Heegaard Floer homologies and rational cuspidal curves
joint with Ch. Livingston

Maciej Borodzik
www.mimuw.edu.pl/~mcboro

Institute of Mathematics, University of Warsaw

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For a singularity $x^p - y^q = 0$ with $p, q$ coprime, the semigroup is generated by $p$ and $q$. 

If $p = 4$, $q = 7$, the semigroup is $S_{4,7} = \langle 0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \ldots \rangle$. 

The gap sequence is $G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$. 

We have $\#G_{4,7} = \mu/2$ and $\max\{x \in G_{4,7}\} = 17 = \mu - 1$. 

The gap function is defined as $I(m) := \#\{x \in \mathbb{Z}, x \geq m, x \notin S_{4,7}\}$. 

We have $I_{4,7}(5) = \#\{5, 6, 9, 10, 13, 17\} = 6$. 

Always $I(0) = \mu/2$, $I(x) = 0$ for $x \geq \mu$ and $I(-n) = n + \mu/2$ for $n > 0$. 

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Semigroups of singular points

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- We have $\#G_{4,7} = \mu/2$ and $\max\{x \in G_{4,7}\} = 17 = \mu - 1$. This is a special property of semigroups of singular points!
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For a semigroup $S$ with a gap sequence $G$ we define

$$\Delta_S(t) = 1 + (t - 1) \sum_{j \in G} t^j.$$ 

For the semigroup $S_{4, 7}$, the gap sequence is $\{1, 2, 3, 5, 6, 9, 10, 13, 17\}$, so we have

$$\Delta_{4, 7}(t) = 1 + (t - 1) (t + t^2 + t^3 + t^5 + t^6 + t^9 + t^{10} + t^{13} + t^{17}).$$

This is the Alexander polynomial of the knot of the singularity.

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\[ \Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^{9} + t^{7} - t^{5} + t^{4} - t + 1. \]
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\[(0,9) \quad 9 = 18/2\]
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Symmetry reflects symmetry of \( \Delta \).
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The staircase complex
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- Place $\mathbb{Z}_2$ for each vertex.

Type A vertices.

Type B vertices.

Bifiltration is given by coordinates.

Absolute grading of a type A vertex is 0, of type B is 1.

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Now there comes something really scary

We will tensor the staircase complex by \( \mathbb{Z}_2[U, U^{-1}] \).
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We will tensor the staircase complex by $\mathbb{Z}_2[U, U^{-1}]$. Are you ready for the challenge?
Tensoring

Tensor $\text{St}(K)$ by $\mathbb{Z}_2[U, U^{-1}]$. 

$U$ changes the filtration level by $(-1, -1)$ and the absolute grading by $-2$.

The resulting complex is $\text{CFK}^\infty(K)$ if $K$ is an algebraic knot. Actually, it is enough that $K$ is so-called an L–space knot.
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The function $J(m)$

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\[
J(m) = l(m - g).
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Now you may start wondering:
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Oh where, oh where has the true mathematics gone?
Proposition

Let $K$ be an $L$–space knot.
Proposition

Let $K$ be an algebraic knot.
Proposition

Let $K$ be an L–space knot. Let $q > 2g(K)$ and $m \in [-q/2, q/2]$. Then

$$d(S^3_q(K), s_m) = \frac{(q - 2m)^2 - q}{4q} - 2J(m).$$
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Theorem

If $M^3$ bounds a smooth negative definite manifold $W^4$ and $s$ is a spin$^c$ structure on $M^3$, that is a restriction of a spin$^c$ structure $t$ on $W$, then

$$d(M, s) \geq \frac{c_1^2(t) - 2\chi(W) - 3\sigma(W)}{4}.$$
Applications. The FLMN conjecture.

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- $N$ neighbourhood of $C$, $M = \partial N$, $W = \mathbb{C}P^2 \setminus N$. $W$ is rational homology ball.
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- Then $M = S^3_{d^2}(K)$, $K$ is connected sum of links of singularities.
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- By Ozsváth and Szabó, $d$–invariant must vanish for $s_m$.

Theorem (—,Livingston, 2013)

If $l$ is the gap function associated with the single singular point, $j = 0, \ldots, d - 3$. Then

$$l(jd + 1) = \frac{(d - j - 1)(d - j - 2)}{2}.$$
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Generalizations apply for many singular points.
Applications. Semigroup semicontinuity.

- $K_1, K_2$ two knots. Suppose there is a PSI cobordism from $K_1$ to $K_2$ with $k$ double points.

Theorem (—, Livingston 2013)

If $K_1$ and $K_2$ are two L–space knots, $g_1$ and $g_2$ their genera, $I_1$ and $I_2$ gap functions, then for any $m \in \mathbb{Z}$:

$$I_2(m + g_2 + k) \leq I_1(m + g_1).$$

Example

Set $K_1$ unknot, $K_2 = T_{p, q}$, $m = 0$, $k = g_2 - 1$. Then

$$I_2(2g_2 - 1) = 1 \neq I_1(g_1) = 0.$$

This detects the unknotting number of torus knots.
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- Another proof by Javier Fernandez de Bobadilla.
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Another proof by Javier Fernandez de Bobadilla.

In general weak, but it uses smooth structure, unlike semicontinuity of spectrum.
In the case of $\delta$–constant deformation, $k = g_2 - g_1$.

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$(6; 7)$ cannot be perturbed to $(4; 9)$, even though the spectrum allows it.
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Perspectives

- Generalize for curves with higher genus (joint project with Ch. Livingston).
- Generalize for curves in Hirzebruch surfaces (joint project with K. Moe).
- Relate staircases to lattice homology by András Némethi.
- Can one classify all the rational unicuspidal curves in $\mathbb{C}P^2$? For many cusps other tools are more useful.