MORSE THEORY FOR PLANE ALGEBRAIC CURVES

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Abstract. We use Morse theory arguments to study links of algebraic curves. Looking at the signature of links of intersection of an algebraic curve $C$ with spheres of growing radii we find some new criteria showing that a curve with given geometric genus and certain singularities does not exist.

1. Introduction

By a plane algebraic curve we understand a set

$$C = \{(w_1, w_2) \in \mathbb{C}^2 : F(w_1, w_2) = 0\},$$

where $F$ is an irreducible polynomial. Let $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$, and $r \in \mathbb{R}$ be positive. If the intersection of $C$ with a 3-sphere $S(\xi, r)$ is transverse, it is a link in $S(\xi, r) \simeq S^3$. We denote it by $L_r$.

If $\xi$ happens to be a singular point of $C$ and $r$ is sufficiently small, $L_r$ is a link of a plane curve singularity of $C$ at $\xi$. On the other hand, for any $\xi \in \mathbb{C}^2$ and for any sufficiently large $r$, $L_r$ is the link of $C$ at infinity.

Links of plane curve singularities have been perfectly understood for almost thirty years (see [EN] for topological or [Wall] for algebro-geometrical approach). Possible links at infinity are also well described (see [Neu3, NeRu]). The most difficult case to study, as it was pointed out in a beautiful survey [Rud1], is the intermediate step, i.e. possible links $L_r$ for $r$ is neither very small nor very large.

The main idea of this paper is to study the differences between the links of singularities of a curve and its link at infinity via Morse theory: we begin with $r$ small and let it grow to infinity. The isotopy type of the link changes, when we pass through critical points. If $C$ is smooth, the theory is classical (see e.g. [Ka, Chapter V] or [Mil]), yet if $C$ has singular points, the analysis requires more care and is a new element in the theory.

To obtain numerical relations we apply some knot invariants. Namely, we study changes of Murasugi’s signature in detail and then pass to Levine–Tristram signatures, which give new set of informations. Our choice is dictated by the fact, that these invariants are well behaved under the one handle

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addition (this is Murasugi’s Lemma, see Lemma 4.3). From knot theoretical
point of view, Morse theory provides inequalities between signatures, which
are very closely related to those in [KSS1, KSS2] (cf. Corollary 5.18 and a
discussion below it). What is important, are the applications in algebraic
geometry. In this paper we show only a few of them. First of all, we present
an elementary proof of Corollary 5.15. The only known up to now proof
[BZ3, BZ4] relies heavily on algebraic geometry techniques. This result is
of interest not only for algebraic geometers, but also in theory of bifurca-
tions of ODE’s (see [ChL, BZ4] and references therein). We also reprove
Varchenko’s estimate on the number of cusps of a degree $d$ curve in $\mathbb{C}P^2$
(see Corollary 6.10). Corollary 5.17 and Lemma 6.9 show a different, com-
pletely new, application of our approach. We refer to [Bo] for a brand new
application in studying deformations of singularities of plane curves.

We are convinced that application of other knot cobordism invariants in
this setting will lead to much deeper understanding of topology of plane
curves.

\textit{Convention 1.1.} Throughout the paper we use standard, Euclidean, metric
on $\mathbb{C}^2$. $B(\xi, r)$ denotes the ball with centre $\xi$ and radius $r$. We may assume,
to be precise, that it is a closed ball, but we never appeal to this fact. The
boundary of the ball $B(\xi, r)$ is the sphere denoted $S(\xi, r)$.

2. Handles related to singular points

Let $C$ be a plane algebraic curve given by equation $F = 0$, where $F$ is a
reduced polynomial. Let $\xi \in \mathbb{C}^2$. Let $z_1, \ldots, z_n$ be all the points of $C$
such that either $C$ is not transverse to $S(\xi, ||z_k - \xi||)$ at $z_k$, or $z_k$ is a singular
point of $C$. We shall call them critical points. Let

$$\rho_k = ||z_k - \xi||.$$ 

We order $z_1, \ldots, z_n$ in such a way that $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_n$. We shall call
$\rho_k$’s critical values. We shall pick a generic $\xi$ which means that

(G1) $\rho_1 < \rho_2 < \cdots < \rho_n$, i.e. at each level set of the distance function

$$g = g_\xi(w_1, w_2) = ||w_1 - \xi_1||^2 + ||w_2 - \xi_2||^2$$

restricted to $C$ there is at most one critical point (this is not very
serious restriction and it is put rather for convenience).

(G2) If $z_k$ is a smooth point of $C$, then $g|_C$ is of Morse type near $z_k$.

(G3) If $z_k$ is a singular point of $C$, we assume the condition (2.4) holds.

Generic points always exist. Obviously G3 and G1 are open-dense conditions.
For G2 see e.g. [Mil, Theorem 6.6].

\textit{Remark 2.1.} From the condition G3 we see in particular that if $\xi$ does not lie
on $C$, then $z_1$ is a smooth point of $C$. Indeed, $g|_C$ attains a global minimum
on $C$. 

It is well known that, if \( r_1 \) and \( r_2 \) are in the same interval \((\rho_k, \rho_{k+1})\) then links \( L_{r_1} \) and \( L_{r_2} \) are isotopic, where

\[ L_r = C \cap S(\xi, r) \subset S(\xi, r). \]

Next definition provides very handy language.

**Definition 2.2.** Let \( \rho_k \) be a critical value. The links \( L_{\rho_k^+} \) and \( L_{\rho_k^-} \) (or, if there is no risk of confusion, just \( L^+ \), \( L^- \)) are the links \( L_{\rho_k^+} \) and \( L_{\rho_k^-} \) with \( \varepsilon > 0 \) such that \( \rho_k + \varepsilon < \rho_{k+1} \) and \( \rho_k - \varepsilon > \rho_{k-1} \). We shall say, informally, that the change from \( L^- \) to \( L^+ \) is a crossing or a passing through a singular point \( z_k \).

**Lemma 2.3.** Assume that \( z_k \) is a smooth point of \( C \). Then \( L_{\rho_k^+} \) arises from \( L_{\rho_k^-} \) by addition of a 0–handle, an 1–handle or a 2–handle according to the Morse index at \( z_k \) of the distance function \( g \) restricted to \( C \).

A 0–handle corresponds to adding an unlinked unknot to the link. A 2–handle corresponds to deleting an unlinked unknot. The addition of a 1–handle is a hyperbolic operation as in Definition 4.2.

**Lemma 2.4.** If \( C \) is a complex curve, there are no 2–handles.

**Proof.** A 2–handle corresponds to a local maximum of a distance function (2.1) restricted to \( C \). The functions \( w_1 - \xi_1 \) and \( w_2 - \xi_2 \) are holomorphic on \( C \), hence \( |w_1 - \xi_1|^2 + |w_2 - \xi_2|^2 \) is subharmonic on \( C \), and as such, it does not have any local maxima on \( C \).

1–handles might occur in three forms.

**Definition 2.5.** Let \( C^- = C \cap B(\xi, \rho_k - \varepsilon) \). A 1–handle attached to two different connected components of normalization of \( C^- \) is called a join. A 1–handle attached to a single component of normalization of \( C^- \) but to two different components of \( L^- \), is called a marriage. And finally, if it is attached to a single component of \( L^- \), it is called a divorce.

If the point \( z_k \) is not smooth, the situation is more complicated.

**Definition 2.6.** The multiplicity of a singular point \( z \) of \( C \) is the local intersection index of \( C \) at \( z \) with a generic line passing through \( z \).

**Proposition 2.7.** Let \( z_k \) be a singular point of \( C \) with multiplicity \( p \). Let \( L^{\text{sing}} \) be the link of the singularity at \( z_k \). Then \( L^+ (= L_{\rho_k^+}) \) can be obtained from the disconnected sum of \( L^- (= L_{\rho_k^-}) \) with \( L^{\text{sing}} \) by adding \( p \) 1–handles.

**Proof.** Up to an isometric coordinate change we can assume that \( \xi = (0, 0) \) and \( z_k = (\rho_k, 0) \). Define also

\[ \vec{v} = \frac{z_k}{||z_k||} = (1, 0, 0, 0). \]

Let \( G_1, \ldots, G_b \) be the branches of \( C \) at \( z_k \). By Puiseux theorem (see e.g. [Wall, Section 2]), each branch \( G_j \) can be locally parametrized in a Puiseux
expansion
\[ w_1 = \rho_k - \beta_j \tau^p_j, \quad w_2 = \alpha_j \tau^p_j + \ldots, \quad \tau \in \mathbb{C}, \quad |\tau| \ll 1, \]
i.e. it is a topological disk.

The (generalised) tangent line to \( G_j \) at \( z_k \) is the sum of lines
\[ \alpha_j (w_1 - \rho_k) + \beta_j w_2 = 0, \]
for \( j = 1, \ldots, b \). By genericity of \( \xi \) we may assume that
\[ \beta_j \neq 0 \]
for any \( j \).

This means that the line \( \{(w_1, w_2): w_1 - \rho_k = 0\} \) is not tangent to \( C \) at \( z_k \).
In other words
\[ \kappa = \frac{1}{2} \min_j \frac{|\beta_j|}{|\alpha_j|} > 0. \]
We chose
\[ \varepsilon \ll \kappa \text{ and } \lambda = \max(\kappa^{-1}, 2). \]

From (2.2) we see that the branch \( G_j \) winds \( p_j \) times around its tangent line at \( z_k \) given by (2.3). As the order of tangency of each branch to the tangent line at \( z_k \) is at least two, we get the following simple but important result.

**Lemma 2.8** (Accumulation principle). For \( \varepsilon \) sufficiently small and \( z \in C \cap B(z_k, \varepsilon) \setminus \{z_k\} \), the tangent space \( T_z C \) is arbitrary close to one of the lines given by \( \alpha_j w_1 + \beta_j w_2 = 0 \). \( \square \)

In particular, in order to check transversality of \( C \) to some subset, we can often show the transversality of \( T_{z_k} C \) and then claim the transversality of \( C \) by applying Lemma 2.8.

Let us define following sets
\[ B_{\pm} = B(0, \rho_k \pm \varepsilon) \quad R_s = C \cap \partial (B_- \cup B(z_k, s\varepsilon)) \]
\[ S_{\pm} = \partial B_{\pm} \quad K = C \cap \partial (B(0, r - \delta \varepsilon) \cup B(z_k - (\lambda - \delta)\varepsilon\vec{v}, \lambda \varepsilon)). \]
Here \( \delta > 0 \) is a small number that will be fixed later, \( s \in [1/2, \lambda] \) is a parameter.

The proof of the proposition will consist of the following steps.

**Step 1.** \( R_{1/2} \) is a disconnected sum of \( L_- \) and the link of singularity \( L_s^{\text{sing}} \);

**Step 2.** \( R_{\lambda} \) arises from \( R_{1/2} \) by adding \( m \) 1-handles;

**Step 3.** \( K \) is isotopic to \( R_{\lambda} \);

**Step 4.** \( L_+ \) is isotopic to \( K \).

The most important part is Step 2, all others are technical.

In proving the isotopies we shall often use an important argument.

**Lemma 2.9** (Transverse isotopy). Let \( \phi_s : S^3 \to \mathbb{C}^2, s \in [0, 1] \) be a smooth family of embeddings such that \( \phi_s(S^3) \) is transverse to \( C \) for any \( s \). Then the links \( \phi_0^{-1}(C) \) and \( \phi_1^{-1}(V) \) are isotopic.
In our setting, the maps $\phi_s$ are not necessarily smooth. More precisely, we can decompose $S^3 = S^3_{up} \cup S^3_{down}$ into a sum of closed half-spheres with $S^3_{mid} = S^3_{up} \cap S^3_{down}$. Then (in Steps 2, 3 and 4) we can choose $\phi_s$ continuous on $S^3$ and smooth on $S^3_{up}$ and $S^3_{down}$. As long as $\phi_s(S^2_{equ})$ is transverse to $C$, the statement of Lemma 2.9 holds. Indeed, we can mimic the standard proof of Lemma 2.9, i.e. we can still integrate the vector field $\frac{\partial \phi_s}{\partial s}$ along $S^2_{equ}$, $S^3_N$ and $S^3_{N/S}$ to provide local isotopies of $\phi_s^{-1}(C \cap \phi_s(S^2_{equ}))$ and $\phi_s^{-1}(C \cap \phi_s(S^3_{N/S}))$. These diffeomorphism glue to a piecewise smooth isotopy. We omit technical details, stressing once more that the assumption on the transversality of $C$ to the 'equator' is crucial, see Figure 2 (and Step 2) for counterexample if this assumption is not satisfied.

**Step 1.** Spheres $S_-$ and $S(z_k, \varepsilon/2)$ are disjoint and $C \cap S(z_k, \varepsilon/2)$ is the link of singularity of $C$ at $z_k$.

**Step 2.** For any $s \in [1/2, \lambda]$, $C$ is transverse to $B(z_k, s\varepsilon)$. If $\partial(B_- \cup B(z_k, s\varepsilon))$ was smooth, it would follow from Lemma 2.9 that $R_\lambda$ is isotopic to $R_{1/2}$. Hence, in order to study the changes of isotopy type of $R_\lambda$ we need to study the intersection of $C$ with the set of non–smooth points of $\partial(B_- \cap B(z_k, s\varepsilon))$, i.e. with $S_- \cap S(z_k, s\varepsilon)$.

To study the above intersection in detail, consider a branch $G_j$ of $C$ (see (2.2)). The annulus

$$G_j \cap (B(z_k, \lambda\varepsilon) \setminus B(z_k, \varepsilon/2))$$

is described by

$$\{ \tau \in \mathbb{C} : \frac{1}{4} \varepsilon^{-2} \leq (|\beta_j|^2 + |\alpha_j|^2)|\tau|^{2p_j} + \cdots \leq \lambda^2 \varepsilon^{-2} \},$$

where $\ldots$ denote higher order terms in $\tau$. The hyperspace $\{\text{Re } w_1 = \rho_k\}$ intersects this annulus in $2p_j$ rays given by

$$\arg \tau = -\frac{1}{p_j} (\arg \beta_j + n\pi) + \ldots, \quad n = 0, \ldots, 2p_j - 1.$$
Figure 2. Toy model in three dimensions, which should help to understand Step 2. Two balls $B_1$ and $B_2$. A plane $C$ intersects the boundary of $\partial(B_1 \cup B_2)$ in two disjoint circles (left picture). If we push the ball $B_2$ inside $B_1$, this intersection becomes a one circle. This is precisely a one handle attachment that occurs in Step 2.

There are $p_j$ sectors of the above annulus, separated by rays with $m = 2l - 1$ and $m = 2l$ that lie in the halfspace $\{\text{Re } w_1 \leq \rho_k\}$. Let us denote these sectors $H_{j1}, \ldots, H_{jp_j}$.

Of course
\[ C \cap (S_- \cap S(z_k, s\varepsilon)) = \bigcup_{j,l} H_{jl} \cap (S_- \cap S(z_k, s\varepsilon)). \]

Let us pick $j$ and $l$ with $1 \leq j \leq b$ and $1 \leq l \leq p_j$. For $s$ close to $1/2$, the intersection
\[ S_- \cap S(z_k, s\varepsilon) \]
is empty and
\[ (2.6) \quad H_{jl} \cap \partial(B_- \cup B(z_k, s\varepsilon)) \]
consists of two separate arcs. One of them lies on $S_-$, the other on $S(z_k, s\varepsilon)$ (see $H_{j1}$ on Figure 3 below).

On the other hand for $s = \lambda$
\[ (2.7) \quad H_{jl} \cap S_- \cap S(z_k, s\varepsilon) \]
consists of two points. Indeed, this intersection is isotopic to the intersection of the tangent line to the branch to $S_- \cap S(z_k, s\varepsilon)$ and we use the fact that $\lambda \geq 2|\beta_j|$. Thus, the intersection (2.6) consists of two arcs: each of them runs for a while along $S_-$ and then on $S(z_k, s\varepsilon)$ (see $H_{j3}$ on the picture below, we present there the intersection with $s\varepsilon$ for $s$ close to $\lambda$ lest we make a false impression that (2.6) is smooth, while it is only piecewise smooth).

There exists an intermediate $s = s_{jl}$ such that (2.7) consists of one point, where $S(z_k, s\varepsilon) \cap H_{jl}$ is tangent to $S_-$. For this $s_{jl}$ the intersection (2.6) is topologically a letter ‘X’. On crossing $s_{jl}$ the topology of $R_s$ changes. The change can be interpreted as gluing an 1–handle. To be more precise, this 1–handle can be chosen to be
\[ \tilde{H}_{jl} = H_{jl} \setminus B_-, \]
and we glue it to $G_j \cap (B(z_k, \varepsilon/2) \cup B_-)$.
Step 3. The isotopy in question shall be proved by using twice Lemma 2.9. First we enlarge the ball \( B(0, \rho_k - \varepsilon) \) to \( B(0, \rho_k - \delta \varepsilon) \). As \( C \) is transverse to all spheres \( S(0, \rho_k - \theta) \) and the intersections \( S(0, \rho_k - \theta) \cap S(z_k, \lambda \varepsilon) \) for \( 0 < \theta < \varepsilon \) (this statement can be trivially checked if we replace \( C \) by its tangent space at \( z_k \), then we use the fact that \( \varepsilon \) is small enough) we get the equivalence of links

\[
C \cap \partial(B(0, \rho_k - \varepsilon) \cup B(z_k, \lambda \varepsilon))
\]

and

\[
C \cap \partial(B(0, \rho_k - \delta \varepsilon) \cup B(z_k, \lambda \varepsilon)).
\]

Now we move the ball \( B(z_k, \lambda \varepsilon) \) to \( B(z_k - (\lambda - \delta)\varepsilon \vec{v}, \lambda \varepsilon) \). We claim that for all \( s \in [0, \lambda - \delta] \), \( C \) is transverse to

\[
S(z_k - s\varepsilon \vec{v}, \lambda \varepsilon) \quad \text{and} \quad S(z_k - s\varepsilon \vec{v}, \lambda \varepsilon) \cap S(z_k - \delta \varepsilon).
\]
Figure 4. Another look on branch $G_j$. As a bold arc we see the intersection of $G_j$ with $\partial (B_- \cup B(z_k, \varepsilon/2))$ on the left (i.e. a part of the link $R_{1/2}$), on the right the intersection of $G_j$ with $\partial (B_- \cup B(z_k, s\varepsilon))$ for $s$ close to $\lambda$. There are three (i.e. the multiplicity of $G_j$) hyperbolic operations. For $s = \lambda$ the part of the link on $S(z_k, \lambda \varepsilon)$ lies entirely on the outer circle.

If $C$ is union of the tangent lines (2.3), the statement is obvious. In general case we observe that $C$ is sufficiently close to this union for small $\varepsilon$, and we use Lemma 2.8.

Observe, that $C$ is definitely not transverse to $S(z_k - \lambda \varepsilon \vec{v}, \lambda \varepsilon)$ at $z_k$, because it is a singular point of $C$, which belongs to this sphere. Thus we have to stop sufficiently close to $z_k$. This will be important in proving transversality in the final step.

**Step 4.** Let

$$X = B(0, p_k - \delta \varepsilon) \cup B(z_k - (\lambda - \delta)\varepsilon \vec{v}, \lambda \varepsilon).$$

Naturally, $X \subset B_+$. We define a deformation retraction of $B_+$ onto $X$:

$$\pi_s : B_+ \to B_+, \ s \in [0, 1]$$

in the following way.

Let $e \in B_+$. If $e \in X$ we put $\pi_s(e) = e$. Otherwise let us take a ray (real halfline) steaming from $(0, 0)$ and passing through $e$. If $\delta$ is small enough, this ray intersects $\partial X$ at a unique point. Let us call it $c$. Then we define

$$\pi_s(e) = sc + (1 - s)e.$$

Step 4 shall be accomplished when we show that $C$ is transverse to $\partial \pi_s(B_+)$ for $s \in [0, 1]$. To this end decompose the boundary

$$\partial \pi_s(B_+) = Y_s^0 \cup Y_s^1,$$

where

$$Y_s^0 = \pi_s \pi_1^{-1}(\partial X \cap S(0, p_k - \delta \varepsilon))$$

$$Y_s^1 = \pi_s \pi_1^{-1}(\partial X \cap S(z_k - (\lambda - \delta)\varepsilon \vec{v}, \lambda \varepsilon)).$$
As $Y^0_s$ is a subset of $\mathcal{S}(0, \rho_k - s\delta\varepsilon + (1 - s)\varepsilon)$, and $z_k \notin Y^0_s$, $C$ is always transverse to $Y^0_s$, since we assumed that $z_k$ is the only non-transversality point of $C$ to $\mathcal{S}(0, \rho_k + \gamma)$ for $\gamma \in (-\varepsilon, \varepsilon)$.

To check transversality of $Y^1_s$ to $C$, observe, that for any $\theta > 0$, we can take $\delta > 0$ so small, that all normal vectors to $Y^1_s$ for $s \in [0, 1]$ make an (unoriented) angle less than $\theta$ with the vector $\vec{v}$. If $\theta < \arctan \kappa$

(which, by assumption (2.4) is always possible), we know that all the tangent lines (2.3) are transverse to $Y^1_s$ and to $\partial Y^1_s$. By Lemma 2.8 this holds also for the curve $C$ provided $\varepsilon$ is small enough. ⌜

Let us fix an arbitrary ordering of handles $\tilde{H}_{11}, \ldots, \tilde{H}_{bp}$ once and for all. We shall then denote them $\tilde{H}_1, \ldots, \tilde{H}_p$. We can think of the procedure described in Proposition 2.7 as follows: first we take the disconnected sum of $L^-$ with $L^{sing}$. After that we glue the handle $\tilde{H}_1$, then $\tilde{H}_2$ and so on (cf. Lemma 5.6 below). In this setting $\tilde{H}_1$ is a join handle and others are either divorces or joins or marriages. Such handles will be called fake joins, fake divorces and fake marriages respectively. The total number of such handles at a point $z_k$ will be denoted $f^j_k$, $f^d_k$ and $f^m_k$. These numbers can be computed by studying changes of the number of components and the Euler characteristics between $C^-$ and $C^+$ and between $L^-$ and $L^+$ (see the proof of Proposition 5.8 below) and as such, they are independent of the ordering of handles.

**Example 2.10.** If $z_k$ is an ordinary double point (locally defined by $\{xy = 0\}$), then $L_+$ arises from $L_-$ by changing a negative crossing on some link diagram to a positive crossing (see Figure 5 and left part of Figure 7).

3. **Number of non–transversality points**

Let us consider a curve $C = \{F = 0\}$ in $\mathbb{C}^2$, such that $F$ is a reduced polynomial of degree $d$. Let $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$ be a fixed point (a ball centre). Let $S_r = S(\xi, r)$ be a three–sphere of radius $r$ centered at $\xi$. Let $w = (w_1, w_2)$ be an arbitrary point in $C \cap S_r$. Assume that $C$ is smooth at $w$. 
Figure 6. Swallowtail curve (given in parametric form by $x(t) = t^3 - 3t$, $y(t) = t^4 - 2t^2$) intersected with a sphere $S((0,0), 2.15)$ on the left and $S((0,0), 2.5)$ on the right. We cross two $A_2$ singularities at $r = \sqrt{5}$. The two external circles on left picture twist around the middle one, after the crossing a singular point.

Figure 7. The transformation of links shown on Figures 5 and 6 explained as taking a sum with a Hopf link (resp. torus knot $T_{2,3}$) and gluing two 1–handles to the result. The bold parts of links represent places, where the handles are attached. Remark that on Figure 6 the procedure is applied twice, because we cross two singular points at one time.

Lemma 3.1. The intersection $C \cap S_r$ is transverse at $w$ if and only if the determinant

$$J_\xi(w) = \det \begin{pmatrix} \frac{\partial F}{\partial w_1}(w) & \frac{\partial F}{\partial w_2}(w) \\ w_1 - \xi_1 & w_2 - \xi_2 \end{pmatrix}$$

does not vanish.

Proof. Assume that $C$ is not transverse to $S_r$ at $w$. This means that

$$T_w C + T_w S_r \neq \mathbb{C}^2.$$ 

Since $T_w S_r$ is real three dimensional, $T_w C + T_w S_r = T_w S_r$, thus

$$T_w C \subset T_w S_r.$$ 

Taking the orthogonal complements of these spaces we see that

$$N_w S_r \subset N_w C.$$ 

But $N_w C$ is a complex space. Thus $i \cdot N_w S_r \subset N_w C$ and by dimension arguments we get that

$$N_w S_r \otimes \mathbb{C} = N_w C.$$
Now $N_w S_r \otimes \mathbb{C}$ is spanned over $\mathbb{C}$ by a vector $(\bar{w}_1 - \bar{\xi}_1, \bar{w}_2 - \bar{\xi}_2)$. The lemma follows (the above reasoning can be reversed to show the "if" part).

If $w$ is a singular point of $C$, $J_\xi(w) = 0$ by the definition.

**Corollary 3.2.** For a curve $C$ of degree $d$ and a point $\xi \in \mathbb{C}^2$ there are $d(d-2)$ such points (counted with multiplicities) $w \in C$ that the intersection $C \cap S(\xi, ||w - \xi||)$, is not transverse at $w$.

**Proof.** For a fixed $\xi$, $J_\xi(w)$ is a polynomial of degree $d-1$ in $w$ and 1 in $\bar{w}$. Intersecting $\{J_\xi = 0\}$ with $C$ of degree $d$ yields $d^2 - 2d$ points by generalised Bézout theorem (see e.g. [Chen]).

**Remark 3.3.** The number of intersection points can be effectively larger than $d^2 - 2d$: as the curve $\{J_\xi = 0\}$ is not complex, there might occur intersection points of multiplicity $-1$.

The local intersection index of $C$ with $\{J_\xi(w) = 0\}$ at a singular point $z$ can be effectively calculated. We have the following lemma.

**Lemma 3.4.** Assume that $0 \in \mathbb{C}^2$ is a singular point of $C$. The local intersection index of $C$ with $\{J_\xi = 0\}$ at 0 is equal to the Milnor number $\mu$ of $C$ at 0 minus 1.

**Proof.** This follows from Teissier lemma (see [Pl] or [GP]), which states that

$$(f, J(f,g))_0 = \mu(f) + (f,g)_0 - 1,$$

where $(a,b)_0$ denotes the local intersection index of curves $\{a = 0\}$ and $\{b = 0\}$ at 0 and $J(f,g)$ is the Jacobian

$$\frac{\partial f}{w_1} \frac{\partial g}{w_2} - \frac{\partial f}{w_2} \frac{\partial g}{w_1}.$$ 

We shall apply this lemma to the case when $f = F$ is the polynomial defining the curve $C$, whereas $g$ is the distance function:

$$g(w_1, w_2) = |w_1 - \xi_1|^2 + |w_2 - \xi_2|^2$$

Then $(f,g)_0 = 0$. In fact, intersection of $\{f = 0\}$ and $\{g = 0\}$ is real one dimensional. But if we perturb $g$ to $g - i\varepsilon$ the intersection set becomes empty.

The issue is that the Teissier lemma holds when $f$ and $g$ are holomorphic. To see that nothing bad happens, if $g$ is as above, we have to skim through a part of the proof of Teissier lemma (see e.g. [Pl]). Assume for a while that the curve $\{f = 0\}$ can be parametrised near 0 by

$$w_1 = t^n, \quad w_2 = w_2(t),$$
where \( w_2(t) \) is holomorphic and \( n \) is the local multiplicity of \( \{ f = 0 \} \) at 0. (The case of many branches does not present new difficulties.) Then

\[
\frac{\partial f}{\partial w_1}(t^n, w_2(t)) \cdot nt^{n-1} + \frac{\partial f}{\partial w_2}(t^n, w_2(t)) = 0
\]

(3.1)

\[
\frac{\partial g}{\partial w_1}(t^n, w_2(t)) \cdot nt^{n-1} + \frac{\partial g}{\partial w_2}(t^n, w_2(t)) = \frac{d}{dt}g(t^n, w_2(t)).
\]

The first equation follows from differentiating the identity \( f(t^n, w_2(t)) \equiv 0 \). The second is simply the chain rule applied to its r.h.s. On its l.h.s. we could have terms with \( \frac{\partial g}{\partial \bar{w}_2} \frac{\partial \bar{w}_2}{\partial t} \). But they vanish, as \( w_2 \) is holomorphic.

From (3.1) we get

\[
nt^{n-1}J(f, g)(t^n, w_2(t)) = -\frac{dg(t^n, w_2(t))}{dt} \cdot \frac{\partial f}{\partial w_2}(t^n, w_2(t)).
\]

(3.2)

Now we can compare orders with respect to \( t \). On the l.h.s. of (3.2) we have

\[ (n - 1) + (f, J(f, g))_0. \]

Whereas on the r.h.s. we get

\[ (f, g)_0 - 1 + (f, \frac{\partial f}{\partial w_2})_0, \]

And we use another lemma, due also to Teissier, that \( (f, \frac{\partial f}{\partial w_2})_0 = \mu(f) + n - 1 \). This can be done directly as \( f \) is holomorphic. \( \square \)

4. Signature of a link and its properties

Let \( L \subset S^3 \) be a link and \( V \) a Seifert matrix of \( L \) (see e.g. [Ka] for necessary definitions).

**Definition 4.1.** Let us consider the symmetric form

\[
V + V^T.
\]

(4.1)

The signature \( \sigma(L) \) of \( L \) is the signature of the above form. The nullity (denoted \( n(L) \)) is 1 plus the dimension of a maximal null-space of the form (4.1).

The signature is an important knot cobordism invariant. To state its properties we need one more definition.

**Definition 4.2** (see [Kaw, Definition 12.3.3]). Let \( L \) be a link with components \( K_1, \ldots, K_{n-1}, K_n \). Let us join the knots \( K_{n-1} \) and \( K_n \) by a band, so as to obtain a knot \( K' \). Let \( L' = K_1 \cup \cdots \cup K_{n-2} \cup K' \). We shall then say, that \( L' \) is obtained from \( L \) by a hyperbolic transformation.

The hyperbolic transformation depends heavily on the position of the band. Signature is one of those link invariants, the changes of which upon this operation can be well controlled. More precisely we have

**Lemma 4.3.** (see [Mur])
(a) Let $L$ and $L'$ be two links such that $L'$ can be obtained from $L$ by a hyperbolic transformation. Then
\[ |n(L) - n(L')| = 1 \text{ and } \sigma(L) = \sigma(L'); \text{ or} \]
\[ |\sigma(L) - \sigma(L')| = 1 \text{ and } n(L) = n(L'). \]

(b) Signature is additive under the connected sum. The nullity of a connected sum of links $L_1$ and $L_2$ is equal to $n(L_1) + n(L_2) - 1$.

(c) Let $L$ be a link and $L'$ be a link resulting in the change from an undercrossing to an overcrossing on some planar diagram of $L$. Then either
\[ \sigma(L') - \sigma(L) \in \{0, -2\} \text{ and } n(L) = n(L'); \text{ or} \]
\[ \sigma(L') = \sigma(L) - 1 \text{ and } |n(L) - n(L')| = 1. \]

(d) $n$ does not exceed the number of components of the link.

(e) The signature and nullity are additive under the disconnected sum.

The signature of a torus knot was computed for example in [Ka, Li].

Lemma 4.4. Let $p, q > 1$ be coprime numbers and $T_{p,q}$ be the $(p, q)$-torus knot. Let us consider a set
\[ \Sigma = \left\{ \frac{i}{p} + \frac{j}{q}, 1 \leq i < p, 1 \leq j < q \right\}, \]
(note in passing that this is the spectrum of the singularity $x^p - y^q = 0$).

Then
\[ \sigma(T_{p,q}) = \#\Sigma - 2\#\Sigma \cap (1/2, 3/2). \]

This means that $\sigma$ counts the elements in $\Sigma$ with a sign $-1$ or $+1$ according to whether the element lies in $(1/2, 3/2)$ or not.

Example 4.5. We have
\[ \sigma(T_{2,2n+1}) = -2n; \]
\[ \sigma(T_{3,n}) = 4 \left\lfloor \frac{n}{3} \right\rfloor - 2(n - 1); \]
\[ \sigma(T_{4,n}) = 4 \left\lfloor \frac{n}{4} \right\rfloor - 3(n - 1). \]

Moreover, for $p$ and $q$ large, $\sigma(T_{p,q}) = -\frac{pq}{2} + \ldots$, where $\ldots$ denote lower order terms in $p$ and $q$.

Lemma 4.4 holds even if $p$ and $q$ are not coprime (see [Ka]): then we have a torus link instead of a knot.

The following result of A. Némethi [Nem2] will also be useful

Proposition 4.6. Let $f$ be a reduced polynomial in two variables such that the curve $\{f = 0\}$ has an isolated singularity at $(0,0)$. Let $f = f_1 \cdot f_2$ be the decomposition of $f$ locally near $(0,0)$, such that $f_1(0,0) = f_2(0,0) = 0$. Let $L$, $L_1$ and $L_2$ be the links of singularities of $\{f = 0\}$, $\{f_1 = 0\}$ and $\{f_2 = 0\}$ at $(0,0)$ and $\sigma$, $\sigma_1$, $\sigma_2$ its signatures. Then we have
\[ \sigma \leq \sigma_1 + \sigma_2 - 1. \]
Lemma 4.7. Let \( L \) be a link of plane curve singularity with \( r \) branches. Then \( \sigma(L) \leq 1 - r \). Moreover the equality holds only for the Hopf link and a trivial knot.

Proof. Let \( G \) be a germ of a singular curve bounding \( L \). Let \( \mu \) be the Milnor number of the singularity of \( G \) and \( \delta = \frac{1}{2}(\mu + r - 1) \) be the \( \delta \)-invariant of the singular point. There is a classical result (see e.g. [Nem3]) that \( -\sigma(L) \geq \delta \). This settles the case if \( r = 1 \). If \( r > 2 \) we use the inequality \( \delta \geq \frac{1}{2}r(r-1) \) and we are done. If \( r = 2 \) we know that \( \delta \geq 1 \), with equality only for an ordinary double point. 

Corollary 4.8. Let \( L = K_1 \cup \cdots \cup K_{n+1} \) be a link of a plane curve singularity with \( n + 1 \) branches. Then

\[
\sigma(L) \leq \sigma(K_{n+1}) - n.
\]

Proof. Let \( L' = K_1 \cup \cdots \cup K_n \). By Lemma 4.6 \( \sigma(L') \leq \sigma(L') + \sigma(K_{n+1}) - 1 \). By Lemma 4.7 \( \sigma(L') \leq 1 - n \).
5. Changes of signature upon an addition of a handle

In order to study the behaviour of some invariants of knots let us introduce
the following notation. Here \( r \in \mathbb{R} \), \( r > 0 \) and \( r \notin \{ \rho_1, \ldots, \rho_n \} \).

- \( L_r \) the link \( C \cap S(\xi, r) \);
- \( C_r \) the surface \( C \cap B(\xi, r) \) and \( \hat{C}_r \) is its normalization;
- \( k(C_r) \) number of connected components of \( \hat{C}_r \);
- \( c(C_r) \) or \( c(L_r) \) number of boundary components of \( C_r \);
- \( \chi(C_r) \) the Euler characteristic of \( C_r \);
- \( p_g(C_r) \) the genus of \( C_r \), which for smooth \( C_r \) satisfies
  \[ 2k - 2p_g = \chi + c \];
- \( \sigma(L_r) \) the signature of \( L_r \);
- \( n(L_r) \) the nullity of \( L_r \).

If \( C_r \) is singular, we are interested in the geometric genus of \( C_r \), i.e. the
genus of normalisation of \( C_r \). This explains the notation \( p_g \) for a genus.

The following table describes the change of the above quantities upon
attaching a handle.

<table>
<thead>
<tr>
<th>name</th>
<th>index</th>
<th>( \Delta c )</th>
<th>( \Delta k )</th>
<th>( \Delta \chi )</th>
<th>( \Delta p_g )</th>
<th>( \Delta \sigma )</th>
<th>( \Delta n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>birth</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>death</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>join</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>s</td>
<td>s'</td>
</tr>
<tr>
<td>divorce</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>s</td>
<td>s'</td>
</tr>
<tr>
<td>marriage</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>s</td>
<td>s'</td>
</tr>
</tbody>
</table>

Here \( s, s' \in \{-1, 0, 1\} \) and \(|s| + |s'| = 1\) by Lemma 4.3 (a).

Let

\[
\begin{align*}
  w(L) &= -\sigma(L) + n(L) - c(L) \\
  u(L) &= -\sigma(L) - n(L) + c(L)
\end{align*}
\]

(5.1)

**Lemma 5.1.** If \( L \) is a link of singularity then \( u(L) > 0 \) and \( w(L) \geq 0 \).
Moreover, \( w(L) = 0 \) if and only if \( L \) is an unknot or a Hopf link.

**Proof.** We use Lemma 4.7 to prove this for \( w(L) \). For \( u(L) \) we use the fact that the signature is negative and Lemma 4.3(d). \( \square \)

For a knot, by Lemma 4.3(d) we have \( w(L) = u(L) = -\sigma(L) \). In general case of links we have

\[
\begin{align*}
  -\sigma(L) + (c(L) - 1) &\geq u(L) \geq -\sigma(L) \geq \\
  &\geq w(L) \geq -\sigma(L) - (c(L) - 1).
\end{align*}
\]

(5.2)

**Lemma 5.2.** The invariants \( w(L) \) and \( u(L) \) are additive under the disconnected sum. \( \square \)

**Lemma 5.3.** Attaching a birth, death, marriage or join handle does not decrease \( w(L) \).
Proof. Only the case of 1–handles requires some attention. The number of component decreases by 1 and either the nullity or the signature can change, and only by 1. \(\square\)

Remark 5.4. The divorce handle might decrease the quantity \(w(L)\) at most by 2.

Lemma 5.5. Attaching a birth, death, marriage or join handle does not increase \(u(L)\). The divorce might increase \(u(L)\) at most by 2. \(\square\)

Lemma 5.6. Let \(z_k\) be a singular point of \(C\), \(L_k^{\text{sing}}\) the link of its singularity and \(f_d^k\) the number of fake divorces at \(z_k\). Let, for \(\varepsilon > 0\) small enough \(L_\pm = L_{\rho_k \pm \varepsilon}\), where \(\rho_k = \|z_k - \xi\|\). Then
\[
w(L_+) \geq w(L_-) + w(L_k^{\text{sing}}) - 2f_d^k
\]
\[
u(L_+) \leq u(L_-) + u(L_k^{\text{sing}}) + 2f_d^k
\]

Proof. We use the notation from the proof of Proposition 2.7. We have
\[
w(R_{1/2}) = w(L_-) + w(L_k^{\text{sing}}) \quad \text{step 1}
\]
\[
w(R_\lambda) \geq w(R_{1/2}) - 2f_d^k \quad \text{step 2}
\]
\[
w(L_+) = w(R_\lambda) \quad \text{steps 3 and 4}.
\]
In the middle equations we have used the fact that a fake divorce can lower the invariant at most by 2. The proof for \(u\) is identical. \(\square\)

Lemma 5.7. Assume that \(C\) is smooth. Let \(p_g\) be the genus of the curve \(C\) and \(d\) number of its components at infinity. Let also \(a_b\), \(a_m\), \(a_d\), and \(a_j\) denote the number of birth, marriage, divorce and join handles. The following formulae hold
\[
a_m = p_g
\]
\[
a_b + a_d - a_j - a_m = d
\]
\[
a_b - a_j = 1.
\]
In particular
\[
a_d = d + p_g - 1.
\]
Proof. For \(r < \rho_1\), \(L_r\) is empty. Thus the first handle must be a birth and for \(r \in (\rho_1, \rho_2)\), \(L_r\) is an unknot. It has \(p_g = 0, c = 1\) and \(k = 1\). When we cross next critical points, these quantities change according to the table on page 15. For \(r > \rho_n\) we have the link at infinity and \(C_r\) is isotopic to \(C\). \(\square\)

Proposition 5.8. Let \(C\) be an algebraic curve in \(\mathbb{C}^2\), not necessarily smooth. For a generic point \(\xi\), let \(S_0 = S(\xi, r_0)\) and \(S_1 = S(\xi, r_1)\) be two spheres such that \(r_0 < r_1\) intersecting \(C\) transversally. For \(i = 0, 1\), we define \(p_{g_i} = p_g(C_{r_i}), c_i = c(C_{r_i})\) and \(k_i = k(C_{r_i})\).

Let \(a_d^{01}\) and \(f_d^{01}\) be the numbers of divorces (respectively fake divorces) on \(C\), which lie between \(S_0\) and \(S_1\). Then
\[
a_d^{01} + f_d^{01} \leq p_{g1} - p_{g0} + c_1 - c_0 - (k_1 - k_0).
\]
**Proof.** Let $\pi: \tilde{C} \to C$ be the normalisation map. The composition of $\pi$ with the distance function $g$ (see (2.1)) restricted to $C$ yields a function $\tilde{g}: \tilde{C} \to \mathbb{R}$. This function does not have to be a Morse function on $\tilde{C}$, but we can take a small subharmonic perturbation of $\tilde{g}$ on $\tilde{C}_{r_1}$, such that the resulting function is Morse in the preimage $\pi^{-1}B(\xi, r_1)$. This perturbation we shall still denote by $\tilde{g}$. Let $\hat{a}_b, \hat{a}_d, \hat{a}_j$ and $\hat{a}_m$ be the number of births, divorces, joins and marriages of $\tilde{g}$ in $U = \pi^{-1}(B(\xi, r_1) \setminus B(\xi, r_0))$. We need the following result

**Lemma 5.9.** There is a bound

\begin{equation}
\hat{a}_d \geq \hat{a}_d^{01} + f_d^{01}.
\end{equation}

**Proof.** If $z_k \in C$ is a smooth point of $C$ and critical point of $g$ then $\pi^{-1}(z_k)$ is a critical point of $\tilde{g}$ of the same index. Moreover, if $z_k$ is a divorce, join, or marriage then $\pi^{-1}(z_k)$ will also be, respectively, a divorce, join or a marriage.

Next we show that any fake divorce on $C$ corresponds to a divorce on $\tilde{C}$. This is done by comparing the changes of topology when crossing a singular point with the changes of topology of normalisation. So let $z_k$ be a singular point of $C$. Let us define

$$C_{\pm} = C \cap B(\xi, \rho_k \pm \varepsilon) \text{ and } L_{\pm} = \partial C_{\pm}.$$ 

Let $\tilde{C}_{\pm}$ be the normalization. Define also

$$\Delta_g = p_g(C_+) - p_g(C_-), \quad \Delta_k = k(C_+) - k(C_-), \quad \Delta_c = c(L_+) - c(L_-).$$

Observe that from topological (as opposed to smooth) point of view, passing through a singular point of multiplicity $p$ and $r$ branches amounts to picking $r$ disks and attaching them to $\tilde{C}_-$ with $p$ 1–handles. Analogously to (5.3) we get then $f_m^k = \Delta_g, f_d^k - f_j^k = \Delta_c$ and $f_j^k = \Delta_k$. Hence

$$f_d^k = \Delta_c + \Delta_g - \Delta_k.$$

The number of divorces on $\tilde{C}$ that are close to $\pi^{-1}(z_k)$ (denote this number by $\hat{a}_d^k$) can be computed in the same way. Since the number of boundary components of $\tilde{C}_{\pm}$ is the same as $c(C_{\pm})$, and $\Delta_g$ measures also the change of genus between $\tilde{C}_+$ and $\tilde{C}_-$, we have

$$\hat{a}_d^k = \Delta_c + \Delta_g - \Delta_k = f_d^k.$$

**Finishing the proof of Proposition 5.8.** Let us consider the changes of the topology of $\tilde{C} \cap \hat{g}^{-1}((\infty, r^2])$ as $r$ changes from $r_0$ to $r_1$. The number of components of the boundary changes by $c_1 - c_0$, while the genus by $g_1 - g_0$ and the number of connected components of normalization by $k_1 - k_0$. Using table on page 15 (compare the argument in the proof of Lemma 5.7) we get $\hat{a}_d = g_1 - g_0 + c_1 - c_0 - (k_1 - k_0)$.

\[\square\]
Remark 5.10. In most applications we will have $k_0 = k_1 = 1$, for example in the case when $L_1$ is a link at infinity of a reduced curve and $L_0$ is trivial knot.

Example 5.11. Let $C$ be a curve given by $x^3 - x^2 - y^2 = 0$ (see Figure 5 above), $\xi = (-1, 0)$, $r_0 = 1.05$ and let us take $r_1$ large enough. Then $L_0$ is a Hopf link, $L_1$ is the trefoil, $g_1 = g_0 = 0$ ($C$ is rational), $c_0 = 2$, $c_1 = 1$, $k_1 = 1$ but $k_0 = 2$ ($\hat{C}_0$ consists of two disks). Then the number of divorces is bounded by 0 and indeed, there is only one critical value between $r_0$ and $r_1$ and the corresponding handle is a join.

Corollary 5.12. If $C \subset \mathbb{C}^2$ is a reduced plane algebraic curve and its link at infinity has $d$ components, then the total number of divorces on $C$ (including the fake divorces) satisfies

$$a_d + f_d \leq p_g(C) + d - 1.$$ 

Proof. Let us pick a generic $\xi$ and choose $r_0 \in (\rho_1, \rho_2)$ while $r_1$ is sufficiently large. Then $S_0$ is an unknot, because the first handle that occurs when coming from $r = 0$, is always a birth. Moreover, $S_1 \cap C$ is the link of $C$ at infinity and so it has $d$ components. The statement follows from Proposition 5.8 \hfill \Box

Theorem 5.13. Let $C$ be a curve with link at infinity $L_\infty$ and with singular points $z_1, \ldots, z_n$, such that the link at the singular point $z_k$ is $L_k^{\text{sing}}$. Then

$$w(L_\infty) \geq \sum_{k=1}^{n} \left(w(L_k^{\text{sing}}) - 2(p_g(C) + d - 1)\right),$$

$$u(L_\infty) \leq \sum_{k=1}^{n} \left(u(L_k^{\text{sing}}) + 2(p_g(C) + d - 1)\right),$$

where $d$ is the number of components of $L_\infty$.

Proof. The proof now is straightforward. For $r \in (\rho_1, \rho_2)$, $L_r$ is an unknot, so $w(L_r) = u(L_r) = 0$. Then, as we cross subsequent singular points, $w(L_r)$ and $u(L_r)$ change (see Lemmas 5.3, 5.4, 5.5 and 5.6). We obtain

$$w(L_\infty) \geq \sum_{k=1}^{n} \left(w(L_k^{\text{sing}}) - 2f_d \right) - 2a_d$$

and similar expression for $u$. Theorem follows now from Corollary 5.12. \hfill \Box

Remark 5.14. Observe that Theorem 5.13 ’does not see’ ordinary double points, because if $z_k$ is an ordinary double point then $w(L_k^{\text{sing}}) = u(L_k^{\text{sing}}) = 0$.

From this theorem we can deduce many interesting corollaries. First of all we use it in showing than some curves with given singularities might not exist. The point (a) of the corollary below is almost a restatement of the result of Petrov [Pet], which can be interpreted as in [BZ3] as a bound for $k$ with $p = 3$. The point (c) gives the same estimate as in [BZ4], but we use here only elementary facts, not the BMY inequality.
Corollary 5.15. Let \( x(t), y(t) \) be polynomials of degree \( p \) and \( q \) with \( p, q \) coprime. Let \( C \) be the curve given in parametric form by

\[
\{ w_1 = x(t), w_2 = y(t), t \in \mathbb{C} \}.
\]

Assume that the singularity of \( C \) at the origin has a branch with singularity \( A_{2k} \) (i.e. \( A_{2k} \) is a singularity of a parametrisation). Then \( 2k \) is less or equal than the signature of the toric knot \( T_{p,q} \). In particular

(a) \( k \leq q - 1 - 2\left\lfloor \frac{q}{6} \right\rfloor \) if \( p = 3 \);

(b) \( k \leq \frac{3}{2}(q - 1) - 2\left\lfloor \frac{q}{4} \right\rfloor \) if \( p = 4 \);

(c) \( k \leq \sim \frac{pq}{4} \) in general.

Proof. Let \( L_0 \) be the link of singularity of \( C \) at \( 0 \). Let \( c(L_0) \) be the number of its components. By assumption, one of its components is a link \( T_{2,2k+1} \) with signature \(-2k\). By Corollary 4.8

\[-\sigma(L_0) \geq 2k + c(L_0) - 1.\]

Hence

\[w(L_0) \geq 2k.\]

The link at infinity \( L_\infty \) is a knot \( T_{p,q} \). Hence \( w(L_\infty) = \sigma(L_\infty) = \sigma(T_{p,q}) \).

This, in turn, is computed in Lemma 4.4. The result is then a direct consequence of Theorem 5.13, since \( p_g(C) = 0 \) by assumption (see (5.6)). \( \square \)

Remark 5.16. Corollary 5.15(c) holds even if \( p \) and \( q \) are not coprime. We can compute the signature of the knot at infinity by Lemma 6.6 below.

Next result is somehow unexpected, especially if we compare it to [Rud2, Proposition 87] stating that no invariant coming from a Seifert matrix of the knot, including the signature, can tell whether a link is a \( \mathbb{C} \)-link.

Corollary 5.17. If a \( \mathbb{C} \)-link \( L \) with \( m \) components bounds an algebraic curve of geometric genus \( p_g \) then

\[-\sigma(L) \geq 2 - 2m - 2p_g.\]

In particular, if a knot bounds a rational curve, its signature is non-positive.

Now we can rephrase Theorem 5.13 in a Kawauchi–like inequality.

Corollary 5.18. Let \( C \) be as in Theorem 5.13. Let \( b \) be the first Betti number of \( C \). Then

\[|\sigma(L_\infty) - \sum_{k=1}^{n} \sigma(L_k^{\text{sing}})| \leq b + n(L_\infty) - 1.\]

Proof. Let \( r_k \) be the number of branches of the link \( L_k^{\text{sing}} \) and \( d \) be the number of branches at infinity. By Theorem 5.13 and the fact that \( w(L_k^{\text{sing}}) \geq -\sigma(L_k^{\text{sing}}) - (r_k - 1) \):

\[-\sigma(L_\infty) - d + n(L_\infty) \geq -\sum \sigma(L_k^{\text{sing}}) - \sum (r_k - 1) - 2(p_g(C) + d - 1).\]
Denoting $R = \sum (r_k - 1)$ we get
\[
\sigma(L_\infty) - \sum \sigma(L_k^{\text{sing}}) \leq 2p_g + R + d + n(L_\infty) - 2 = b + n(L_\infty) - 1,
\]
as $b = 2p_g + R + d - 1$. The inequality in other direction is proved in an identical way, using invariant $u$ instead of $w$. □

With not much work, Corollary 5.18 can be deduced from [KSS1, KSS2] (see [Kaw, Theorem 12.3.1]), without ever using the holomorphicity of $C$. Roughly speaking, we pick up a ball $B \subset C^2$ disjoint from $C$ and pull (by an isotopy) all the singular points of $C$ inside $B$, so as to get a real surface $C'$ with the property that $C' \cap \partial B$ is a disjoint union of links $L_1^{\text{sing}}, \ldots, L_n^{\text{sing}}$. Then $C' \setminus B$ realizes a cobordism between this sum and the link of $C$ at infinity. Then [Kaw, Theorem 12.3.1] provides Corollary 5.18.

The main drawback of that approach is that $C'$ is no longer holomorphic. In short, it works for the signature (and Tristram–Levine signatures as well), but if we want at some moment to go beyond and use some more subtle invariant, holomorphicity of $C$ might be crucial. At present we do not know any such invariant, but we are convinced that without exploiting thoroughly the holomorphicity of $C$ we cannot get a full understanding of the relation between the link at infinity and the links of singularities of $C$.

6. Application of Tristram–Levine signatures

The notion of signature was generalised by Tristram and Levine [Tr, Le]. The Tristram–Levine signature turns out to be a very strong tool in the theory of plane algebraic curves. In what follows $\zeta$ will denote a complex number of module 1.

**Definition 6.1.** Let $L$ be a link and $S$ be a Seifert matrix. Consider the Hermitian form
\[
(1 - \zeta)V + (1 - \overline{\zeta})V^T.
\]
The *Tristram–Levine signature* $\sigma_\zeta(L)$ is the signature of the above form. The *nullity* $n_\zeta(L)$ is the nullity of above form increased by 1.

The addition of 1 is a matter of convention. This makes the nullity additive under disconnected and not connected sum.

**Remark 6.2.** For a link $L$, let us define $n_0(L)$ as a minimal number such that the $n_0(L)$-th Alexander polynomial is non-zero. Let $\Delta_{\min}(L) = \Delta_{n_0(L)}(L)$. Then, it is a matter of elementary linear algebra to prove that $n_\zeta(L) \geq n_0(L) + 1$ and $n_\zeta > n_0(L) + 1$ iff $\Delta_{\min}(\zeta) = 0$ (we owe this remark to A. Stoimenow, see [BN] for deeper discussion).

**Example 6.3.** For $\zeta = -1$ we obtain classical signature and nullity.

We have, in general, scarce control on the values of $n_\zeta$ if $\zeta$ is a root of Alexander polynomial. However, many interesting results can be obtained already by studying invariants $\sigma_\zeta$ and $n_\zeta$ when $\zeta$ is not a root of Alexander
polynomial. To simplify the formulation of these results let us define the functions $\sigma^*_\zeta$ and $n^*_\zeta$ as

\[
\sigma^*_\zeta = \begin{cases} 
\sigma_\zeta & \text{if } \zeta \text{ is not a root of } \Delta_{\text{min}} \\
\lim_{\rho \to \zeta^+} \sigma_\rho & \text{otherwise.} 
\end{cases}
\]

(6.2)

Here $\rho \to \zeta^+$ if we can write $\rho = \exp(2\pi iy)$, $\zeta = \exp(2\pi ix)$ and $y \to x^+$. Similarly we can define $n^*_\zeta$. By Remark 6.2, $n^*_\zeta \equiv n_0(L) + 1$, but we keep this function in order to make notation consistent with previous sections.

Tristram–Levine signatures share similar properties as classical signature.

Lemma 6.4 (see [Tr, Le], compare also [St]). Lemma 4.3 holds if we exchange $\sigma(L)$ and $n(L)$ with $\sigma^*_\zeta(L)$ and $n^*_\zeta(L)$.

Litherland [Li] computes also the signature of torus knot $T_{p,q}$:

Lemma 6.5. Let $p, q$ be coprime and $\Sigma$ as in Lemma 4.4. Let $\zeta = \exp(2\pi ix)$ with $x \in (0,1)$. Then

\[
\sigma^*_\zeta(T_{p,q}) = \#\Sigma - 2\#\Sigma \cap (x, 1+x).
\]

(6.3)

The openness of the interval $(x, 1+x)$ in formula (6.5) agrees with taking the right limit in formula (6.2).

The signature of an iterated torus knot can be computed inductively from the result of [Li].

Lemma 6.6. Let $K$ be a knot and $K_{p,q}$ be the $(p,q)$–cable on $K$. Then for any $\zeta$ we have

\[
\sigma_\zeta(K_{p,q}) = \sigma_\zeta(K) + \sigma_\zeta(T_{p,q}).
\]

This allows recursive computation of signatures of all possible links of unbranched singularities. In general case one uses results of [Neu1, Neu2].

Because of Lemma 6.4 we can repeat the reasoning from Section 5 to obtain a reformulation of Theorem 5.13, Corollary 5.17 and Corollary 5.18.

Theorem 6.7. Let $C$ be an algebraic curve with singular points $z_1, \ldots, z_n$, with links of singularities $L_1^{\text{sing}}, \ldots, L_n^{\text{sing}}$. Let $L_\infty$ be the link of $C$ at infinity. Let also $b$ be the first Betti number of $C$. Then

\[
\left| \sigma^*_\zeta(L_\infty) - \sum \sigma^*_\zeta(L_k^{\text{sing}}) \right| \leq b + n_0(L_\infty).
\]

(6.4)

The proof goes along the same line as the proof of Corollary 5.18. We introduce the quantities $w_\zeta = -\sigma^*_\zeta(L) + n^*_\zeta(L) - c(L)$ and $u_\zeta = -\sigma^*_\zeta(L) - n^*_\zeta(L) + c(L)$ and study their changes on crossing different singular handles. We remark only that $n^*_\zeta(L_\infty) = n_0(L_\infty) + 1$.

Using the same argument as in Proposition 5.8 we obtain a result which relates the signatures at two intermediate steps.

Proposition 6.8. For any generic parameter $\xi$, let $r_0$ and $r_1$ be two non-critical parameters. For $i = 0, 1$ let $L_i, c_i$ be, respectively, the link $C \cap S(\xi, r_i)$...
and its number of components. Let $\Delta p_g$ be the difference of genera of $C \cap B(\xi, r_1)$ and $C \cap B(\xi, r_0)$ and $\Delta k$ the difference between number of connected components of corresponding normalizations. We have then

$$w(\xi) - \sum w(L_i^{\text{sing}}) - w(\xi) \geq -2(\Delta p_g + c_1 - c_0 - \Delta k),$$

$$-(u(\xi) - \sum u(L_i^{\text{sing}}) - u(\xi)) \geq -2(\Delta p_g + c_1 - c_0 - \Delta k),$$

where we sum only over critical points that lie in $B(\xi, r_1) \setminus B(\xi, r_0)$.

Corollary 5.17 generalises immediately to the following, apparently new result.

**Lemma 6.9.** If $K$ is a $C$-knot bounding a rational curve, then $\sigma(K) \leq 0$ for any $\zeta$.

Another application of Theorem 6.7 is in the classical problem of bounding the number of cusps of a plane curve of degree $d$, see [Hir] for discussion of this problem. Our result is a topological proof of Varchenko’s bound.

**Corollary 6.10.** Let $s(d)$ be a maximal number of $A_2$ singularities on an algebraic curve in $\mathbb{C}P^2$ of degree $d$. Then

$$\limsup \frac{s(d)}{d^2} \leq \frac{23}{72}.$$

**Proof (sketch).** Let $C$ be a curve of degree $d$ in $\mathbb{C}P^2$. Let us pick up a line $H$ intersecting $C$ in $d$ distinct points. We chose an affine coordinate system on $\mathbb{C}P^2$ such that $H$ is the line at infinity. Let $C_0$ be the affine part of $C$. Then $C_0$ can be defined as a zero set of a polynomial $F$ of degree $d$. Let $z_1, \ldots, z_s$ be the singular points of $C_0$ of type $A_2$.

**Case 1.** $C_0$ has no other singular points.

Then $b = d^2 - 2s + O(d)$. Let us take $\zeta = e^{\pi i/6}$. Then $\sigma(\xi^{L_i^{\text{sing}}}) = 2$. On the other hand, the link of $C_0$ at infinity is toric link $T_{d,d}$ and its signature

$$\sigma(T_{d,d}) = 2d^2 \cdot \frac{1}{6} \left(1 - \frac{1}{6}\right) + O(d) = \frac{5}{18}d^2 + O(d).$$

(For $\zeta = e^{2\pi ix}$ we have an asymptotics $\sigma(T_{d,d}) = 2d^2x(1-x) + O(d)$ by results [Neu1, Neu2].) Then (6.4) provides

$$2s - \frac{5}{18}d^2 \leq d^2 - 2s + O(d).$$

**Case 2.** $C_0$ has other singular points. Let $\xi \in \mathbb{C}^2$ be a generic point of $\mathbb{C}^2$, and let $r_\infty$ be sufficiently large, so that the intersection of $C_0$ with a sphere $S(\xi, r_\infty)$. Let $G$ be a generic polynomial of very high degree vanishing at each of $z_k$ with up to order at least 4 (i.e. generic among polynomials sharing this property). For $\varepsilon > 0$ small enough this guarantees that the curve

$$C_\varepsilon = \{F + \varepsilon G = 0\}$$
has singularities of type $A_2$ at each $z_k$, is smooth in $B(\xi, r_\infty)$ away from $z_k$’s and its intersection with sphere $S(\xi, r_\infty)$ is the same as the intersection of $C_0$. Now we can repeat the proof in Case 1. □

The above estimate is very close to the best known to the author, that the limit is bounded from above by $(125 + \sqrt{73})/432$ ([Lan]).

Theorem 6.7 can be used together with results (especially Lemma 3 and Theorem 3) in [Li]. We can get another proof of classical Zajdenberg–Lin theorem (see [LZ]), if we put $b = 0$ (we defer the details to a subsequent paper). It is, presumably, possible to go beyond this theorem and classify all plane curves with small first Betti number (compare [BZ1] and [BZ2]). We can also hope to prove some results concerning the maximal possible number of singular point of algebraic curve with given first Betti number, the problem that is known as Lin conjecture.

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References

[ChL] C. Christopher and S. Lynch, Small–amplitude limit cycle bifurcations for Liénard systems with quadratic damping or restoring forces, Nonlinearity 12 (1999), 1099–1112.


