Khovanov invariants for knots

Maciej Borodzik

Institute of Mathematics, University of Warsaw

Warsaw, 2018
A knot is a possibly tangled circle in $\mathbb{R}^3$: 

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**Definition**

A *knot* in $\mathbb{R}^3$ is an image of a smooth embedding $\phi: S^1 \to \mathbb{R}^3$. A *link* is “a knot with more than one component”.

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![Knot diagram](image)
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![Diagram of a knot and a link](image-url)
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Khovanov invariants for knots
A knot invariant assigns a simpler (more tractable) object to a knot;
A *knot invariant* assigns a simpler (more tractable) object to a knot; 
Should be the same no matter how the knot is drawn;
Distinguishing knots

- A knot invariant assigns a simpler (more tractable) object to a knot;
- Should be the same no matter how the knot is drawn;
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A knot invariant assigns a simpler (more tractable) object to a knot;
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Should have a meaning;
A knot invariant assigns a simpler (more tractable) object to a knot;

Should be the same no matter how the knot is drawn;

Should be computable;

Should have a meaning;

Should really distinguish knots.
Polynomial invariants

- Assign a polynomial to a knot.
Assign a polynomial to a knot.

- **Alexander polynomial** defined in 1928.
Polynomial invariants

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HOMFLYPT polynomial constructed in 1985 by Hoste, Ocneanu, Millet, Freyd, Yetter, and independently by Przytycki and Traczyk in 1986.
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Alexander and Jones polynomials are polynomials in one variable (formally in $t^{1/2}$ and $t^{-1/2}$, so Laurent polynomials. HOMFLYPT is a two-variable polynomial.
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There are many more polynomial invariants, but these are the most basic. They have a special property.
Definition (Informal)

A skein relation is a relation between the polynomials for links differing at a single place of the diagram.
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Let $A$ be the Alexander polynomial and $J$ be the Jones polynomial.

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Mathematical expressions:

\[ A(L_+ + (t) - A(L_0 - (t)) = (t^{1/2} - t^{-1/2}) A(L_0(t)) \]

\[ t - 1 J(L_+ + (t) - t J(L_0 - (t)) = (t^{1/2} - t^{-1/2}) J(L_0(t)) \]

Remark: There are various normalizations of the Alexander and Jones polynomials, which lead to different looking formulas.
Let $A$ be the Alexander polynomial and $J$ be the Jones polynomial.

- We have: $A_{L^+}(t) - A_{L^-}(t) = (t^{1/2} - t^{-1/2})A_{L_0}(t)$. 

![Diagram of knots](image)
Let $A$ be the Alexander polynomial and $J$ be the Jones polynomial.

- We have: $A_{L_+}(t) - A_{L_-}(t) = (t^{1/2} - t^{-1/2})A_{L_0}(t)$.

- For Jones: $t^{-1}J_{L_+}(t) - tJ_{L_-}(t) = (t^{1/2} - t^{-1/2})J_{L_0}(t)$. 

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Khovanov invariants for knots
### Alexander polynomial

$\Delta_K(t) = \pm 1$ for knots

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We know very little beyond combinatorics
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**Note:**
- $\Delta_K(t)$ is the Alexander polynomial.
- $J$ is the Jones polynomial.
- $A(t)$ is another polynomial related to knots.
- The Alexander polynomial's condition is a general property that holds for all knots.
- The Jones polynomial's condition is specific to knots determined on roots of unity of certain orders.
- The existence of knots satisfying certain polynomial conditions is an open question.
- The topological meaning of these invariants is understood to a limited extent.
- Computationally, these invariants can be computed efficiently.
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We specify *resolutions* of a knot diagram.
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Take a knot.
We specify *resolutions* of a knot diagram.

Take a knot. Enumerate its crossings.
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0-resolution of the first crossing.
We specify *resolutions* of a knot diagram.

Take a knot. Enumerate its crossings.

1-resolution of the first crossing.
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Take a knot. Enumerate its crossings.

0-resolution of the second crossing.
We specify *resolutions* of a knot diagram.

Take a knot. Enumerate its crossings.

010 resolution.

Any triple \( \{0, 1\} \) gives a resolution.
We specify *resolutions* of a knot diagram.

Take a knot. Enumerate its crossings.

010 resolution.

Any triple \( \{0, 1\}^3 \) gives a resolution.
Cube of resolution

000

001
2 circles

010
2 circles

100
2 circles

110
1 circle

111
2 circles

011
1 circle

101
1 circle

111
2 circles
Cube of resolution

000
3 circles

001
2 circles

010
2 circles

011
1 circle

100
2 circles

101
1 circle

110
1 circle

111
2 circles

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Cube of resolution

000 \( (q + q^{-1})^3 \)
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110 \( (q + q^{-1})^2 \)
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111 \( (q + q^{-1})^2 \)

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Khovanov invariants for knots
Cube of resolution

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Cube of resolution

\[(q + q^{-1})^3\]  \(3(q + q^{-1})^2\)  \(3(q + q^{-1})\)  \((q + q^{-1})^2\)
Cube of resolution

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Khovanov invariants for knots
Cube of resolution

\[ (q + q^{-1})^3 q^0 - 3(q + q^{-1})^2 q^1 + 3(q + q^{-1})q^2 - (q + q^{-1})^2 q^3 \]
We have

$$(q^{-1} + q)^3 - 3q(q^{-1} + q)^2 + 3q^2(q^{-1} + q) - q^3(q^{-1} + q) =$$

$$- q^6(q^{-2} - q^{-3} + q^{-4} - q^{-9})$$
We have

\[(q^{-1} + q)^3 - 3q(q^{-1} + q)^2 + 3q^2(q^{-1} + q) - q^3(q^{-1} + q) =
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\[- q^6(q^{-2} - q^{-3} + q^{-4} - q^{-9})\]

In this way we obtain the Jones polynomial for the (negative) trefoil. Factor \(-q^{-6}\) is a normalization.
Khovanov’s approach
Khovanov’s approach

Main Idea

Replace factor $q + q^{-1}$ in the cube of resolution by a two-dimensional vector space $V$. 
Khovanov’s approach

Khovanov invariants for knots

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1 circle

1 circle

2 circles

2 circles

3 circles

2 circles

1 circle

2 circles

1 circle

2 circles

1 circle

1 circle

2 circles

1 circle

2 circles
Khovanov’s approach

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Khovanov invariants for knots
**Explanation**

The meaning of $V^3$ is the tensor product. An element in $V^3$ is a linear combination of triples $(a, b, c)$ (written usually $a \otimes b \otimes c$).

We have $a_1 \otimes b \otimes c + a_2 \otimes b \otimes c = (a_1 + a_2) \otimes b \otimes c$, but not $a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 = (a_1 + a_2) \otimes (b_1 + b_2) \otimes (c_1 + c_2)$.

$\dim V^\otimes 3 = (\dim V)^3$ and not $3 \dim V$!
Khovanov’s approach

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Khovanov invariants for knots
Khovanov’s approach

$V \otimes^3 V \otimes^2 V \oplus V \otimes^2 V \oplus V \otimes^2 V \oplus V \oplus V \oplus V \oplus V \otimes^2$
Khovanov’s approach

\[ V^3 \oplus V^2 \oplus V^2 \oplus V^2 \oplus V \oplus V \oplus V \oplus V \]
Khovanov’s approach

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An arrow can either merge two circles into one.
Maps in Khovanov’s approach

- An arrow can either merge two circles into one.
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In the first case we need a map $V \otimes V \rightarrow V$. 
An arrow can either merge two circles into one.
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In the second case we need a map $V \to V \otimes V$. 
Maps in Khovanov’s approach

- An arrow can either merge two circles into one.
- Or split one into two circles.
- In the first case we need a map $V \otimes V \to V$.
- In the second case we need a map $V \to V \otimes V$.
- Without extra structure, it is hard to define such maps consistently.
Think of $V$ as a space of affine functions $ax + b$ with $a, b \in \mathbb{Z}$. 
Maps in Khovanov homology

Think of $V$ as a space of affine functions $ax + b$ with $a, b \in \mathbb{Z}$.

The map $V \otimes V \rightarrow V$ is the linear part of the product:

$1 \otimes 1 \mapsto 1$, $x \otimes 1 \mapsto x$, $1 \otimes x \mapsto x$, $x \otimes x \mapsto 0$. 
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Combining these maps (and after some sign adjustments) we obtain maps replacing $+$ and $-$ signs.
Global maps. Revised

\[ V^3 \xrightarrow{d_0} V^2 \oplus V^2 \oplus V^2 d_1 \xrightarrow{d_2} V^2 \]
Theorem (Khovanov 2000)

The maps $d_0$, $d_1$ and $d_2$ satisfy $d_2 \circ d_1 = 0$ and $d_1 \circ d_0 = 0$. The abelian groups $\ker d_i / \im d_{i-1}$ are independent of the knot diagram.
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Remark

In mathematics, a sequence of vector spaces $V_0, \ldots, V_s$ together with linear maps $d_i : V_i \rightarrow V_{i+1}$ satisfying $d_i \circ d_{i-1} = 0$ for all $i$ is called a cochain complex. The groups $\ker d_i / \im d_{i-1}$ are called cohomology groups.
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Yes, I know, saying ‘a vector space over $\mathbb{Z}$’ is an abuse.
Properties of Khovanov invariant

- Detects the unknot (Kronheimer, Mrowka 2011).
- Detects the Hopf link and the trefoil.
- Specifies to and generalizes the Jones polynomial.
- Can be used to prove the Milnor’s conjecture (on the unknotting number of torus knots).
- Computational complexity is daunting.
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- Can be used to prove the Milnor’s conjecture (on the unknotting number of torus knots).
- Computational complexity is daunting.
We said that Khovanov invariant is cohomology of a chain complex.
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Many people are familiar with cohomology of topological spaces.
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**Question**

*Given a knot $K$ can one construct a topological space $X$ such that the cohomology of $X$ is the Khovanov invariant of $K$? Is there a consistent construction?*
First construction of Khovanov homotopy type using flow categories and Cohen-Jones-Segal (2012).
- New invariants of knots coming from cohomological operations (2013).
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New invariants of knots coming from cohomological operations (2013).

Another construction of flow categories using cubical flow categories and Burnside categories (2014, jointly with Lawson).
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Invited to the ICM in 2018.
A knot is $p$-periodic if it admits a diagram invariant under rotation by $\mathbb{Z}/p$. 

Periodic knots
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Which knots are $p$-periodic?
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Equivariant Khovanov invariants

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- Borodzik, Politarczyk 2018: Another, much stronger, periodicity criterion based on equivariant Khovanov invariants.
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**Question**

*Does there exists equivariant Khovanov homotopy type?*
Equivariant Khovanov homotopy type

Theorem (B. — Politarczyk — Silvero 2018, Stoffregen — Zhang 2018)

There exists equivariant Khovanov homotopy type.
Theorem (B. — Politarczyk — Silvero 2018, Stoffregen — Zhang 2018)

There exists equivariant Khovanov homotopy type.

BPS approach proves also that equivariant cohomology of this space is Politarczyk’s equivariant Khovanov invariant.
Construct HOMLYPT homotopy type;
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Construct a homotopy type that reflects and intertwines the quantum grading.
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Understand, why Khovanov invariants work.
Perspectives

- Construct HOMLYPT homotopy type;
- Construct a homotopy type that reflects and intertwines the quantum grading.
- Understand, why Khovanov invariants work.
- Find a simpler way to calculate Khovanov invariants.