Real Seifert forms 20 years after

Maciej Borodzik
www.mimuw.edu.pl/~mcboro

Institute of Mathematics, University of Warsaw

Budapest, May 2019

Milnor fibration

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial map with $0 \in \mathbb{C}^{n+1}$ an isolated critical point.

Milnor fibration

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial map with $0 \in \mathbb{C}^{n+1}$ an isolated critical point.

Theorem (Milnor)

For $\varepsilon > 0$ sufficiently small, the map $\Psi\colon S_\varepsilon^{2n+1} \setminus f^{-1}(0) \to S^1$ given by $\Psi(z) = f(z)/|f(z)|$ is a locally trivial fibration.

Milnor fibration

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial map with $0 \in \mathbb{C}^{n+1}$ an isolated critical point.

Theorem (Milnor)

For $\varepsilon > 0$ sufficiently small, the map $\Psi\colon S^{2n+1}_{\varepsilon} \setminus f^{-1}(0) \to S^1$ given by $\Psi(z) = f(z)/|f(z)|$ is a locally trivial fibration. The fiber $\Psi^{-1}(1)$ has homotopy type of a wedge sum of some finite number (μ) of spheres S^n.

Milnor fibration

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial map with $0 \in \mathbb{C}^{n+1}$ an isolated critical point.

Theorem (Milnor)

For $\varepsilon > 0$ sufficiently small, the map $\Psi\colon S^{2n+1}_{\varepsilon} \setminus f^{-1}(0) \to S^1$ given by $\Psi(z) = f(z)/|f(z)|$ is a locally trivial fibration. The fiber $\Psi^{-1}(1)$ has homotopy type of a wedge sum of some finite number (μ) of spheres S^n.

Definition

Let F_t be the fiber $\Psi^{-1}(t)$. The *geometric monodromy* h_t (for $t \in S^1$) is a diffeomorphism $h_t\colon F_1 \to F_t$, smoothly depending on t.

Milnor fibration

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial map with $0 \in \mathbb{C}^{n+1}$ an isolated critical point.

Theorem (Milnor)

For $\varepsilon > 0$ sufficiently small, the map $\Psi\colon S_{\varepsilon}^{2n+1} \setminus f^{-1}(0) \to S^1$ given by $\Psi(z) = f(z)/|f(z)|$ is a locally trivial fibration. The fiber $\Psi^{-1}(1)$ has homotopy type of a wedge sum of some finite number (μ) of spheres S^n.

Definition

Let F_t be the fiber $\Psi^{-1}(t)$. The *geometric monodromy* h_t (for $t \in S^1$) is a diffeomorphism $h_t\colon F_1 \to F_t$, smoothly depending on t.

Remark

We sometimes consider Ψ on $S^{2n+1} \setminus N(f^{-1}(0)) \to S^1$.

Milnor fibration

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial map with $0 \in \mathbb{C}^{n+1}$ an isolated critical point.

Theorem (Milnor)

For $\varepsilon > 0$ sufficiently small, the map $\Psi\colon S^{2n+1}_{\varepsilon} \setminus f^{-1}(0) \to S^1$ given by $\Psi(z) = f(z)/|f(z)|$ is a locally trivial fibration. The fiber $\Psi^{-1}(1)$ has homotopy type of a wedge sum of some finite number (μ) of spheres S^n.

Definition

Let F_t be the fiber $\Psi^{-1}(t)$. The *geometric monodromy* h_t (for $t \in S^1$) is a diffeomorphism $h_t\colon F_1 \to F_t$, smoothly depending on t.

Remark

We sometimes consider Ψ on $S^{2n+1} \setminus N(f^{-1}(0)) \to S^1$. Then F_t are manifolds with boundary.

Milnor fibration

Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial map with $0 \in \mathbb{C}^{n+1}$ an isolated critical point.

Theorem (Milnor)

For $\varepsilon > 0$ sufficiently small, the map $\Psi\colon S^{2n+1}_\varepsilon \setminus f^{-1}(0) \to S^1$ given by $\Psi(z) = f(z)/|f(z)|$ is a locally trivial fibration. The fiber $\Psi^{-1}(1)$ has homotopy type of a wedge sum of some finite number (μ) of spheres S^n.

Definition

Let F_t be the fiber $\Psi^{-1}(t)$. The *geometric monodromy* h_t (for $t \in S^1$) is a diffeomorphism $h_t\colon F_1 \to F_t$, smoothly depending on t.

Remark

We sometimes consider Ψ on $S^{2n+1} \setminus N(f^{-1}(0)) \to S^1$. Then F_t are manifolds with boundary.We have $h_1|_{\partial F_t} = id$.

Homological invariants

Definition

The *homological monodromy* is the map
$h\colon H_n(F_1;\mathbb{Z}) \to H_n(F_1;\mathbb{Z})$ induced by the monodromy.

Homological invariants

Definition

The *homological monodromy* is the map
$h\colon H_n(F_1;\mathbb{Z}) \to H_n(F_1;\mathbb{Z})$ induced by the monodromy.

Take a cycle $\alpha \in H_n(F_1, \partial F_1;\mathbb{Z})$. The image $h_1(\alpha)$ has the same boundary. Hence $h_1(\alpha) - \alpha$ is an absolute cycle.

Homological invariants

Definition

The *homological monodromy* is the map $h\colon H_n(F_1;\mathbb{Z}) \to H_n(F_1;\mathbb{Z})$ induced by the monodromy.

Take a cycle $\alpha \in H_n(F_1, \partial F_1;\mathbb{Z})$. The image $h_1(\alpha)$ has the same boundary. Hence $h_1(\alpha) - \alpha$ is an absolute cycle.

Definition

The map defined just above is called the *variation map* and denoted $\mathrm{var}\colon H_n(F_1, \partial F_1;\mathbb{Z}) \to H_n(F_1;\mathbb{Z})$.

Seifert form

Definition

The *Seifert form* is the map $H_n(F_1;\mathbb{Z}) \times H_n(F_1;\mathbb{Z}) \to \mathbb{Z}$ given by $L(\alpha, \beta) \mapsto \mathrm{lk}(\alpha, h_{1/2}\beta)$.

Seifert form

Definition

The *Seifert form* is the map $H_n(F_1;\mathbb{Z}) \times H_n(F_1;\mathbb{Z}) \to \mathbb{Z}$ given by $L(\alpha,\beta) \mapsto \operatorname{lk}(\alpha, h_{1/2}\beta)$.

Theorem (Picard–Lefschetz package)

We have $L(\operatorname{var}\alpha,\beta) = \langle\alpha,\beta\rangle$, where $\langle\cdot,\cdot\rangle$ is the Poincaré–Lefschetz duality pairing.

Hermitian Variation Structure. Definition

We gather these objects (variation, intersection form, monodromy, Seifert form) into a structure.

Hermitian Variation Structure. Definition

We gather these objects (variation, intersection form, monodromy, Seifert form) into a structure.

Definition (Hermitian Variation Structure, Némethi 1995)

An $\varepsilon = \pm 1$ *hermitian variation structure* is a quadruple $(U; b, h, V)$, where

Hermitian Variation Structure. Definition

We gather these objects (variation, intersection form, monodromy, Seifert form) into a structure.

Definition (Hermitian Variation Structure, Némethi 1995)

An $\varepsilon = \pm 1$ *hermitian variation structure* is a quadruple $(U; b, h, V)$, where

- U is a finite dimensional vector space

Hermitian Variation Structure. Definition

We gather these objects (variation, intersection form, monodromy, Seifert form) into a structure.

Definition (Hermitian Variation Structure, Némethi 1995)

An $\varepsilon = \pm 1$ *hermitian variation structure* is a quadruple $(U; b, h, V)$, where

- U is a finite dimensional vector space ($U = H_n(F_1; \mathbb{C})$);

Hermitian Variation Structure. Definition

We gather these objects (variation, intersection form, monodromy, Seifert form) into a structure.

Definition (Hermitian Variation Structure, Némethi 1995)

An $\varepsilon = \pm 1$ *hermitian variation structure* is a quadruple $(U; b, h, V)$, where

- U is a finite dimensional vector space ($U = H_n(F_1; \mathbb{C})$);
- $b \colon U \to U^*$ is a $\mathbb{C}$-linear endomorphism with $\overline{b^* \circ \theta} = \varepsilon b$, where $\theta \colon U \to U^{**}$ is a natural isomorphism

Hermitian Variation Structure. Definition

We gather these objects (variation, intersection form, monodromy, Seifert form) into a structure.

Definition (Hermitian Variation Structure, Némethi 1995)

An $\varepsilon = \pm 1$ *hermitian variation structure* is a quadruple $(U; b, h, V)$, where

- U is a finite dimensional vector space ($U = H_n(F_1; \mathbb{C})$);
- $b\colon U \to U^*$ is a $\mathbb{C}$-linear endomorphism with $\overline{b^* \circ \theta} = \varepsilon b$, where $\theta\colon U \to U^{**}$ is a natural isomorphism ($b\colon H_n(F_1; \mathbb{C}) \to H_n(F_1, \partial F_1; \mathbb{C})$);

Hermitian Variation Structure. Definition

We gather these objects (variation, intersection form, monodromy, Seifert form) into a structure.

Definition (Hermitian Variation Structure, Némethi 1995)

An $\varepsilon = \pm 1$ *hermitian variation structure* is a quadruple $(U; b, h, V)$, where

- U is a finite dimensional vector space ($U = H_n(F_1; \mathbb{C})$);
- $b \colon U \to U^*$ is a $\mathbb{C}$-linear endomorphism with $\overline{b^* \circ \theta} = \varepsilon b$, where $\theta \colon U \to U^{**}$ is a natural isomorphism ($b \colon H_n(F_1; \mathbb{C}) \to H_n(F_1, \partial F_1; \mathbb{C})$);
- $h \colon U \to U$ is b-orthogonal, that is $\overline{h}^* \circ b \circ h = b$

.

Hermitian Variation Structure. Definition

We gather these objects (variation, intersection form, monodromy, Seifert form) into a structure.

Definition (Hermitian Variation Structure, Némethi 1995)

An $\varepsilon = \pm 1$ *hermitian variation structure* is a quadruple $(U; b, h, V)$, where

- U is a finite dimensional vector space ($U = H_n(F_1; \mathbb{C})$);
- $b\colon U \to U^*$ is a $\mathbb{C}$-linear endomorphism with $\overline{b^* \circ \theta} = \varepsilon b$, where $\theta\colon U \to U^{**}$ is a natural isomorphism ($b\colon H_n(F_1; \mathbb{C}) \to H_n(F_1, \partial F_1; \mathbb{C})$);
- $h\colon U \to U$ is b-orthogonal, that is $\overline{h}^* \circ b \circ h = b$ (h is the homological monodromy).

Hermitian Variation Structure. Definition

We gather these objects (variation, intersection form, monodromy, Seifert form) into a structure.

Definition (Hermitian Variation Structure, Némethi 1995)

An $\varepsilon = \pm 1$ *hermitian variation structure* is a quadruple $(U; b, h, V)$, where

- U is a finite dimensional vector space ($U = H_n(F_1; \mathbb{C})$);
- $b\colon U \to U^*$ is a $\mathbb{C}$-linear endomorphism with $\overline{b^* \circ \theta} = \varepsilon b$, where $\theta\colon U \to U^{**}$ is a natural isomorphism ($b\colon H_n(F_1; \mathbb{C}) \to H_n(F_1, \partial F_1; \mathbb{C})$);
- $h\colon U \to U$ is b-orthogonal, that is $\overline{h}^* \circ b \circ h = b$ (h is the homological monodromy).
- $V\colon U^* \to U$ is a $\mathbb{C}$-linear endomorphism with $\overline{\theta^{-1} \circ V^*} = -\varepsilon V \circ \overline{h^*}$ and $V \circ b = h - I$

Hermitian Variation Structure. Definition

We gather these objects (variation, intersection form, monodromy, Seifert form) into a structure.

Definition (Hermitian Variation Structure, Némethi 1995)

An $\varepsilon = \pm 1$ *hermitian variation structure* is a quadruple $(U; b, h, V)$, where

- U is a finite dimensional vector space ($U = H_n(F_1; \mathbb{C})$);
- $b \colon U \to U^*$ is a $\mathbb{C}$-linear endomorphism with $\overline{b^* \circ \theta} = \varepsilon b$, where $\theta \colon U \to U^{**}$ is a natural isomorphism ($b \colon H_n(F_1; \mathbb{C}) \to H_n(F_1, \partial F_1; \mathbb{C})$);
- $h \colon U \to U$ is b-orthogonal, that is $\overline{h}^* \circ b \circ h = b$ (h is the homological monodromy).
- $V \colon U^* \to U$ is a $\mathbb{C}$-linear endomorphism with $\overline{\theta^{-1} \circ V^*} = -\varepsilon V \circ \overline{h^*}$ and $V \circ b = h - I$ (V is the variation map.)

Hermitian Variation Structure. Remarks

Hermitian Variation Structure. Remarks

Exercise

Deduce the axioms of V from the properties on previous slides.

Hermitian Variation Structure. Remarks

Exercise

Deduce the axioms of V from the properties on previous slides.

Lemma

If b is an isomorphism (such a structure is called non-degenerate), then $V = (h - I)b^{-1}$. The HVS is determined by the triple $(U; h, b)$.

Hermitian Variation Structure. Remarks

Exercise

Deduce the axioms of V from the properties on previous slides.

Lemma

If b is an isomorphism (such a structure is called non-degenerate), then $V = (h - I)b^{-1}$. The HVS is determined by the triple $(U; h, b)$.

Lemma

If V is an isomorphism (such a structure is called simple), then $h = -\varepsilon V\overline{(\theta^{-1} \circ V^*)^{-1}}$ and $b = -V^{-1} - \varepsilon\overline{(\theta^{-1} \circ V^*)^{-1}}$, so V determines the HVS.

Hermitian Variation Structure. Remarks

Exercise

Deduce the axioms of V *from the properties on previous slides.*

Lemma

If b *is an isomorphism (such a structure is called non-degenerate), then* $V = (h - I)b^{-1}$. *The HVS is determined by the triple* $(U; h, b)$.

Lemma

If V *is an isomorphism (such a structure is called simple), then* $h = -\varepsilon V \overline{(\theta^{-1} \circ V^*)^{-1}}$ *and* $b = -V^{-1} - \varepsilon \overline{(\theta^{-1} \circ V^*)^{-1}}$, *so* V *determines the HVS.*

Remark

Complexity of formulas gives us sometimes possibility to deal with degenerate/non-simple cases.

Hermitian Variation Structure. Classification

- Classification exists essentially for simple HVS;

Hermitian Variation Structure. Classification

- Classification exists essentially for simple HVS;
- The starting point is the Jordan block decomposition for h;

Hermitian Variation Structure. Classification

- Classification exists essentially for simple HVS;
- The starting point is the Jordan block decomposition for h;
- For $\lambda \in S^1$: two structures $V(\lambda, k, \pm 1)$ for each Jordan block;

Hermitian Variation Structure. Classification

- Classification exists essentially for simple HVS;
- The starting point is the Jordan block decomposition for h;
- For $\lambda \in S^1$: two structures $V(\lambda, k, \pm 1)$ for each Jordan block;
- For $|\lambda| < 1$: one structure $V(\lambda, k)$ corresponding to a Jordan block with e.v. λ and $1/\lambda$.

Hermitian Variation Structure. Classification

- Classification exists essentially for simple HVS;
- The starting point is the Jordan block decomposition for h;
- For $\lambda \in S^1$: two structures $V(\lambda, k, \pm 1)$ for each Jordan block;
- For $|\lambda| < 1$: one structure $V(\lambda, k)$ corresponding to a Jordan block with e.v. λ and $1/\lambda$.
- Each simple structure is a direct sum of structures corresponding to Jordan blocks.

Hermitian Variation Structure. Classification

- Classification exists essentially for simple HVS;
- The starting point is the Jordan block decomposition for h;
- For $\lambda \in S^1$: two structures $V(\lambda, k, \pm 1)$ for each Jordan block;
- For $|\lambda| < 1$: one structure $V(\lambda, k)$ corresponding to a Jordan block with e.v. λ and $1/\lambda$.
- Each simple structure is a direct sum of structures corresponding to Jordan blocks.

Hermitian Variation Structure. Classification

- Classification exists essentially for simple HVS;
- The starting point is the Jordan block decomposition for h;
- For $\lambda \in S^1$: two structures $V(\lambda, k, \pm 1)$ for each Jordan block;
- For $|\lambda| < 1$: one structure $V(\lambda, k)$ corresponding to a Jordan block with e.v. λ and $1/\lambda$.
- Each simple structure is a direct sum of structures corresponding to Jordan blocks.

Definition

Given a simple HVS, *Hodge numbers* $p_\lambda^k(\pm 1)$ (for $\lambda \in S^1$) and q_λ^k indicate how many times the given basic structure enters the HVS as a summand.

Hermitian Variation Structure. Properties

- Hodge numbers determine the mod-2 spectrum of singularity;

Hermitian Variation Structure. Properties

- Hodge numbers determine the mod-2 spectrum of singularity;
- They determine monodromy over $\mathbb{C}$;

Hermitian Variation Structure. Properties

- Hodge numbers determine the mod-2 spectrum of singularity;
- They determine monodromy over $\mathbb{C}$;
- Good way to state theorems, like monodromy theorems.

HVS for knots

- Let S be a Seifert form for a knot K;

HVS for knots

- Let S be a Seifert form for a knot K;
- Make it invertible by S-equivalence (possible by Kawauchi);

HVS for knots

- Let S be a Seifert form for a knot K;
- Make it invertible by S-equivalence (possible by Kawauchi);
- Associate a HVS with $V = S^{-1}$;

HVS for knots

- Let S be a Seifert form for a knot K;
- Make it invertible by S-equivalence (possible by Kawauchi);
- Associate a HVS with $V = S^{-1}$;
- If K is a link of singularity, then the HVS is the same as the one given by Picard–Lefschetz package;

HVS for knots

- Let S be a Seifert form for a knot K;
- Make it invertible by S-equivalence (possible by Kawauchi);
- Associate a HVS with $V = S^{-1}$;
- If K is a link of singularity, then the HVS is the same as the one given by Picard–Lefschetz package;
- We obtain Hodge numbers for knots (and more generally for links).

Classical invariants for links

Theorem (—, Némethi, 2011)

Let K be a knot and $p^k_\lambda(\epsilon)$, q^k_λ the Hodge numbers.

- *The Alexander polynomial and higher Alexander polynomials are determined from the Hodge numbers.*

Classical invariants for links

Theorem (—, Némethi, 2011)

Let K be a knot and $p^k_\lambda(\epsilon)$, q^k_λ the Hodge numbers.

- *The Alexander polynomial and higher Alexander polynomials are determined from the Hodge numbers.*
- *For example, Δ is the characteristic polynomial of $h = S^{-1}S^T$.*

Classical invariants for links

Theorem (—, Némethi, 2011)

Let K be a knot and $p^k_\lambda(\epsilon)$, q^k_λ the Hodge numbers.

- *The Alexander polynomial and higher Alexander polynomials are determined from the Hodge numbers.*
- *For example, Δ is the characteristic polynomial of $h = S^{-1}S^T$.*
- *The Tristram–Levine signature of K is determined by $p^k_\lambda(\varepsilon)$. More precisely $p^k_\lambda(\varepsilon)$ for odd k determine the jumps at λ and for k even determine the peek at λ.*

Example

Consider a slice knot 8_{20}.

- It has Alexander polynomial $(t-\lambda)^2(t-\overline{\lambda})^2$ with $\lambda = \frac{1}{2}(1+i\sqrt{3})$.

Example

Consider a slice knot 8_{20}.

- It has Alexander polynomial $(t-\lambda)^2(t-\overline{\lambda})^2$ with $\lambda = \frac{1}{2}(1+i\sqrt{3})$.
- Hence either $p^1_\lambda(+1) + p^1_\lambda(-1) = 2$ or $p^2_\lambda(\varepsilon) = 1$ for some ε.

Example

Consider a slice knot 8_{20}.

- It has Alexander polynomial $(t-\lambda)^2(t-\overline{\lambda})^2$ with $\lambda = \frac{1}{2}(1+i\sqrt{3})$.
- Hence either $p^1_\lambda(+1) + p^1_\lambda(-1) = 2$ or $p^2_\lambda(\varepsilon) = 1$ for some ε.
- In the first case the Alexander module is not cyclic, but we know that 8_{20} has cyclic Alexander module (Nakanishi index is 1).

Example

Consider a slice knot 8_{20}.

- It has Alexander polynomial $(t-\lambda)^2(t-\overline{\lambda})^2$ with $\lambda = \frac{1}{2}(1+i\sqrt{3})$.
- Hence either $p^1_\lambda(+1) + p^1_\lambda(-1) = 2$ or $p^2_\lambda(\varepsilon) = 1$ for some ε.
- In the first case the Alexander module is not cyclic, but we know that 8_{20} has cyclic Alexander module (Nakanishi index is 1).
- Hence $p^2_\lambda(\varepsilon) = 1$. The signature function is constantly zero for $t \neq \lambda, \overline{\lambda}$ and equal to ε for $t = \lambda, \overline{\lambda}$.

Applications

Theorem (Murasugi's inequality)

Let $K \subset S^3$ be a knot bounding a surface $S \subset B^4$. Then $|\sigma_t(K)| \leq 2g(S)$ for almost all $t \in S^1$.

- Hodge numbers relate signature to the spectrum of a singular point (if K is algebraic).

Applications

Theorem (Murasugi's inequality)

Let $K \subset S^3$ *be a knot bounding a surface* $S \subset B^4$*. Then* $|\sigma_t(K)| \leq 2g(S)$ *for almost all* $t \in S^1$*.*

- Hodge numbers relate signature to the spectrum of a singular point (if K is algebraic).
- Murasugi's inequality translates into semicontinuity of spectrum.

Applications

Theorem (Murasugi's inequality)

Let $K \subset S^3$ be a knot bounding a surface $S \subset B^4$. Then $|\sigma_t(K)| \leq 2g(S)$ for almost all $t \in S^1$.

- Hodge numbers relate signature to the spectrum of a singular point (if K is algebraic).
- Murasugi's inequality translates into semicontinuity of spectrum.
- We (—, Némethi 2013) obtain not only another proof of spectrum semicontinuity, but various other statements on semicontinuity.

Applications

Theorem (Murasugi's inequality)

Let $K \subset S^3$ be a knot bounding a surface $S \subset B^4$. Then $|\sigma_t(K)| \leq 2g(S)$ for almost all $t \in S^1$.

- Hodge numbers relate signature to the spectrum of a singular point (if K is algebraic).
- Murasugi's inequality translates into semicontinuity of spectrum.
- We (—, Némethi 2013) obtain not only another proof of spectrum semicontinuity, but various other statements on semicontinuity.
- In particular, semicontinuity of spectrum of a plane curve singularity depends on topological data only.

Applications

Theorem (Murasugi's inequality)

Let $K \subset S^3$ be a knot bounding a surface $S \subset B^4$. Then $|\sigma_t(K)| \leq 2g(S)$ for almost all $t \in S^1$.

- Hodge numbers relate signature to the spectrum of a singular point (if K is algebraic).
- Murasugi's inequality translates into semicontinuity of spectrum.
- We (—, Némethi 2013) obtain not only another proof of spectrum semicontinuity, but various other statements on semicontinuity.
- In particular, semicontinuity of spectrum of a plane curve singularity depends on topological data only.
- . . . unlike semigroup semicontinuity established by Gorsky and Némethi in 2013, which depends on the smooth data.

Blanchfield pairings

A more popular point of view is via Blanchfield pairings.

Blanchfield pairings

A more popular point of view is via Blanchfield pairings.

Theorem (Blanchfield 1959)

Let K be a knot and X the knot exterior.

Blanchfield pairings

A more popular point of view is via Blanchfield pairings.

Theorem (Blanchfield 1959)

Let K be a knot and X the knot exterior. Consider the universal abelian cover $\pi\colon \widetilde{X} \to X$.

Blanchfield pairings

A more popular point of view is via Blanchfield pairings.

Theorem (Blanchfield 1959)

Let K be a knot and X the knot exterior. Consider the universal abelian cover $\pi\colon \widetilde{X} \to X$. Let $H = H_1(\widetilde{X}; \mathbb{Z})$ regarded as a $\mathbb{Z}[t, t^{-1}]$ module.

Blanchfield pairings

A more popular point of view is via Blanchfield pairings.

Theorem (Blanchfield 1959)

Let K be a knot and X the knot exterior. Consider the universal abelian cover $\pi\colon \widetilde{X} \to X$. Let $H = H_1(\widetilde{X};\mathbb{Z})$ regarded as a $\mathbb{Z}[t,t^{-1}]$ module. Then there exists a non-degenerate sesquilinear pairing $H \times H \to \mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]$, which is a knot invariant. The pairing is determined by the Seifert form of K and it determines the S-equivalence class of Seifert forms.

Blanchfield pairings

A more popular point of view is via Blanchfield pairings.

Theorem (Blanchfield 1959)

Let K be a knot and X the knot exterior. Consider the universal abelian cover $\pi\colon \widetilde{X} \to X$. Let $H = H_1(\widetilde{X}; \mathbb{Z})$ regarded as a $\mathbb{Z}[t, t^{-1}]$ module. Then there exists a non-degenerate sesquilinear pairing $H \times H \to \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]$, which is a knot invariant. The pairing is determined by the Seifert form of K and it determines the S-equivalence class of Seifert forms.

Remark

The pairing is a more sophisticated version of the linking pairing on a rational homology 3-sphere.

Blanchfield pairings

A more popular point of view is via Blanchfield pairings.

Theorem (Blanchfield 1959)

Let K be a knot and X the knot exterior. Consider the universal abelian cover $\pi\colon \widetilde{X} \to X$. Let $H = H_1(\widetilde{X}; \mathbb{Z})$ regarded as a $\mathbb{Z}[t, t^{-1}]$ module. Then there exists a non-degenerate sesquilinear pairing $H \times H \to \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]$, which is a knot invariant. The pairing is determined by the Seifert form of K and it determines the S-equivalence class of Seifert forms.

Remark

The pairing is a more sophisticated version of the linking pairing on a rational homology 3-sphere.

Remark

If S is a Seifert matrix, then $H = \mathbb{Z}[t, t^{-1}]^n/(tS - S^T)\mathbb{Z}[t, t^{-1}]^n$ and the pairing is given by $(a, b) \mapsto \overline{a}^T(tS - S^T)^{-1}(t - 1)b$.

Blanchfield pairings over $\mathbb{R}$

- Let $\Lambda = \mathbb{R}[t, t^{-1}]$. Consider a Λ-module H and a pairing $H \times H \to \mathbb{R}(t)/\Lambda$.

Blanchfield pairings over $\mathbb{R}$

- Let $\Lambda = \mathbb{R}[t, t^{-1}]$. Consider a Λ-module H and a pairing $H \times H \to \mathbb{R}(t)/\Lambda$.
- Assume there are no $(t \pm 1)$ torsion parts of H.

Blanchfield pairings over $\mathbb{R}$

- Let $\Lambda = \mathbb{R}[t, t^{-1}]$. Consider a Λ-module H and a pairing $H \times H \to \mathbb{R}(t)/\Lambda$.
- Assume there are no $(t \pm 1)$ torsion parts of H.

Blanchfield pairings over $\mathbb{R}$

- Let $\Lambda = \mathbb{R}[t, t^{-1}]$. Consider a Λ-module H and a pairing $H \times H \to \mathbb{R}(t)/\Lambda$.
- Assume there are no $(t \pm 1)$ torsion parts of H. (This means that $t \pm 1$ does not divide the Alexander polynomial,

Blanchfield pairings over $\mathbb{R}$

- Let $\Lambda = \mathbb{R}[t, t^{-1}]$. Consider a Λ-module H and a pairing $H \times H \to \mathbb{R}(t)/\Lambda$.
- Assume there are no $(t \pm 1)$ torsion parts of H. (This means that $t \pm 1$ does not divide the Alexander polynomial, or h has no eigenvalues ± 1.)
- Then H decomposes as an orthogonal sum of modules of form $\Lambda/b_\xi^k\Lambda$, $\Lambda/c_\xi^\ell\Lambda$, where

Blanchfield pairings over $\mathbb{R}$

- Let $\Lambda = \mathbb{R}[t, t^{-1}]$. Consider a Λ-module H and a pairing $H \times H \to \mathbb{R}(t)/\Lambda$.
- Assume there are no $(t \pm 1)$ torsion parts of H. (This means that $t \pm 1$ does not divide the Alexander polynomial, or h has no eigenvalues ± 1.)
- Then H decomposes as an orthogonal sum of modules of form $\Lambda/b_\xi^k\Lambda$, $\Lambda/c_\xi^\ell\Lambda$, where
- For $\xi \in S^1$, $\operatorname{Im} \xi > 0$, $b_\xi = (t - \xi)(t - \overline{\xi})t^{-1}$;

Blanchfield pairings over $\mathbb{R}$

- Let $\Lambda = \mathbb{R}[t, t^{-1}]$. Consider a Λ-module H and a pairing $H \times H \to \mathbb{R}(t)/\Lambda$.
- Assume there are no $(t \pm 1)$ torsion parts of H. (This means that $t \pm 1$ does not divide the Alexander polynomial, or h has no eigenvalues ± 1.)
- Then H decomposes as an orthogonal sum of modules of form $\Lambda/b_\xi^k\Lambda$, $\Lambda/c_\xi^\ell\Lambda$, where
- For $\xi \in S^1$, $\operatorname{Im} \xi > 0$, $b_\xi = (t-\xi)(t-\overline{\xi})t^{-1}$;
- For $\xi \notin (S^1 \cup \mathbb{R})$, $|\xi| < 1$, $c_\xi = (t-\xi)(t-\overline{\xi})(t^{-1}-\xi)(t^{-1}-\overline{\xi})$;

Blanchfield pairings over $\mathbb{R}$

- Let $\Lambda = \mathbb{R}[t, t^{-1}]$. Consider a Λ-module H and a pairing $H \times H \to \mathbb{R}(t)/\Lambda$.
- Assume there are no $(t \pm 1)$ torsion parts of H. (This means that $t \pm 1$ does not divide the Alexander polynomial, or h has no eigenvalues ± 1.)
- Then H decomposes as an orthogonal sum of modules of form $\Lambda/b_\xi^k\Lambda$, $\Lambda/c_\xi^\ell\Lambda$, where
- For $\xi \in S^1$, $\operatorname{Im} \xi > 0$, $b_\xi = (t - \xi)(t - \overline{\xi})t^{-1}$;
- For $\xi \notin (S^1 \cup \mathbb{R})$, $|\xi| < 1$, $c_\xi = (t - \xi)(t - \overline{\xi})(t^{-1} - \xi)(t^{-1} - \overline{\xi})$;
- For $\xi \in \mathbb{R} \setminus \{0\}$, $|\xi| < 1$, $c_\xi = (t - \xi)(t^{-1} - \xi)$.

Blanchfield pairings over $\mathbb{R}$

- Let $\Lambda = \mathbb{R}[t, t^{-1}]$. Consider a Λ-module H and a pairing $H \times H \to \mathbb{R}(t)/\Lambda$.
- Assume there are no $(t \pm 1)$ torsion parts of H. (This means that $t \pm 1$ does not divide the Alexander polynomial, or h has no eigenvalues ± 1.)
- Then H decomposes as an orthogonal sum of modules of form $\Lambda/b_\xi^k\Lambda$, $\Lambda/c_\xi^\ell\Lambda$, where
- For $\xi \in S^1$, $\operatorname{Im}\xi > 0$, $b_\xi = (t-\xi)(t-\overline{\xi})t^{-1}$;
- For $\xi \notin (S^1 \cup \mathbb{R})$, $|\xi| < 1$, $c_\xi = (t-\xi)(t-\overline{\xi})(t^{-1}-\xi)(t^{-1}-\overline{\xi})$;
- For $\xi \in \mathbb{R} \setminus \{0\}$, $|\xi| < 1$, $c_\xi = (t-\xi)(t^{-1}-\xi)$.

Blanchfield pairings over $\mathbb{R}$

- Let $\Lambda = \mathbb{R}[t, t^{-1}]$. Consider a Λ-module H and a pairing $H \times H \to \mathbb{R}(t)/\Lambda$.
- Assume there are no $(t \pm 1)$ torsion parts of H. (This means that $t \pm 1$ does not divide the Alexander polynomial, or h has no eigenvalues ± 1.)
- Then H decomposes as an orthogonal sum of modules of form $\Lambda/b_\xi^k\Lambda$, $\Lambda/c_\xi^\ell\Lambda$, where
- For $\xi \in S^1$, $\operatorname{Im} \xi > 0$, $b_\xi = (t-\xi)(t-\overline{\xi})t^{-1}$;
- For $\xi \notin (S^1 \cup \mathbb{R})$, $|\xi| < 1$, $c_\xi = (t-\xi)(t-\overline{\xi})(t^{-1}-\xi)(t^{-1}-\overline{\xi})$;
- For $\xi \in \mathbb{R} \setminus \{0\}$, $|\xi| < 1$, $c_\xi = (t-\xi)(t^{-1}-\xi)$.

Remark

This decomposition corresponds to the Jordan block decomposition of the monodromy operator.

Pairings over cyclic modules

Theorem

Every non-degenerate sesquilinear pairing over $\Lambda/b_\xi^k\Lambda$ is equivalent to a pairing

$$(a, b) \mapsto \frac{\epsilon \bar{a} b}{b_\xi^k},$$

where $\epsilon = \pm 1$ and the pairings with different sign are not isometric.

Pairings over cyclic modules

Theorem

Every non-degenerate sesquilinear pairing over $\Lambda/b_\xi^k\Lambda$ is equivalent to a pairing

$$(a, b) \mapsto \frac{\epsilon \bar{a} b}{b_\xi^k},$$

where $\epsilon = \pm 1$ and the pairings with different sign are not isometric.

Theorem

All non-degenerate sesquilinear pairings over $\Lambda/c_\xi^k\Lambda$ are isometric.

Pairings over cyclic modules

Theorem

Every non-degenerate sesquilinear pairing over $\Lambda/b_\xi^k\Lambda$ is equivalent to a pairing

$$(a, b) \mapsto \frac{\epsilon \bar{a} b}{b_\xi^k},$$

where $\epsilon = \pm 1$ and the pairings with different sign are not isometric.

Theorem

All non-degenerate sesquilinear pairings over $\Lambda/c_\xi^k\Lambda$ are isometric.

This reminds of classification of HVS.

A step further. Twisted Alexander polynomials

- Let N be a 3-dimensional manifold

A step further. Twisted Alexander polynomials

- Let N be a 3-dimensional manifold

A step further. Twisted Alexander polynomials

- Let N be a 3-dimensional manifold and $\phi\colon \pi_1(N) \to GL(\mathbb{C}^n[t, t^{-1}])$ be a unitary representation.
- Let $\widetilde{N}$ be the universal cover.

A step further. Twisted Alexander polynomials

- Let N be a 3-dimensional manifold and $\phi\colon \pi_1(N) \to GL(\mathbb{C}^n[t,t^{-1}])$ be a unitary representation.
- Let $\widetilde{N}$ be the universal cover.
- The chain complex $C_*(\widetilde{N};\mathbb{Z})$ has a structure of a $\mathbb{Z}[\pi_1(N)]$ module.

A step further. Twisted Alexander polynomials

- Let N be a 3-dimensional manifold and $\phi\colon \pi_1(N) \to GL(\mathbb{C}^n[t, t^{-1}])$ be a unitary representation.
- Let $\widetilde{N}$ be the universal cover.
- The chain complex $C_*(\widetilde{N}; \mathbb{Z})$ has a structure of a $\mathbb{Z}[\pi_1(N)]$ module.
- Via ϕ, $\mathbb{C}[t, t^{-1}]^n \otimes C_*(\widetilde{N}; \mathbb{Z})$ has a structure of a $\mathbb{C}[t, t^{-1}]$-module.

A step further. Twisted Alexander polynomials

- Let N be a 3-dimensional manifold and $\phi\colon \pi_1(N) \to GL(\mathbb{C}^n[t,t^{-1}])$ be a unitary representation.
- Let $\widetilde{N}$ be the universal cover.
- The chain complex $C_*(\widetilde{N};\mathbb{Z})$ has a structure of a $\mathbb{Z}[\pi_1(N)]$ module.
- Via ϕ, $\mathbb{C}[t,t^{-1}]^n \otimes C_*(\widetilde{N};\mathbb{Z})$ has a structure of a $\mathbb{C}[t,t^{-1}]$-module.
- The homology of this complex is denoted by $H_*(N;\mathbb{C}[t,t^{-1}]^n_\phi)$ and it is called the *twisted homology*.

A step further. Twisted Alexander polynomials

- Let N be a 3-dimensional manifold and $\phi\colon \pi_1(N) \to GL(\mathbb{C}^n[t, t^{-1}])$ be a unitary representation.
- Let $\widetilde{N}$ be the universal cover.
- The chain complex $C_*(\widetilde{N}; \mathbb{Z})$ has a structure of a $\mathbb{Z}[\pi_1(N)]$ module.
- Via ϕ, $\mathbb{C}[t, t^{-1}]^n \otimes C_*(\widetilde{N}; \mathbb{Z})$ has a structure of a $\mathbb{C}[t, t^{-1}]$-module.
- The homology of this complex is denoted by $H_*(N; \mathbb{C}[t, t^{-1}]^n_\phi)$ and it is called the *twisted homology*.
- For a knot K, if $N = M(K)$, the zero-surgery, the order of $H_*(N; \mathbb{C}[t, t^{-1}]^n_\phi)$ is called the *twisted Alexander polynomial*, see Kirk–Livingston.

Twisted Blanchfield pairing

- Miller and Powell defined rigorously twisted Blanchfield pairing for $H_*(N; \mathbb{C}[t, t^{-1}]^n_\phi)$.

Twisted Blanchfield pairing

- Miller and Powell defined rigorously twisted Blanchfield pairing for $H_*(N; \mathbb{C}[t, t^{-1}]^n_\phi)$.
- We can use the 'Hodge' decomposition of the pairing and obtain 'Hodge numbers'.

Twisted Blanchfield pairing

- Miller and Powell defined rigorously twisted Blanchfield pairing for $H_*(N; \mathbb{C}[t, t^{-1}]^n_\phi)$.
- We can use the 'Hodge' decomposition of the pairing and obtain 'Hodge numbers'.
- We work over $\mathbb{C}[t, t^{-1}]$ and not $\mathbb{R}[t, t^{-1}]$ (it is harder!).

Twisted Blanchfield pairing

- Miller and Powell defined rigorously twisted Blanchfield pairing for $H_*(N; \mathbb{C}[t, t^{-1}]^n_\phi)$.
- We can use the 'Hodge' decomposition of the pairing and obtain 'Hodge numbers'.
- We work over $\mathbb{C}[t, t^{-1}]$ and not $\mathbb{R}[t, t^{-1}]$ (it is harder!).
- From the 'Hodge numbers' we can recover 'twisted signatures'.

Twisted Blanchfield pairing

- Miller and Powell defined rigorously twisted Blanchfield pairing for $H_*(N;\mathbb{C}[t,t^{-1}]^n_\phi)$.
- We can use the ‘Hodge’ decomposition of the pairing and obtain ‘Hodge numbers’.
- We work over $\mathbb{C}[t,t^{-1}]$ and not $\mathbb{R}[t,t^{-1}]$ (it is harder!).
- From the ‘Hodge numbers’ we can recover ‘twisted signatures’.

Twisted Blanchfield pairing

- Miller and Powell defined rigorously twisted Blanchfield pairing for $H_*(N; \mathbb{C}[t, t^{-1}]^n_\phi)$.
- We can use the 'Hodge' decomposition of the pairing and obtain 'Hodge numbers'.
- We work over $\mathbb{C}[t, t^{-1}]$ and not $\mathbb{R}[t, t^{-1}]$ (it is harder!).
- From the 'Hodge numbers' we can recover 'twisted signatures'.

Example (—, Conway, Politarczyk 2018)

Using a specific representation of $\pi_1(M(K))$ we can recover Casson–Gordon signatures.

Twisted Blanchfield pairing

- Miller and Powell defined rigorously twisted Blanchfield pairing for $H_*(N; \mathbb{C}[t, t^{-1}]^n_\phi)$.
- We can use the 'Hodge' decomposition of the pairing and obtain 'Hodge numbers'.
- We work over $\mathbb{C}[t, t^{-1}]$ and not $\mathbb{R}[t, t^{-1}]$ (it is harder!).
- From the 'Hodge numbers' we can recover 'twisted signatures'.

Example (—, Conway, Politarczyk 2018)

Using a specific representation of $\pi_1(M(K))$ we can recover Casson–Gordon signatures.

Remark

Using the abstract algebraic approach we obtain a very general cabling formula for twisted Blanchfield pairings, which specifies to the cabling formula of Litherland.

Hedden–Kirk–Livingston knot

- Let
  $K = T(2;15)\#T(2,3;2,13)\#-T(2;13)\#-T(2,3;2,15)$;

Hedden–Kirk–Livingston knot

- Let $K = T(2;15)\#T(2,3;2,13)\# - T(2;13)\# - T(2,3;2,15)$;
- K is algebraically slice, i.e., in particular Tristram–Levine signature is zero a.e.;

Hedden–Kirk–Livingston knot

- Let $K = T(2;15) \# T(2,3;2,13) \# -T(2;13) \# -T(2,3;2,15)$;
- K is algebraically slice, i.e., in particular Tristram–Levine signature is zero a.e.;
- No known invariant from HF or HFI or Kh can detect non-sliceness of K;

Hedden–Kirk–Livingston knot

- Let $K = T(2;15)\#T(2,3;2,13)\# - T(2;13)\# - T(2,3;2,15)$;
- K is algebraically slice, i.e., in particular Tristram–Levine signature is zero a.e.;
- No known invariant from HF or HFI or Kh can detect non-sliceness of K; I haven't checked yet the Alfieri–Kang–Stipsicz invariant. Sorry.

Hedden–Kirk–Livingston knot

- Let $K = T(2;15)\#T(2,3;2,13)\# - T(2;13)\# - T(2,3;2,15)$;
- K is algebraically slice, i.e., in particular Tristram–Levine signature is zero a.e.;
- No known invariant from HF or HFI or Kh can detect non-sliceness of K;
- Hedden, Kirk and Livingston used Casson-Gordon invariants to show non-sliceness. The proof is rather specific;

Hedden–Kirk–Livingston knot

- Let $K = T(2;15) \# T(2,3;2,13) \# - T(2;13) \# - T(2,3;2,15)$;
- K is algebraically slice, i.e., in particular Tristram–Levine signature is zero a.e.;
- No known invariant from HF or HFI or Kh can detect non-sliceness of K;
- Hedden, Kirk and Livingston used Casson-Gordon invariants to show non-sliceness. The proof is rather specific;
- We can algorithmically compute the 'Hodge numbers' related to the Casson-Gordon invariants and show non-sliceness in a simple way.

Remarks

- We do not know if all algebraic knots are linearly independent in the concordance group;

Remarks

- We do not know if all algebraic knots are linearly independent in the concordance group;
- We expect that there are non-trivial combinations for which Casson-Gordon obstruction vanishes;

Remarks

- We do not know if all algebraic knots are linearly independent in the concordance group;
- We expect that there are non-trivial combinations for which Casson-Gordon obstruction vanishes;
- Can we use more general L^2-invariants?

Remarks

- We do not know if all algebraic knots are linearly independent in the concordance group;
- We expect that there are non-trivial combinations for which Casson-Gordon obstruction vanishes;
- Can we use more general L^2-invariants?
- Another question: the spectrum of a plane curve singularity is topological. Can we recover the spectrum of a general hypersurface singularity from some topological data?