Heegaard Floer homologies and rational cuspidal curves

joint with Ch. Livingston

Maciej Borodzik
www.mimuw.edu.pl/~mcboro

Institute of Mathematics, University of Warsaw

Edinburgh, September 2013

Semigroups of singular points

- For a singularity $x^p - y^q = 0$ with p, q coprime, the semigroup is generated by p and q.

Semigroups of singular points

- For a singularity $x^p - y^q = 0$ with p, q coprime, the semigroup is generated by p and q.
- If $p = 4$, $q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \dots)$.

Semigroups of singular points

- For a singularity $x^p - y^q = 0$ with p, q coprime, the semigroup is generated by p and q.
- If $p = 4$, $q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \ldots)$.
- The gap sequence is $G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$.

Semigroups of singular points

- For a singularity $x^p - y^q = 0$ with p, q coprime, the semigroup is generated by p and q.
- If $p = 4$, $q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \dots)$.
- The gap sequence is $G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$.
- We have $\#G_{4,7} = \mu/2$ and $\max\{x \in G_{4,7}\} = 17 = \mu - 1$.

Semigroups of singular points

- For a singularity $x^p - y^q = 0$ with p, q coprime, the semigroup is generated by p and q.
- If $p = 4$, $q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \dots)$.
- The gap sequence is $G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$.
- We have $\#G_{4,7} = \mu/2$ and $\max\{x \in G_{4,7}\} = 17 = \mu - 1$. this is a special property of semigroups of singular points!

Semigroups of singular points

- For a singularity $x^p - y^q = 0$ with p, q coprime, the semigroup is generated by p and q.
- If $p = 4$, $q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \dots)$.
- The gap sequence is $G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$.
- We have $\# G_{4,7} = \mu/2$ and $\max\{x \in G_{4,7}\} = 17 = \mu - 1$.
- The gap function is defined as

$$I(m) := \#\{x \in \mathbb{Z},\ x \geq m,\ x \notin S_{4,7}\}.$$

Semigroups of singular points

- For a singularity $x^p - y^q = 0$ with p, q coprime, the semigroup is generated by p and q.
- If $p = 4$, $q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \dots)$.
- The gap sequence is $G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$.
- We have $\#G_{4,7} = \mu/2$ and $\max\{x \in G_{4,7}\} = 17 = \mu - 1$.
- The gap function is defined as

$$I(m) := \#\{x \in \mathbb{Z},\ x \geq m,\ x \notin S_{4,7}\}.$$

- We have

$$I_{4,7}(5) = \#\{5, 6, 9, 10, 13, 17\} = 6.$$

Semigroups of singular points

- For a singularity $x^p - y^q = 0$ with p, q coprime, the semigroup is generated by p and q.
- If $p = 4$, $q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \dots)$.
- The gap sequence is $G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$.
- We have $\# G_{4,7} = \mu/2$ and $\max\{x \in G_{4,7}\} = 17 = \mu - 1$.
- The gap function is defined as

$$I(m) := \#\{x \in \mathbb{Z},\ x \geq m,\ x \notin S_{4,7}\}.$$

- We have

$$I_{4,7}(5) = \#\{5, 6, 9, 10, 13, 17\} = 6.$$

- Always $I(0) = \mu/2$, $I(x) = 0$ for $x \geq \mu$ and $I(-n) = n + \mu/2$ for $n > 0$.

The Alexander polynomial

For a semigroup S with a gap sequence G we define

The Alexander polynomial

For a semigroup S with a gap sequence G we define

$$\Delta_S(t) = 1 + (t-1)\sum_{j\in G} t^j.$$

The Alexander polynomial

For a semigroup S with a gap sequence G we define

$$\Delta_S(t) = 1 + (t-1)\sum_{j \in G} t^j.$$

For the semigroup $S_{4,7}$, the gap sequence is $\{1, 2, 3, 5, 6, 9, 10, 13, 17\}$, so we have

The Alexander polynomial

For a semigroup S with a gap sequence G we define

$$\Delta_S(t) = 1 + (t-1)\sum_{j\in G} t^j.$$

For the semigroup $S_{4,7}$, the gap sequence is $\{1, 2, 3, 5, 6, 9, 10, 13, 17\}$, so we have

$$\Delta_{4,7}(t) = 1 + (t-1)\left(t + t^2 + t^3 + t^5 + t^6 + t^9 + t^{10} + t^{13} + t^{17}\right)$$

The Alexander polynomial

For a semigroup S with a gap sequence G we define

$$\Delta_S(t) = 1 + (t-1)\sum_{j \in G} t^j.$$

For the semigroup $S_{4,7}$, the gap sequence is $\{1, 2, 3, 5, 6, 9, 10, 13, 17\}$, so we have

$$\Delta_{4,7}(t) = 1 + (t-1)\left(t + t^2 + t^3 + t^5 + t^6 + t^9 + t^{10} + t^{13} + t^{17}\right)$$

or:

$$\Delta_{4,7} = 1 - t + t^4 - t^5 + t^7 - t^9 + t^{11} - t^{13} + t^{14} - t^{17} + t^{18}.$$

The Alexander polynomial

For a semigroup S with a gap sequence G we define

$$\Delta_S(t) = 1 + (t-1)\sum_{j \in G} t^j.$$

For the semigroup $S_{4,7}$, the gap sequence is $\{1, 2, 3, 5, 6, 9, 10, 13, 17\}$, so we have

$$\Delta_{4,7}(t) = 1 + (t-1)\left(t + t^2 + t^3 + t^5 + t^6 + t^9 + t^{10} + t^{13} + t^{17}\right)$$

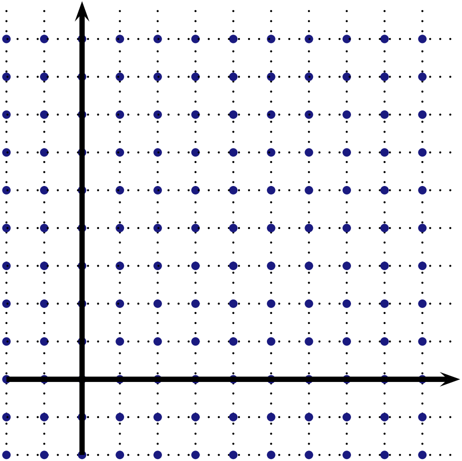
or:

$$\Delta_{4,7} = 1 - t + t^4 - t^5 + t^7 - t^9 + t^{11} - t^{13} + t^{14} - t^{17} + t^{18}.$$

This is the Alexander polynomial of the knot of the singularity.

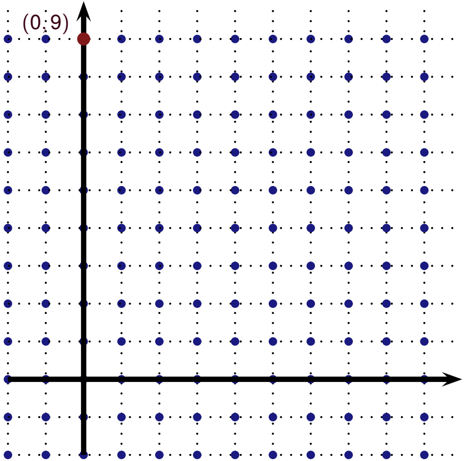
The staircase

$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^{9} + t^{7} - t^{5} + t^{4} - t + 1.$$



The staircase

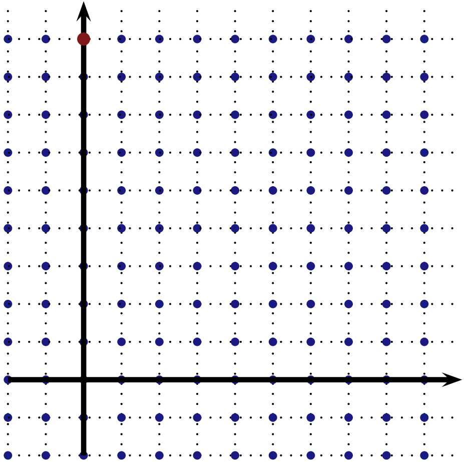
$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^{9} + t^{7} - t^{5} + t^{4} - t + 1.$$



- $9 = 18/2$

The staircase

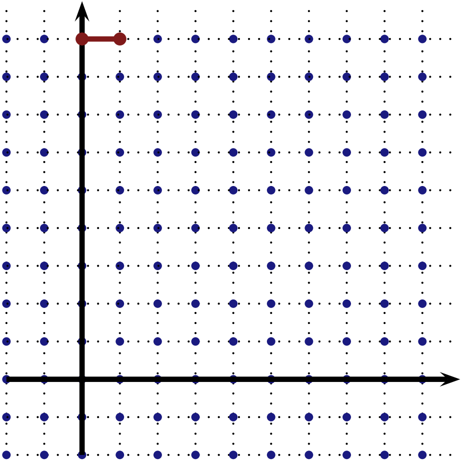
$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1.$$



- $9 = g(T_{4,7})$

The staircase

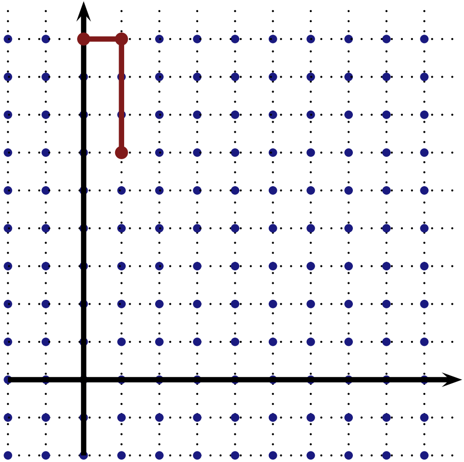
$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1.$$



- $9 = g(T_{4,7})$
- $18 - 17 = 1$

The staircase

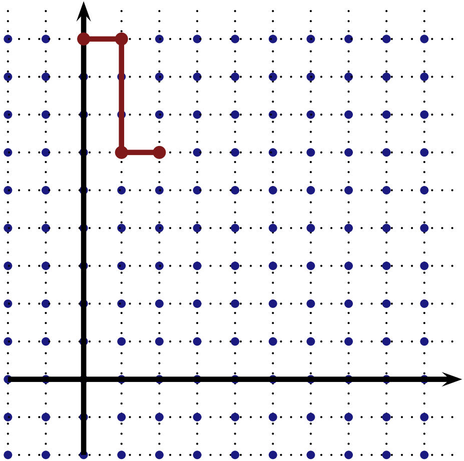
$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1.$$



- $9 = g(T_{4,7})$
- $18 - 17 = 1$
- $17 - 14 = 3$

The staircase

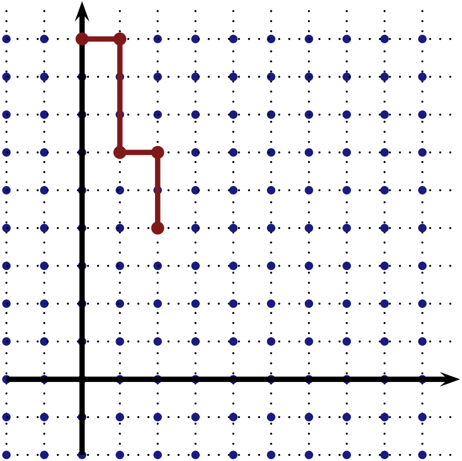
$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^{9} + t^{7} - t^{5} + t^{4} - t + 1.$$



- $9 = g(T_{4,7})$
- $18 - 17 = 1$
- $17 - 14 = 3$
- $14 - 13 = 1$

The staircase

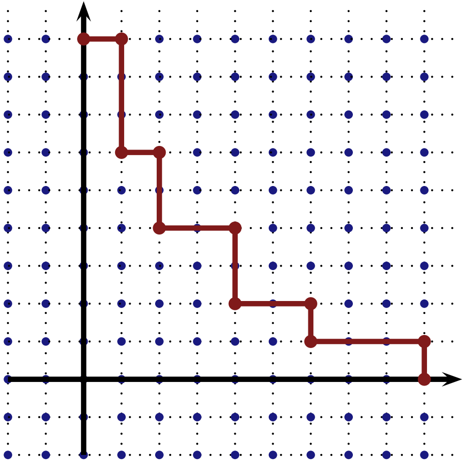
$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^{9} + t^{7} - t^{5} + t^{4} - t + 1.$$



- $9 = g(T_{4,7})$
- $18 - 17 = 1$
- $17 - 14 = 3$
- $14 - 13 = 1$
- $13 - 11 = 2$

The staircase

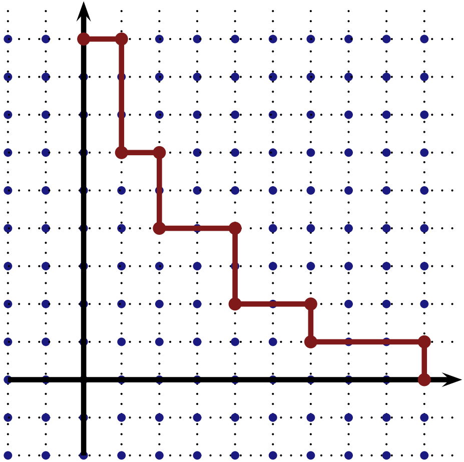
$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^{9} + t^{7} - t^{5} + t^{4} - t + 1.$$



- $9 = g(T_{4,7})$
- $18 - 17 = 1$
- $17 - 14 = 3$
- $14 - 13 = 1$
- $13 - 11 = 2$
- …and so on

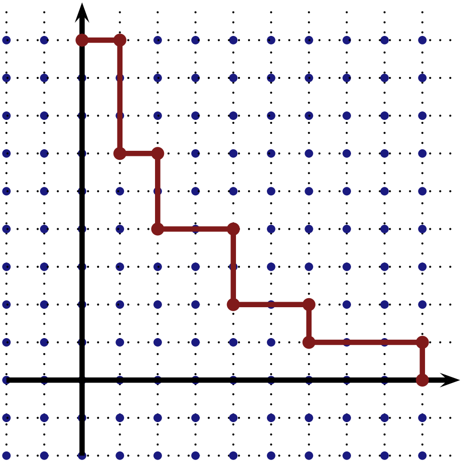
The staircase

$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^{9} + t^{7} - t^{5} + t^{4} - t + 1.$$

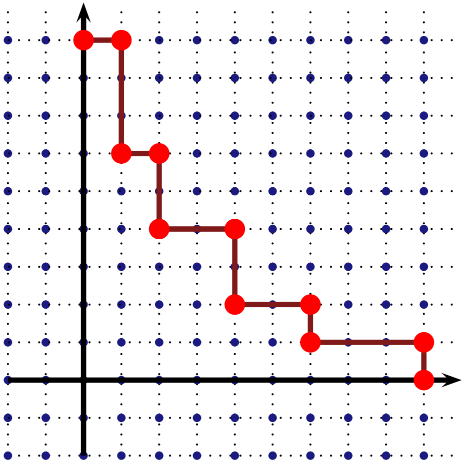


- $9 = g(T_{4,7})$
- $18 - 17 = 1$
- $17 - 14 = 3$
- $14 - 13 = 1$
- $13 - 11 = 2$
- ...and so on
- Symmetry reflects symmetry of Δ

The staircase complex

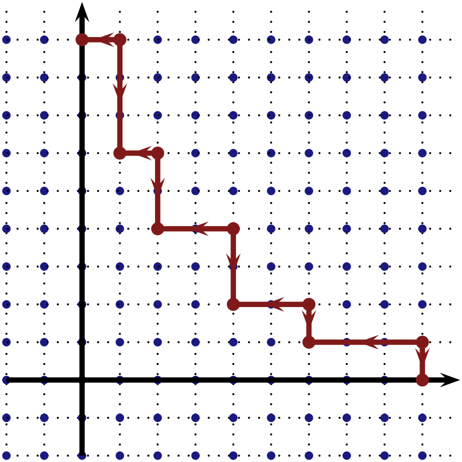


The staircase complex



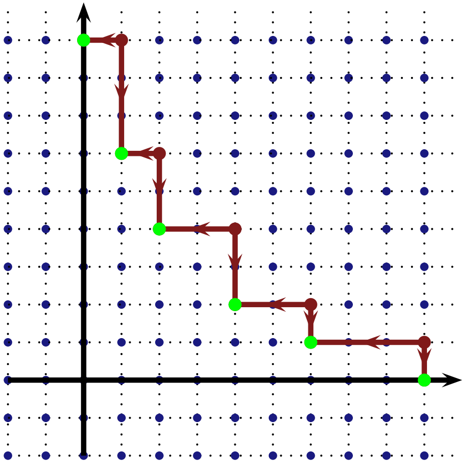
- Place $\mathbb{Z}_2$ for each vertex.

The staircase complex



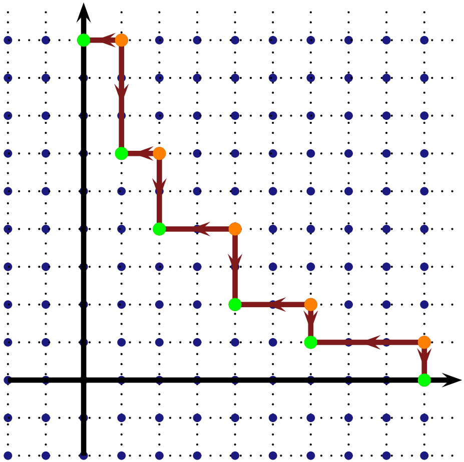
- Place $\mathbb{Z}_2$ for each vertex.
- Differential is given by lines as depicted.

The staircase complex



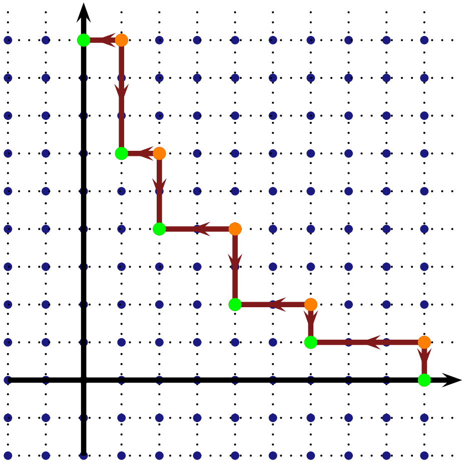
- Place $\mathbb{Z}_2$ for each vertex.
- Differential is given by lines as depicted.
- Type A vertices.

The staircase complex



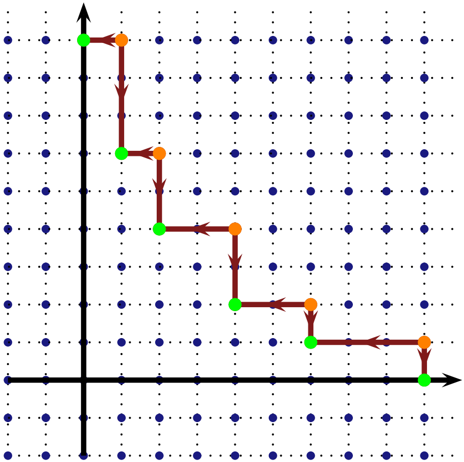
- Place $\mathbb{Z}_2$ for each vertex.
- Differential is given by lines as depicted.
- Type A vertices.
- Type B vertices.

The staircase complex



- Place $\mathbb{Z}_2$ for each vertex.
- Differential is given by lines as depicted.
- Type A vertices.
- Type B vertices.
- Bifiltration is given by coordinates.

The staircase complex



- Place $\mathbb{Z}_2$ for each vertex.
- Differential is given by lines as depicted.
- Type A vertices.
- Type B vertices.
- Bifiltration is given by coordinates.
- Absolute grading of a type A vertex is 0, of type B is 1.

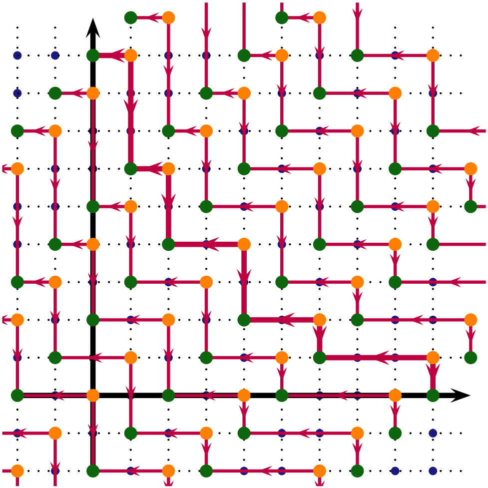
Now there comes something really scary

We will tensor the staircase complex by $\mathbb{Z}_2[U, U^{-1}]$.

Now there comes something really scary

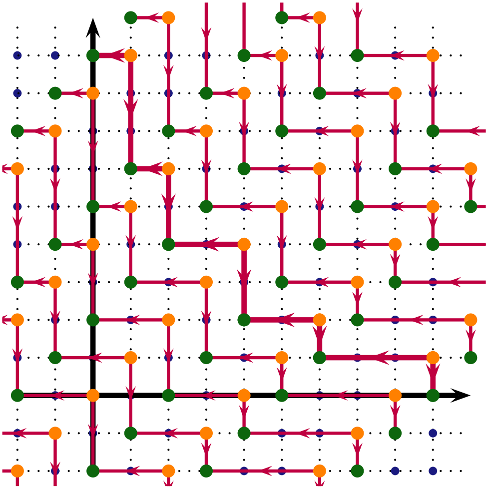
We will tensor the staircase complex by $\mathbb{Z}_2[U, U^{-1}]$.
Are you ready for the challenge?

Tensoring



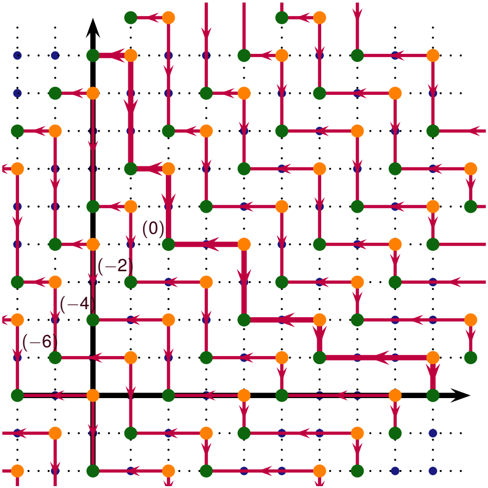
- Tensor $\mathrm{St}(K)$ by $\mathbb{Z}_2[U, U^{-1}]$.

Tensoring



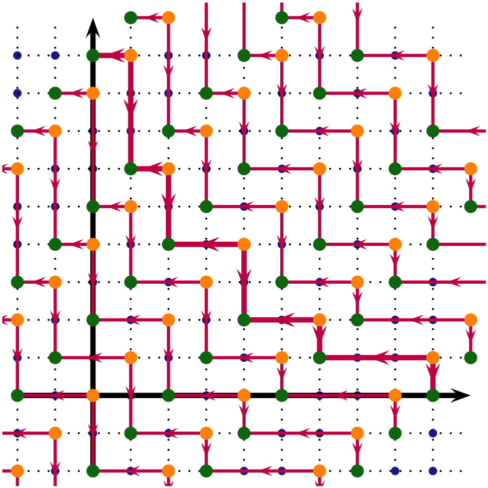
- Tensor St(K) by $\mathbb{Z}_2[U, U^{-1}]$.
- U changes the filtration level by $(-1, -1)$ and the absolute grading by -2.

Tensoring



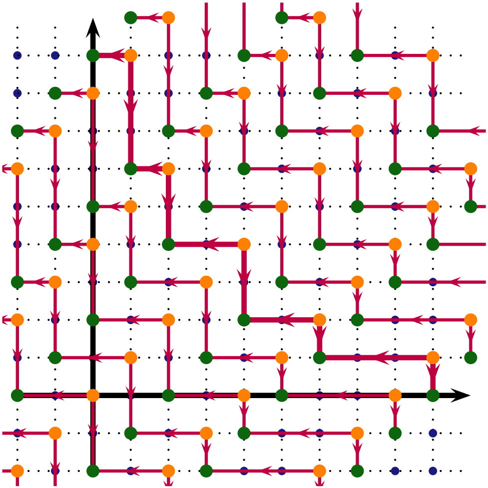
- Tensor St(K) by $\mathbb{Z}_2[U, U^{-1}]$.
- U changes the filtration level by $(-1, -1)$ and the absolute grading by -2.

Tensoring



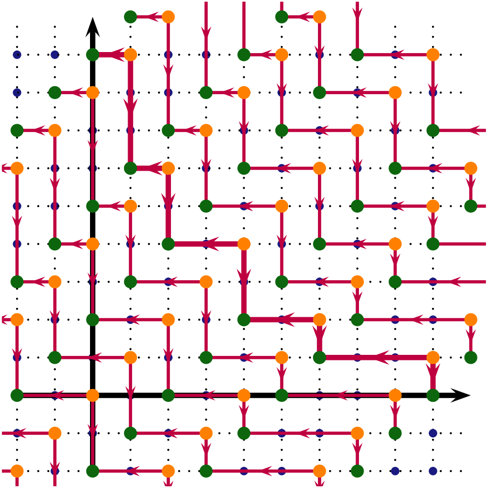
- Tensor $\mathrm{St}(K)$ by $\mathbb{Z}_2[U, U^{-1}]$.
- U changes the filtration level by $(-1, -1)$ and the absolute grading by -2.
- The resulting complex is $CFK^\infty(K)$ if K is an algebraic knot.

Tensoring



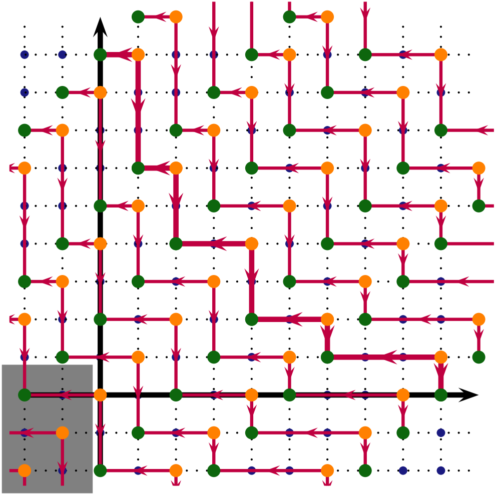
- Tensor St(K) by $\mathbb{Z}_2[U, U^{-1}]$.
- U changes the filtration level by $(-1, -1)$ and the absolute grading by -2.
- The resulting complex is $CFK^\infty(K)$ if K is an algebraic knot.
- Actually, it is enough that K is so called an L–space knot.

The function $J(m)$



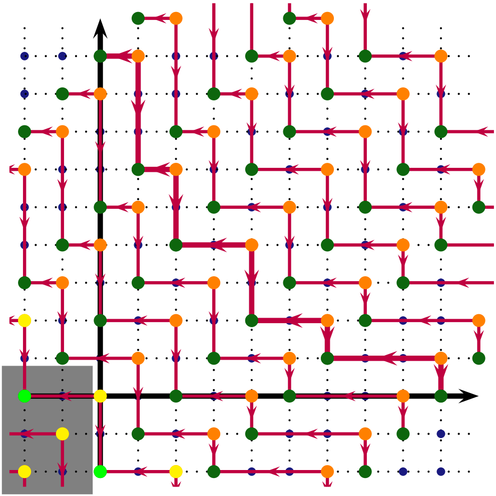
- $m \in \mathbb{Z}$. Here $m = 1$.

The function $J(m)$



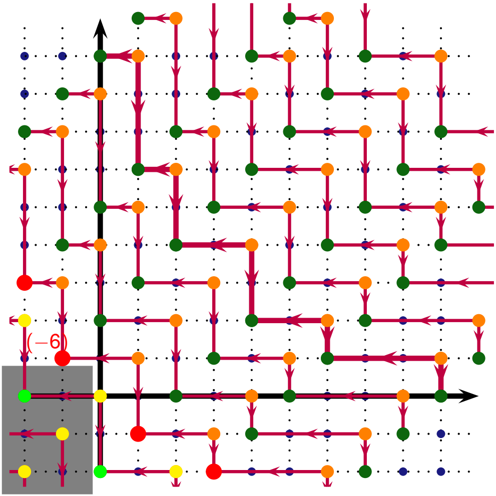
- $m \in \mathbb{Z}$. Here $m = 1$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient C_+.

The function $J(m)$



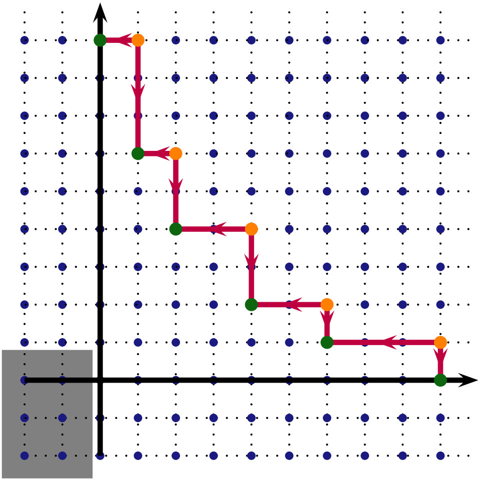
- $m \in \mathbb{Z}$. Here $m = 1$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient C_+.
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.

The function $J(m)$



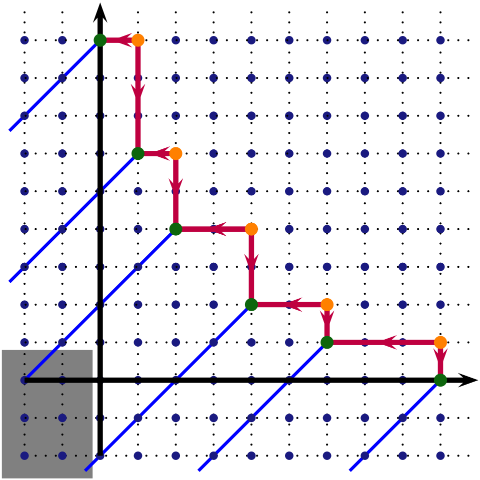
- $m \in \mathbb{Z}$. Here $m = 1$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient C_+.
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.

The function $J(m)$



- $m \in \mathbb{Z}$. Here $m = 1$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient C_+.
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.

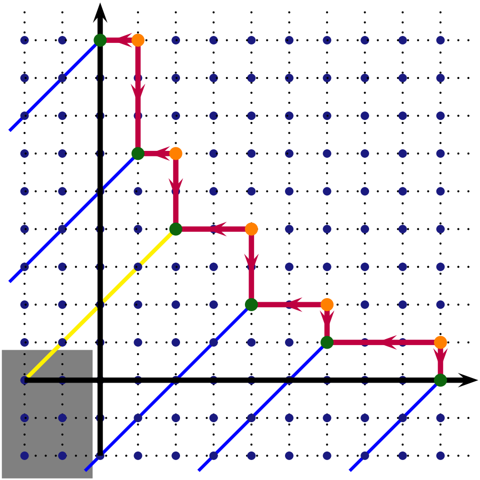
The function $J(m)$



- $m \in \mathbb{Z}$. Here $m = 1$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient C_+.
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.

$$J(m) = \min_{(v_1, v_2) \in \mathrm{Vert}_A} \max(v_1, v_2 - m)$$

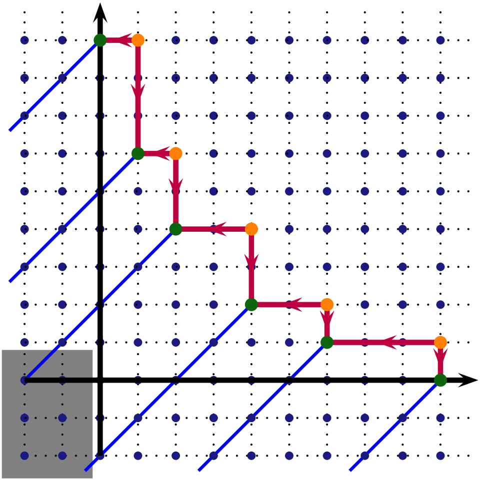
The function $J(m)$



- $m \in \mathbb{Z}$. Here $m = 1$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient C_+.
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.

$$J(m) = \min_{(v_1,v_2)\in \mathrm{Vert}_A} \max(v_1, v_2 - m)$$

The function $J(m)$



$$J(m) = \min_{(v_1,v_2)\in \mathrm{Vert}_A} \max(v_1, v_2 - m)$$

- $m \in \mathbb{Z}$. Here $m = 1$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient C_+.
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.
- $J(m) = I(m - g)$.

Question

Now you may start wondering:

Question

Now you may start wondering:

Oh where, oh where has the true mathematics gone?

d-invariants of Ozsváth and Szabó

Proposition

Let K be an L–space knot.

d-invariants of Ozsváth and Szabó

Proposition

Let K be an algebraic knot.

d-invariants of Ozsváth and Szabó

Proposition

Let K be an L–space knot. Let $q > 2g(K)$ and $m \in [-q/2, q/2]$. Then

$$d(S^3_q(K), \mathfrak{s}_m) = \frac{(q-2m)^2 - q}{4q} - 2J(m).$$

d-invariants of Ozsváth and Szabó

Proposition

Let K be an L–space knot. Let $q > 2g(K)$ and $m \in [-q/2, q/2]$. Then

$$d(S^3_q(K), \mathfrak{s}_m) = \frac{(q-2m)^2 - q}{4q} - 2I(m+g).$$

Theorem

If M^3 bounds a smooth negative definite manifold W^4 and $\mathfrak{s}$ is a spin^c structure on M^3, that is a restriction of a spin^c structure $\mathfrak{t}$ on W, then

$$d(M, \mathfrak{s}) \geq \frac{c_1^2(\mathfrak{t}) - 2\chi(W) - 3\sigma(W)}{4}.$$

Applications. The FLMN conjecture.

- C degree d rational cuspidal curve in $\mathbb{C}P^2$.

Applications. The FLMN conjecture.

- C degree d rational cuspidal curve in $\mathbb{C}P^2$.
- N neighbourhood of C, $M = \partial N$, $W = \mathbb{C}P^2 \setminus N$. W is rational homology ball.

Applications. The FLMN conjecture.

- C degree d rational cuspidal curve in $\mathbb{C}P^2$.
- N neighbourhood of C, $M = \partial N$, $W = \mathbb{C}P^2 \setminus N$. W is rational homology ball.
- Then $M = S^3_{d^2}(K)$, K is connected sum of links of singularities.

Applications. The FLMN conjecture.

- C degree d rational cuspidal curve in $\mathbb{C}P^2$.
- N neighbourhood of C, $M = \partial N$, $W = \mathbb{C}P^2 \setminus N$. W is rational homology ball.
- Then $M = S^3_{d^2}(K)$, K is connected sum of links of singularities. Suppose that C is unicuspidal.

Applications. The FLMN conjecture.

- C degree d rational cuspidal curve in $\mathbb{C}P^2$.
- N neighbourhood of C, $M = \partial N$, $W = \mathbb{C}P^2 \setminus N$. W is rational homology ball.
- Then $M = S^3_{d^2}(K)$, K is connected sum of links of singularities. Suppose that C is unicuspidal.
- The *spin^c* structures that extend over W are those with $m = kd$ for d odd or $kd/2$ for d even, $k \in \mathbb{Z}$.

Applications. The FLMN conjecture.

- C degree d rational cuspidal curve in $\mathbb{C}P^2$.
- N neighbourhood of C, $M = \partial N$, $W = \mathbb{C}P^2 \setminus N$. W is rational homology ball.
- Then $M = S^3_{d^2}(K)$, K is connected sum of links of singularities. Suppose that C is unicuspidal.
- The *spin*^{c} structures that extend over W are those with $m = kd$ for d odd or $kd/2$ for d even, $k \in \mathbb{Z}$.
- By Ozsváth and Szabó, d–invariant must vanish for $\mathfrak{s}_m$.

Theorem (—,Livingston, 2013)

If I is the gap function associated with the single singular point, $j = 0, \dots, d-3$. Then

$$I(jd+1) = \frac{(d-j-1)(d-j-2)}{2}.$$

Applications. The FLMN conjecture.

- C degree d rational cuspidal curve in $\mathbb{C}P^2$.
- N neighbourhood of C, $M = \partial N$, $W = \mathbb{C}P^2 \setminus N$. W is rational homology ball.
- Then $M = S^3_{d^2}(K)$, K is connected sum of links of singularities. Suppose that C is unicuspidal.
- The *spin*^{c} structures that extend over W are those with $m = kd$ for d odd or $kd/2$ for d even, $k \in \mathbb{Z}$.
- By Ozsváth and Szabó, d–invariant must vanish for $\mathfrak{s}_m$.

Theorem (—,Livingston, 2013)

If I is the gap function associated with the single singular point, $j = 0, \dots, d-3$. Then

$$I(jd+1) = \frac{(d-j-1)(d-j-2)}{2}.$$

Generalizations apply for many singular points.

Applications. Semigroup semicontinuity.

- K_1, K_2 two knots. Suppose there is a PSI cobordism from K_1 to K_2 with k double points.

Applications. Semigroup semicontinuity.

- K_1, K_2 two knots. Suppose there is a PSI cobordism from K_1 to K_2 with k double points.
- W_q is a cobordism between $S^3_q(K_1)$ and $S^3_{q+4k}(K_2)$.

Applications. Semigroup semicontinuity.

- K_1, K_2 two knots. Suppose there is a PSI cobordism from K_1 to K_2 with k double points.
- W_q is a cobordism between $S^3_q(K_1)$ and $S^3_{q+4k}(K_2)$.

Theorem (—,Livingston 2013)

If K_1 and K_2 are two L–space knots, g_1 and g_2 their genera, I_1 and I_2 gap functions, then for any $m \in \mathbb{Z}$:

$$I_2(m + g_2 + k) \leq I_1(m + g_1).$$

Applications. Semigroup semicontinuity.

- K_1, K_2 two knots. Suppose there is a PSI cobordism from K_1 to K_2 with k double points.
- W_q is a cobordism between $S^3_q(K_1)$ and $S^3_{q+4k}(K_2)$.

Theorem (—,Livingston 2013)

If K_1 and K_2 are two L–space knots, g_1 and g_2 their genera, I_1 and I_2 gap functions, then for any $m \in \mathbb{Z}$:

$$I_2(m + g_2 + k) \leq I_1(m + g_1).$$

Example

Set K_1 unknot, $K_2 = T_{p,q}$, $m = 0$, $k = g_2 - 1$. Then $I_2(2g_2 - 1) = 1 \not\leq I_1(g_1) = 0$.

Applications. Semigroup semicontinuity.

- K_1, K_2 two knots. Suppose there is a PSI cobordism from K_1 to K_2 with k double points.
- W_q is a cobordism between $S^3_q(K_1)$ and $S^3_{q+4k}(K_2)$.

Theorem (—,Livingston 2013)

If K_1 and K_2 are two L–space knots, g_1 and g_2 their genera, I_1 and I_2 gap functions, then for any $m \in \mathbb{Z}$:

$$I_2(m + g_2 + k) \leq I_1(m + g_1).$$

Example

Set K_1 unknot, $K_2 = T_{p,q}$, $m = 0$, $k = g_2 - 1$. Then $I_2(2g_2 - 1) = 1 \not\leq I_1(g_1) = 0$.

This detects the unknotting number of torus knots.

Applications. Semigroup semicontinuity.

- In the case of δ–constant deformation, $k = g_2 - g_1$.

Applications. Semigroup semicontinuity.

- In the case of δ–constant deformation, $k = g_2 - g_1$.
- We get $\#\Gamma_2 \cap [0, m) \leq \#\Gamma_1 \cap [0, m)$. Semigroup semicontinuity property.

Applications. Semigroup semicontinuity.

- In the case of δ–constant deformation, $k = g_2 - g_1$.
- We get $\#\Gamma_2 \cap [0, m) \leq \#\Gamma_1 \cap [0, m)$. Semigroup semicontinuity property.
- First obtained by Gorsky–Némethi in January 2013 (without δ–constant assumption).

Applications. Semigroup semicontinuity.

- In the case of δ–constant deformation, $k = g_2 - g_1$.
- We get $\#\Gamma_2 \cap [0, m) \leq \#\Gamma_1 \cap [0, m)$. Semigroup semicontinuity property.
- First obtained by Gorsky–Némethi in January 2013 (without δ–constant assumption).
- Another proof by Javier Fernandez de Bobadilla.

Applications. Semigroup semicontinuity.

- In the case of δ–constant deformation, $k = g_2 - g_1$.
- We get $\#\Gamma_2 \cap [0, m) \leq \#\Gamma_1 \cap [0, m)$. Semigroup semicontinuity property.
- First obtained by Gorsky–Némethi in January 2013 (without δ–constant assumption).
- Another proof by Javier Fernandez de Bobadilla.
- In general weak, but it uses smooth structure, unlike semicontinuity of spectrum.

Applications. Semigroup semicontinuity.

- In the case of δ–constant deformation, $k = g_2 - g_1$.
- We get $\#\Gamma_2 \cap [0, m) \leq \#\Gamma_1 \cap [0, m)$. Semigroup semicontinuity property.
- First obtained by Gorsky–Némethi in January 2013 (without δ–constant assumption).
- Another proof by Javier Fernandez de Bobadilla.
- In general weak, but it uses smooth structure, unlike semicontinuity of spectrum.
- $(6; 7)$ cannot be perturbed to $(4; 9)$, even though the spectrum allows it.

. . . you must have been waiting long time for this slide.

. . . you must have been waiting long time for this slide.

Thank you!

. . . you must have been waiting long time for this slide.

Thank you!

... you must have been waiting long time for this slide.

Thank you!

. . . you must have been waiting long time for this slide.

Thank you!

. . . you must have been waiting long time for this slide.

Thank you!

Perspectives

- Generalize for curves with higher genus (joint project with Ch. Livingston).
- Generalize for curves in Hirzebruch surfaces (joint project with K. Moe).
- Relate staircases to lattice homology by András Némethi.
- Can one classify all the rational unicuspidal curves in $\mathbb{C}P^2$? For many cusps other tools are more useful.