$;

(c) $x = t^4, \ y = t^{4k+2} - t^{4k+1}, \ k = 1, 2, \ldots$;

(d) $x = t^4, \ y = t^{4k+3} - t^{4k+2}, \ k = 0, 1, \ldots$;
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(a) $x = t^2, y = (t^2 - 1)^k t^{2l+1}, k = 1, 2, \ldots, l = 0, 1, \ldots$;

(b) $x = t^3, y = t^{3k+2} - t^{3k+1}, k = 1, 2, \ldots$;

(c) $x = t^4, y = t^{4k+2} - t^{4k+1}, k = 1, 2, \ldots$;

(d) $x = t^4, y = t^{4k+3} - t^{4k+2}, k = 0, 1, \ldots$;

(e) $x = t^6, y = t^{6k+3} - t^{6k+2}, k = 1, 2, \ldots$;
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(a) $x = t^2, y = (t^2 - 1)^k t^{2l+1}, k = 1, 2, \ldots, l = 0, 1, \ldots$;
(b) $x = t^3, y = t^{3k+2} - t^{3k+1}, k = 1, 2, \ldots$;
(c) $x = t^4, y = t^{4k+2} - t^{4k+1}, k = 1, 2, \ldots$;
(d) $x = t^4, y = t^{4k+3} - t^{4k+2}, k = 0, 1, \ldots$;
(e) $x = t^6, y = t^{6k+3} - t^{6k+2}, k = 1, 2, \ldots$;
(f) $x = t^6, y = t^{6k+4} - t^{6k+3}, k = 0, 1, \ldots$;
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(a) $x = t^2, \quad y = (t^2 - 1)^k t^{2l+1}, \quad k = 1, 2, \ldots, \quad l = 0, 1, \ldots$;

(b) $x = t^3, \quad y = t^{3k+2} - t^{3k+1}, \quad k = 1, 2, \ldots$;

(c) $x = t^4, \quad y = t^{4k+2} - t^{4k+1}, \quad k = 1, 2, \ldots$;

(d) $x = t^4, \quad y = t^{4k+3} - t^{4k+2}, \quad k = 0, 1, \ldots$;

(e) $x = t^6, \quad y = t^{6k+3} - t^{6k+2}, \quad k = 1, 2, \ldots$;

(f) $x = t^6, \quad y = t^{6k+4} - t^{6k+3}, \quad k = 0, 1, \ldots$;

(g) $x = t^a(t - 1)^{kb}, \quad y = t^c(t - 1)^{kd}, \quad \kappa = |ad - bc| = 1, \quad k = 1, 2, \ldots, \quad 2 < a + kb < c + kd$;
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(h) $x = t^{2a}(t - 1)^{2b}, y = t^{2c}(t - 1)^{2d}, \kappa = 1, 2 < ka < kc;$
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(h) $x = t^{2a}(t - 1)^{2b}, \quad y = t^{2c}(t - 1)^{2d}, \quad \kappa = 1,$
    $2 < ka < kc$;

(i) $x = t^{ka-b}(t - 1)^{b}, \quad y = t^{kc-d}(t - 1)^{d}, \quad \kappa = 1,$
    $k = 1, 2, \ldots ;$
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(h) $x = t^{2a}(t - 1)^{2b}, y = t^{2c}(t - 1)^{2d}, \kappa = 1, 2 < ka < kc$;

(i) $x = t^{ka-b}(t - 1)^b, y = t^{kc-d}(t - 1)^d, \kappa = 1, k = 1, 2, \ldots$;

(j) $x = t^2(t - 1), y = t^{2k+1}(t - 1)^k(t - \frac{4}{3}), k = 1, 2, \ldots$;
Maps $\mathbb{C} \to \mathbb{C}^2$

(h) \[ x = t^{2a}(t - 1)^{2b}, \quad y = t^{2c}(t - 1)^{2d}, \quad \kappa = 1, \quad 2 < ka < kc; \]

(i) \[ x = t^{ka-b}(t - 1)^{b}, \quad y = t^{kc-d}(t - 1)^{d}, \quad \kappa = 1, \quad k = 1, 2, \ldots ; \]

(j) \[ x = t^2(t - 1), \quad y = t^{2k+1}(t - 1)^{k}(t - \frac{4}{3}), \quad k = 1, 2, \ldots ; \]

(k) \[ x = t^3(t - 1), \quad y = t^{3k+1}(t - 1)^{k}(t - \frac{3}{2}), \quad k = 1, 2, \ldots ; \]
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(h) $x = t^{2a}(t - 1)^{2b}$, $y = t^{2c}(t - 1)^{2d}$, $\kappa = 1$, $2 < k a < k c$;

(i) $x = t^{k a - b}(t - 1)^{b}$, $y = t^{k c - d}(t - 1)^{d}$, $\kappa = 1$, $k = 1, 2, \ldots$;

(j) $x = t^2(t - 1)$, $y = t^{2k+1}(t - 1)^k(t - \frac{4}{3})$, $k = 1, 2, \ldots$;

(k) $x = t^3(t - 1)$, $y = t^{3k+1}(t - 1)^k(t - \frac{3}{2})$, $k = 1, 2, \ldots$;

(l) $x = [t(t - 1)]^{2k}$, $y = [t(t - 1)]^{(2l+1)k}(t - \frac{1}{2})$, $k = 1, 2, \ldots$, $l = 0, 1, \ldots$;
Maps $\mathbb{C} \to \mathbb{C}^2$

(m) $x = [t(t - 1)]^{2k+1}$, $y = x^l[t(t - 1)]^k(t - \frac{1}{2})$,
$k = 0, 1, \ldots, l = 0, 1, \ldots, (k, l) \neq (0, 0), (0, 1);$
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

\( (m) \ x = [t(t - 1)]^{2k+1}, \ y = x^l[t(t - 1)]^k(t - \frac{1}{2}), \)

\( k = 0, 1, \ldots, \ l = 0, 1, \ldots, \ (k, l) \neq (0, 0), (0, 1); \)

\( (n) \ x = t^k(t - 1)^{k+1}(t - \frac{1}{2})y^l, \ y = t^{2k}(t - 1)^{2k+2}, \)

\( k = 1, 2, \ldots, \ l = 0, 1, \ldots; \)
Maps $\mathbb{C} \to \mathbb{C}^2$

(m) $x = [t(t - 1)]^{2k+1} y = x^l [t(t - 1)]^k (t - \frac{1}{2})$,  
$k = 0, 1, \ldots, l = 0, 1, \ldots, (k, l) \neq (0, 0), (0, 1)$;

(n) $x = t^k (t - 1)^{k+1} (t - \frac{1}{2}) y^l$,  
$y = t^{2k} (t - 1)^{2k+2}$,  
$k = 1, 2, \ldots, l = 0, 1, \ldots$;

(o) $x = t^{2k-1} (t - 1)^{2k+1} y = x^l t^{k-1} (t - 1)^k (t - \frac{1}{2})$,  
$k = 1, 2, \ldots, l = 1, \ldots$;
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(m) $x = [t(t - 1)]^{2k+1}, y = x^l[t(t - 1)]^k(t - \frac{1}{2}),$
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(n) $x = t^k(t - 1)^{k+1}(t - \frac{1}{2})y^l, y = t^{2k}(t - 1)^{2k+2},$
$k = 1, 2, \ldots, l = 0, 1, \ldots;$

(o) $x = t^{2k-1}(t - 1)^{2k+1}, y = x^l t^{k-1}(t - 1)^k(t - \frac{1}{2}),$
$k = 1, 2, \ldots, l = 1, \ldots;$

(p) $x = t^3(t - 1)^3, y = t(t - 1)(t - \frac{1}{2} - \frac{1}{6}i\sqrt{3})x^k,$
$k = 1, 2, \ldots.$
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(q) $x = t^3 - 3t, \quad y = t^4 - 2t^2$;
Maps $\mathbb{C} \rightarrow \mathbb{C}^2$

(q) $x = t^3 - 3t$, $y = t^4 - 2t^2$;

(r) $x = t^3 - 3t$, $y = t^5 - 2\sqrt{-2}t^4 + 11\sqrt{-2}t^2 - \frac{37}{4}t$;
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(t) $x = t^3 - 3t, y = t^5 - \frac{5}{2}t^4 + 5t^2 - 5t$;
(u) $x = t^3 - 3t, y = t^5 - \frac{7}{2}t^4 - t^2 + 11t$. 
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

(a) $x = t^m, y = t^n + \gamma_1 t^{-m} + \gamma_2 t^{-2m} + \cdots + \gamma_k t^{-mk}$, where $m > 0$, $\gcd(m, |n|) = 1$, $k = 0, 1, \ldots$, $\gamma_j \in \mathbb{C}$, $\gamma_k = 1$ (if $k > 0$) and $k > 0$ if $n > 0$. and at least one $\gamma_i \neq 0$ if $m > 0$. 
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(a) $x = t^m, y = t^n + \gamma_1 t^{-m} + \gamma_2 t^{-2m} + \cdots + \gamma_k t^{-mk}$, where $m > 0$, $\gcd(m, |n|) = 1$, $k = 0, 1, \ldots$, $\gamma_j \in \mathbb{C}$, $\gamma_k = 1$ (if $k > 0$) and $k > 0$ if $n > 0$. and at least one $\gamma_i \neq 0$ if $m > 0$.

(b) $x = t(t - 1), y = (x + \frac{1}{4})^m x^n R_l(1/t)$, where $m, n = 0, 1, \ldots$ and $R_l$ is a polynomial satisfying $R_l(1/t) - R_l(1/(1 - t)) = (2t - 1)t^{-l}(1 - t)^{-l}$, $l = 1, 2, \ldots$. 
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

(a) $x = t^m, y = t^n + \gamma_1 t^{-m} + \gamma_2 t^{-2m} + \cdots + \gamma_k t^{-mk}$, where $m > 0$, $gcd(m, |n|) = 1$, $k = 0, 1, \ldots$, $\gamma_j \in \mathbb{C}$, $\gamma_k = 1$ (if $k > 0$) and $k > 0$ if $n > 0$. and at least one $\gamma_i \neq 0$ if $m > 0$.

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(c) $x = t^{mn}(t - 1), y = S_{k+1}^+(1/t)$, where $mn \geq 2$, $k = 1, 2, \ldots$. $S_k$ are polynomials defined recursively by $S_0^+(u) = u^n$, $S_{k+1}^+(u) = [S_k^+(u) - S_k^+(1)]u^{mn+1}/(u - 1)$. 
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

(d) $x = t^{mn-1}(t - 1)$, $y = T_k^+(1/t)$, where $mn \geq 2$, $k = 1, 2, \ldots$. $T_k^+$ are polynomials satisfying

$T_0^+(u) = u^{mn}$,

$T_{k+1}^+(u) = [T_k^+(u) - T_k^+(1)]u^{mn}/(u - 1)$. 
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

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(e) $x = t^{mn}(t - 1), y = S_k^-(1/t)$, where $mn \geq 2$, $k = 1, \ldots$, and $S_m^-$ is a polynomial such that $S_0^-(u) = u^{-mn}$, $S_{k+1}^-(u) = [S_k^-(u) - S_k^-(1)]u^{mn+1}/(u - 1)$. 
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

(d) $x = t^{mn-1}(t - 1), \ y = T_k^+(1/t)$, where $mn \geq 2, k = 1, 2, \ldots$. $T_k^+$ are polynomials satisfying

$T_0^+(u) = u^{mn}$,

$T_{k+1}^+(u) = [T_k^+(u) - T_k^+(1)]u^{mn}/(u - 1)$.

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(f) $x = t^{mn-1}(t - 1), \ y = T_k^-(1/t)$, where $mn \geq 2, k = 1, 2, \ldots$ and $T_m^-$ is a polynomial given by

$T_0^-(u) = u^{-mn}$,

$T_{k+1}^-(u) = [T_k^-(u) - T_k^-(1)]u^{mn}/(u - 1)$. 
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

(g) $x = t^2(t - 1)$, $y = U_k(1/t)$, $k = 1, 2, \ldots$, 
$U_1(u) = 3u + u^2$, 
$U_{k+1}(u) = [U_k(u) - U_k(1)]u^3/(u - 1)$. 
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

(g) $x = t^2(t - 1)$, $y = U_k(1/t)$, $k = 1, 2, \ldots,$
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$U_{k+1}(u) = [U_k(u) - U_k(1)]u^3/(u - 1).$

(h) $x = t^3(t - 1)$, $y = V_k(1/t)$, $V_1(u) = 2u^2 - u^3,$
$V_{k+1}(u) = [V_k(u) - V_k(1)]u^4/(u - 1).$
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

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$V_{k+1}(u) = [V_k(u) - V_k(1)]u^4/(u - 1).$

(i) $x = t^3(t - 1), y = W_k(1/t), \text{ where } k = 1, 2, \ldots,$
$W_1(u) = 2u^2 + u^3,$
$W_{k+1}(u) = [W_k(u) - W_k(1)]u^4/(u - 1).$
Maps $\mathbb{C}^* \to \mathbb{C}^2$

(g) $x = t^2(t - 1)$, $y = U_k(1/t)$, $k = 1, 2, \ldots$,
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$W_1(u) = 2u^2 + u^3$,
$W_{k+1}(u) = [W_k(u) - W_k(1)]u^4/(u - 1)$.

(j) $x = t + t^{-1}$, $y = Z(t)$ is a polynomial satisfying
$y(t) + y(1/t) = (t - 1)^{2m+1}(t + 1)^{n+1}/t^{m+n+1}$,
where $0 \leq m \leq n$ and $m + n > 0$. 
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

(k) $x = (t - 1)^3 t^{-2}, \quad y = x^k (t - 1)(t - 4)t^{-1},$
$k = 1, 2, \ldots.$
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

(k) \( x = (t - 1)^3 t^{-2}, \ y = x^k (t - 1)(t - 4) t^{-1}, \)
\( k = 1, 2, \ldots. \)

(l) \( x = (t - 1)^m t^{-pn}, \ y = (t - 1)^k t^{-pl}, \)
\( ml - nk = 1, \ p = 1, 2, \ldots. \)
Maps $\mathbb{C}^* \to \mathbb{C}^2$

(k) $x = (t - 1)^3 t^{-2}, \ y = x^k (t - 1)(t - 4) t^{-1}, \ k = 1, 2, \ldots.$

(l) $x = (t - 1)^m t^{-p} n, \ y = (t - 1)^k t^{-p} l, \ ml - nk = 1, \ p = 1, 2, \ldots.$

(m) $x = (t - 1)^p m t^{-n}, \ y = (t - 1)^p k t^{-l}, \ ml - nk = 1, \ p = 1, 2, \ldots.$
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(o) $x = (t - 1)^{4l} t^{1-2l}$, $y = x^k (t - 1)^{2l} (t + 1) t^{-l}$, $k = 0, 1, \ldots$, $l = 1, 2, 3, \ldots$. 
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

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(p) $x = (t - 1)^{4} t^{-3}, \ y = x^k(t - 1)^{2}(t - 3)t^{-2}, \ k = 1, 2, \ldots$. 
Maps $\mathbb{C}^* \to \mathbb{C}^2$

(q) $x = (t - 1)^{4m-2}t^{1-2m}$,
$y = x^k \cdot (t - 1)^{2m-1}(t + 3)t^{-m}$, $m = 2, 3, \ldots,$
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(u) $x = (t - 1)^2 (t + 2) t^{-1}$, $y = (t - 1)^4 (t + \frac{1}{2}) t^{-2}$.
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

(v) $x = (t - 1)^2(t + 4 + 2\sqrt{5})t^{-1},$
$y = (t - 1)^4(t + \frac{1}{4}(11 + 5\sqrt{5}))t^{-2}.$
Maps $\mathbb{C}^* \rightarrow \mathbb{C}^2$

(v) $x = (t - 1)^2(t + 4 + 2\sqrt{5})t^{-1}$,
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(w) $x = (t - 1)^2(t + 2)t^{-1}$, $y = (t - 1)^2(t + \frac{1}{2})t^{-2}$. 
What next?

- Prove regularity conjecture.
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- Study intersections on the space of curves $\text{Cur}_{a,c}$ and $\text{Cur}_{a,b,c,d}$.
- Applications to XVI Hilbert problem (Liénard vector fields).
THANK YOU