Small amplitude limit cycles for the polynomial Liénard system

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Abstract. We prove a quadratic in m and n estimate for the maximal number of limit cycles bifurcating from a focus for the Liénard equation $\ddot{x}+f(x)\dot{x}+g(x)=0$, where f and g are polynomials of degree m and n respectively. In the proof we use a bound for the number of double points of a rational affine curve.

1. The result

Consider the Liénard vector field

$$\dot{x} = y - F(x), \quad \dot{y} = -G'(x), \tag{1.1}$$

where F and G are polynomials of degree $m+1$ and $n+1$ respectively. It is related with the second order Liénard equation via the formulas $f(x) = F'(x)$, $g(x) = G'(x)$. The principal problem concerning the system (1.1) is to find a maximal number $H(m,n)$ of its limit cycles (a special case of the Hilbert's 16th problem). In this paper we study a weaker problem, we ask about the number of small limit cycles.

We assume that the origin $x = y = 0$ is a singular point of the center or focus type. Therefore

$$F(x) = a_1 x + \ldots + a_{m+1}x^{m+1}, \quad G(x) = b_2 x^2 + \ldots + b_{n+1}x^{n+1}, \tag{1.2}$$

where $a_1^2 < 8b_2$. We can also assume that

$$b_2 = 1. \tag{1.3}$$

When we introduce the local analytic variable

$$u = \sqrt{G(x)} = x + \ldots,$$

then the system (1.1) becomes orbitally equivalent to

$$\dot{u} = y - \Phi(u), \quad \dot{y} = -2u, \quad \Phi = c_1 u + c_2 u^2 + \ldots. \tag{1.4}$$

Here the series

$$X = c_1 Y^{1/2} + c_2 Y + c_3 Y^{3/2} + \ldots \tag{1.5}$$

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is the Puiseux expansion at the point $X = Y = 0$ of the curve

$$C : X = F(x), \ \ Y = G(x). \tag{1.6}$$

It is well known, see [Che], that the system (1.1) (equivalently, (1.4)) has center at the origin if and only if $c_1 = c_3 = \ldots = 0$, i.e. $\Phi(u) = \widetilde{\Phi}(u^2)$ is an even function. From the algebraic point of view this means that the curve (1.6) is *multiply covered* (or *non-primitive*). By the Lüroth theorem (see [GrHa]) we have $F(x) = \widetilde{F} \circ \omega(x)$, $G(x) = \widetilde{G} \circ \omega(x)$ for a polynomial $\omega(x) = x^2 + \ldots$. From the dynamical point of view this means that the system (1.4) is *time–reversible* and the system (1.1) is *rationally reversible*, i.e. it can be pushed forward via the map $(x, y) \to (\omega(x), y)$.

The coefficients $c_1, c_3, c_5, \ldots$ are called the *essential Puiseux quantities* of the singularity $X = Y = 0$ of the curve C (see [BZI]). They are related with the *Poincaré–Lyapunov quantities* $g_1, g_3, \ldots$, which appear in the Taylor expansion of the Poincaré return map

$$r \to P(r) = r + g_1 r(1 + \ldots) + g_3 r^3(1 + \ldots) + \ldots, \quad r \to 0^+, \tag{1.7}$$

from the section $\{(x, y) = (r, 0) : r \geq 0)\}$ to itself. Namely, g_j are proportional to c_j with coefficients depending only on j. We refer the reader to [ChLy] for details.

Since the fixed points of the map (1.7) correspond to the limit cycles of the Liénard vector field, the essential Puiseux quantities of the curve C become responsible for the small amplitude limit cycles of the system (1.1).

The quantities c_j and g_j depend on the coefficients a_k and b_l in the polynomials F and G (see (1.2)). In fact, they are polynomials in $a = (a_1, \ldots, a_{m+1})$ and $b = (b_3, \ldots, b_{n+1})$, e.g. for $b_2 = 1$. So the expansion (1.6) varies with varying (a, b). This variation results in bifurcation of fixed points of the map $P(r)$ from the point $r = 0$ (the generalized Hopf bifurcation). For instance, when $g_{2\nu+1} \neq 0$ and the coefficients $g_1, g_3, \ldots, g_{2\nu-1}$ vary independently, then they can be chosen such that either

$$\begin{array}{ll} 0 < g_1 << -g_3 << g_5 << \ldots \pm g_{2\nu+1}, & \text{or} \\ 0 < -g_1 << g_3 << -g_5 << \ldots \mp g_{2\nu+1}. & \end{array} \tag{1.8}$$

Thus one finds exactly ν limit cycles of small amplitude.

Since $g_j(a, b)$ are real polynomials, one cannot ensure free choice of signs, like in (1.8) (although the functions g_j may be independent).

C. Christopher and S. Lynch in [ChLy] introduced the following quantities:

$\widehat{H}(m, n)$ — the maximal number of limit cycles which can bifurcate from the origin;

$H^*(m, n)$ — the maximal cyclicity of the focus at $x = y = 0$, i.e. $\max\{\nu : c_1 = c_3 = \ldots = c_{2\nu-1} = 0 = c_{2\nu+1}\}$;

$\widehat{H}_{\mathbb{C}}(m, n)$ — the maximal number of limit cycles bifurcating from the origin in the complex sense, i.e. $\frac{1}{2}\times$maximal number of zeroes $r_i \neq 0$ of the function $P(r) - r$ for $r \in (\mathbb{C}, 0)$ (counted with multiplicities);

$H^*_{\mathbb{C}}(m, n)$ — the maximal cyclicity of $x = y = 0$ in the complex sense.

In the definitions of $\widehat{H}_{\mathbb{C}}(m, n)$ and $H^*_{\mathbb{C}}(m, n)$ one assumes complex coefficients a_i, b_j and considers the complex foliation defined by (1.1) in $(\mathbb{C}^2, (0, 0))$.

We have the following simple relations

$$\widehat{H}(m, n) \leq H^*(m, n) \leq H^*_{\mathbb{C}}(m, n) = \widehat{H}_{\mathbb{C}}(m, n). \tag{1.9}$$

Christopher and Lynch stated several conjectures concerning the above quantities. To formulate them we introduce the space $\mathcal{X}$ of curves of the form (1.6) with F, G like in (1.2), thus $\mathcal{X} \simeq \mathbb{C}^{m+n+1}$. This space is acted on by a group $\mathcal{G}$ of equivalences of curves, generated by:

— rescalings $x \to \alpha x, \quad X \to \beta X, \quad Y \to \gamma Y$;

— elementary Cremona transformations $X \to X + const \cdot Y^j, \quad 1 \leq j \leq [(m+1)/(n+1)]$, if $n \leq m$; or of the form $Y \to Y + const \cdot X^j$ if $m < n$.

These changes have no influence to the property of vanishing of successive coefficients c_{2j-1}. Therefore the equations $c_1 = c_3 = \ldots = c_{2\nu-1} = 0$ can be regarded as equations on the quotient space $\mathcal{X}/\mathcal{G}$. They define varietes in $\mathcal{X}$ composed of whole orbits of the action of $\mathcal{G}$ on $\mathcal{X}$. If $n < m$ then there exists one (exceptional) orbit, which contains the quasi-homogeneous curve $F(x) = x^{m+1}$, $G(x) = x^{n+1}$, of dimension $2 + [\frac{m+1}{n+1}]$; other orbits have dimension $3 + [\frac{m+1}{n+1}]$. Also for $n \geq m$ there is such division.

Since we assume that $b_2 \neq 0$, the first case occurs when $n = 1$ (and $G(x) = x^2$). But here $c_j = a_j$ and the problem is elementary: we have $\widehat{H}(m,1) = \widehat{H}_{\mathbb{C}}(m,n) = [\frac{m}{2}]$, where $[\cdot]$ denotes the integer part. When $n \geq 2$ we have the following

Conjecture ([ChLy]) 1. $\widehat{H}_{\mathbb{C}}(m,n) = \widehat{H}_{\mathbb{C}}(n,m) = m + n - 2 - \left[\frac{m+1}{n+1}\right]$ for $2 \leq n \leq m$;

2. $\widehat{H}(m,n) = \widehat{H}(n,m)$;

3. $H^*(m,n) = H^*(n,m)$.

Remark 1. In [ChLy] one finds the conjectured bounds $\widehat{H}_{\mathbb{C}}(m,n) = \left[\frac{n(m+2)}{n+1}\right] + n - 3$ for $2 \leq n < m$ (which agrees with the above) and $\widehat{H}_{\mathbb{C}}(n,n) = 2n - 4 + \left[\frac{2}{n}\right]$ (which is stronger than above).

Christopher and Lynch proved the formula $\widehat{H}(m,2) = [\frac{2m+1}{3}] = m - [\frac{m+1}{3}]$, using some Petrov's [Pet] ideas. They also proved that $\widehat{H}(m,3) = 2[\frac{2(m+2)}{8}]$ when $3 \leq m \leq 50$ and $\widehat{H}_{\mathbb{C}}(m,3) = [\frac{3(m+2)}{4}]$ when $6 \leq m \leq 50$. They found examples where $\widehat{H}_{\mathbb{C}}(m,3) > \widehat{H}(m,3)$ (e.g. $\widehat{H}_{\mathbb{C}}(7,3) = 7$ and $\widehat{H}(7,3) = 6$).

Also other computer calculations confirm the above conjecture.

We do not prove the Christopher–Lynch conjecture in this paper (although initially we aimed at it). We are able to show the following quadratic bound for $H^*_{\mathbb{C}}(m,n)$. Introduce the number

$$\delta_{\max} = \delta_{\max}(m,n) = mn - \gcd(m+1, n+1) + 1; \tag{1.10}$$

in the next section we interpret $\delta_{\max}$ as the maximal number of double points of a curve of the form (1.6).

Main Theorem. *If* $2 \leq n$ *then* $H^*_{\mathbb{C}} \leq \delta_{\max} - 1$.

We prove it in the next section.

2. Double points of a curve via a Hamiltonian vector field

If $A \subset (\mathbb{C}^2, 0)$ is a germ of holomorphic curve defined by $H(X,Y) = 0$, then the (complex) Hamiltonian vector field

$$V_H = H'_Y \partial_X - H'_X \partial_Y$$

is tangent to A. Below we shall regard V_H as a real vector field in $\mathbb{R}^4$ (i.e. with real time). One can check that the real field V_H is also Hamiltonian with $\operatorname{Re} H$ as the Hamilton function, but with respect to the symplectic structure given by $d\operatorname{Re} X \wedge d\operatorname{Re} Y - d\operatorname{Im} X \wedge d\operatorname{Im} Y$.

We denote $W := V_H|_A$. If 0 is an isolated singular point of A, then we consider the normalization $N : \widetilde{A} \to A$; thus each topological component $\widetilde{A}_j$, $j = 1, \dots k$ of $\widetilde{A}$ (preimage of an analytic component A_j of A) is a disc. The pull-back $\widetilde{W} := N^* W = (N_*)^{-1} W \circ N$ of the vector field W is a vector field on the smooth manifold $\widetilde{A}$ with isolated equilibrium points $p_j \in N^{-1}(0)$, $j = 1, \dots, k$. Thus one can define the indices $i_{p_j}\widetilde{W}$.

We call the quantity

$$\delta_0 := \frac{1}{2} \sum_j i_{p_j} \widetilde{W} \tag{2.1}$$

the *number of double points of* A *hidden at* 0. In the literature δ_0 is sometimes called the δ-*invariant* of the singularity or the *virtual number of double points.* The next lemma justifies this definition.

Lemma 1. *The number* δ_0 *equals to the number of simple double points of a typical perturbation* N' *of the normalization map* $N : \widetilde{A}_1 \sqcup \ldots \sqcup \widetilde{A}_k \to \mathbb{C}^2$.

Proof. If, after perturbation, in the disc $\widetilde{A_j}$ there remain only preimages of simple double points then the number of such preimages equals to the sum of indices of the vector field $\widetilde{W}'|_{\widetilde{A}_j} = (N')^* V_{H'}|_{\widetilde{A}_j}$,where H' defines the perturbed curve (see the below lemma). But this is exactly the index of the field $\widetilde{W}'$ along ∂A_j. The latter index equals the index of the field $\widetilde{W}|_{\widetilde{A}_j}$ at p_j.

Summing-up all this over j we get the doubled number of double points of the perturbation. □

Lemma 2. ([Mil], [Lins], [BZI]). *We have*

$$2\delta_0 = \sum_j \mu_0(A_j) + 2 \sum_{i<j} (A_i \cdot A_j)_0, \tag{2.2}$$

where $\mu_0(A_j)$ *is the Milnor numbers of the germ* A_j *at the point* 0 *and* $(A_i \cdot A_j)_0$ *is the intersection number at* 0 *of the components* A_i *and* A_j. *In particular, if* 0 *is a simple double point of* A *then* $i_{p_1}\widetilde{W} + i_{p_2}\widetilde{W} = 2$.

Moreover, the Milnor number of the whole set A *equals*

$$\mu_0(A) = 2\delta_0 - k + 1. \tag{2.3}$$

Proof. The Milnor number $\mu_0(H) = \mu_0(A)$ is the first Betti number of the following manifold with boundary: $A_\lambda = B_\rho \cap \{H = \lambda\}$, where B_ρ is a ball of small radius ρ around 0 and λ is a small non-critical value of H (the Milnor theorem). The manifold A_λ is a surface of genus g with k holes and $\mu_0 = 2g + k - 1$. The vector field $V_H|_{A_\lambda}$ does not vanish and its index at the j-th component of the boundary ∂A_λ equals $i_{p_j}\widetilde{Y}$. Consider the manifold M obtained from A_λ by contracting the boundary circles to points q_j and a vector field $Z = f \cdot V_H$ on M, such that $f > 0$ on $A_\lambda \setminus \partial A_\lambda$ and $f = 0$ on ∂A_λ. We have $i_{q_j} Z = 2 - i_{p_j}\widetilde{W}$; (if $\frac{d}{dt} z = z^\alpha$ on S^1, then $\frac{d}{dt}(1/z) = (1/z)^{2-\alpha}$). The Poincaré–Hopf theorem says that the Euler–Poincaré characteristic $\chi(M)=2-2g$ equals $\sum i_q Z$. Therefore $2-2g=\sum_j(2-i_{p_j}\widetilde{W})=2k-2\delta_0$ from which (2.3) follows.

Let $N_j:(\mathbb{C},0)\to(A_j,0)$, $z\to(X(z),Y(z))$ be the local parametrization (normalization) of A_j. Assume also that the coordinates X,Y are such that A_j does not lie in the line $X=0$. Then we get $\dot z=(dX/dz)^{-1}(\partial H/\partial Y)|_{A_j}$ and $i_{p_j}\widetilde{W}=\operatorname{ord}_{z=0}(dX/dz)^{-1}(\partial H/\partial Y)|_{A_j}$. If $H=H_1\dots H_k$, where H_j define A_j, then $\operatorname{ord}_{z=0}(dX/dz)^{-1}(\partial H/\partial Y)|_{A_j}$ equals

$$\operatorname{ord}_{z=0}(dX/dz)^{-1}(\partial H_j/\partial Y)|_{A_j}+\sum_{i\neq j}\operatorname{ord}_{z=0}H_i|_{A_j}=\mu_0(A_j)+\sum_{i\neq j}(A_i\cdot A_j)_0.$$

This gives (2.2). □

Consider now the curve C of the form (1.6), where we assume that $a_{m+1}b_{n+1}\neq 0$.

Corollary 1. *The quantity ν for the curve (1.6) such that $c_1=c_3=\ldots=c_{2\nu-1}=0\neq c_{2\nu+1}$ (i.e. the codimension of the singularity $x=0$ of a parametrized curve) equals δ_0, the number of double points at the singularity $X=Y=0$ of C (which is of the type $\mathbf{A}_{2\nu+1}$).*

Denote by $\xi=(F,G):\mathbb{C}\to C$ the parametrization of the curve C and let $H(X,Y)=0$ be the equation for C. The extension of the map ξ to a map from $\mathbb{CP}^1$ is the normalization of the closure $\overline{C}=C\cup p_\infty\subset\mathbb{CP}^2$ of the curve C. We define a (real) vector field W on $\overline{C}$, or $\widetilde{W}$ on $\mathbb{CP}^1$, by the formula

$$\widetilde{W}(x)=\chi(x)\cdot(\xi^*V_H)(x),\quad x\in\mathbb{CP}^1\setminus\infty.$$

Here $\chi(x)>0$ is a smooth function tending to 0 as $x\to\infty$ in a way that $\widetilde{W}$ becomes smooth at ∞. Namely, in the variable $z=1/x$ the pull-back vector field ξ^*V_H usually has pole, $\xi^*V_H=z^{-\alpha}(c+\ldots)\frac{d}{dz}$ for $c\neq 0$. Then we put $\chi(x)=|z|^{2\alpha}$ near $z=0$. We find that

$$i_\infty\widetilde{Y}=-\alpha. \tag{2.4}$$

Lemma 3. *If C has only simple double points as singularities, then their number equals*

$$\delta:=1-\frac{1}{2}i_\infty\widetilde{W}.$$

For general curve C the number $\delta=1-\frac{1}{2}i_\infty\widetilde{W}$ equals the sum of the numbers of double points hidden at the (finite) singular points of C.

Proof. It follows from the Poincaré–Hopf formula, equality (2.4) and $\chi(\mathbb{CP}^1)=2$. □

Let us calculate the number $i_\infty\widetilde{W}$ in terms of the Puiseux expansion of the curve C at infinity:

(2.5)
$$\begin{aligned}Y&=X^{u_1/p_1}\left[d_1+\ldots+X^{-u_2/p_1p_2}\left[d_2+\ldots+\ldots X^{-u_r/p_1\ldots p_r}\left[d_r+\ldots\right]\ldots\right]\right]\\&=\left(d_1X^{v_1/(m+1)}+\ldots\right)+\left(d_2X^{v_2/(m+1)}+\ldots\right)+\ldots+\left(d_rX^{v_r/(m+1)}+\ldots\right).\end{aligned}$$

Here $p_j>1$ for $j\geq 2$, $\deg F=m+1=p_1\dots p_r$, $v_j=u_1p_2\dots p_r-u_2p_3\dots p_r-\ldots-u_jp_{j+1}\dots p_r$ and $\gcd(u_j,p_j)=\gcd(v_j,p_j)=1$. The coefficients $d_j\neq 0$ and the dots denote power series in $X^{1/p_1\cdots p_j}$ in the j-th summand. Moreover, $v_1 = \deg G = n+1$. The pairs $(p_1,u_1),(p_2,-u_2),\ldots,(p_r,-u_r)$ are called the *characteristic pairs (at infinity)*. We call the expansion (2.5) as the *topologically arranged Puiseux expansion.*

Proposition 1. *The number $i_\infty\widetilde{W}$ equals*

$$2-[(v_1-1)(p_1-1)p_2\ldots p_r+(v_2-1)(p_2-1)p_3\ldots p_r+\ldots+(v_r-1)(p_r-1)]\,.$$

In particular, the number of double points of C equals

$$\delta=\frac{1}{2}\sum(v_j-1)(p_j-1)p_{j+1}\ldots p_r. \tag{2.6}$$

Proof. The Hamiltonian differential equation on C, i.e. $\dot{X}=H'_Y$, in the local variable $z=1/x$ takes the form $\dot{z}=z^{-m}H'_Y(c+\ldots)$ for some constant $c\neq 0$. So we have to calculate the order of $H'_Y|_C$ at $z=0$.

Formula (2.5) gives one branch $Y=f_{\zeta^*}(x)$ of the multi-valued solution to the equation $H(X,Y)=0$. All branches $Y=f_\zeta(X)$ of this solution take the form

$$\zeta_1\left[d_1X^{v_1/(m+1)}+\ldots+\zeta_2\left[d_2X^{v_2/(m+1)}+\ldots+\zeta_r\left[d_rX^{v_r/(m+1)}+\ldots\right]\ldots\right]\right],$$

where ζ_1 takes p_1 values, ζ_2 takes p_2 values, etc. We have $\zeta^*=(1,\ldots,1)$. The polynomial H can be represented in the form $H=\prod_\zeta(Y-f_\zeta(X))$ near infinity and

$$H'_Y|_C=\prod_{\zeta\neq\zeta^*}(Y-f_\zeta(X)).$$

In the latter product we have $(p_1-1)p_2\ldots p_r$ factors with $\zeta_1\neq 1$ and of order $X^{v_1/(m+1)}\sim z^{-v_1}$ each, we have $(p_2-1)p_3\ldots p_r$ factors of order z^{-v_2}, etc. We find $\mathrm{ord}_{z=0}H'_Y=-\sum v_j(p_j-1)p_{j+1}\ldots p_r$.

Together with $(m+1)-1=\sum(p_j-1)p_{j+1}\ldots p_r$, this gives the thesis of the proposition. □

Note that when $n+1=l(n+1)$, then $p_1=1$, $v_1=l$ and the first term in the sum in (2.6) gives zero contribution to δ.

The number δ is maximal when either $m+1$ and $n+1$ are relatively prime, here $\delta=\delta_{\max}=\frac{1}{2}mn$, or when there are exactly two essential terms in the expansion (2.5): $d_1X^{(n+1)/(m+1)}+d_2X^{n/(m+1)}$, here $\delta_{\max}=\frac{1}{2}[n(p_1-1)p_2+(n-1)(p_2-1)]=\frac{1}{2}[mn-p_2+1]$ where $p_2=\gcd(m+1,n+1)$. These numbers agree with (1.10). We obtain the following bound (which is weaker than in Main Theorem)

Corollary 2. $H^*_{\mathbb{C}}(m,n)\leq\delta_{\max}$.

In order to improve this estimate we use the following theorem of M. Zaidenberg and V. Lin [ZaLi]:

if an algebraic curve of the form (1.6) has only one singular point then it is equivalent to a quasi-homogeneous curve.

Proof of Main Theorem. We know that C is equivalent to a quasi-homogeneous curve only when $n=1$. So, for $n\geq 2$ the Zaidenberg–Lin theorem says that the curve C must have another double point (simple or hidden at another singularity).

Hence the number of double points hidden at the point $X = Y = 0$ does not exceed $\delta_{\max} - 1$. □

Remark 2. The reader can note that the above proof allows to estimate the number of small amplitude limit cycles for the Liénard system (1.1) which bifurcate simultaneously from several foci. This number does not exceed $\delta_{\max}$.

The idea to use the Hamiltonian vector field to study geometry of plane algebraic curves originates from our work [BZI]. In fact, most of the results of this section can be found on that paper.

It is possible to improve the bound $H^* \leq\sim \frac{1}{2}mn$ from Main Theorem. Namely, using so-called Bogomolov–Miyaoka–Yau inequality and calculations for resolutions of singularities performed by S. Orevkov and M. Zaidenberg [Or, OrZa], one can get a bound of the form $H^* \leq\sim \frac{1}{4}mn$. That analysis is nontrivial, so we do not present it here and we refer the reader to our preprint [BZIII] (which is devoted to more complicated singularities of plane curves).

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