DEFORMATIONS OF SINGULARITIES OF PLANE CURVES

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Abstract. In this paper we present some new results in the deformation theory of plane curve singularities. The methods rely on the study of analytic properties of linear non homogeneous ODE’s.

1. Introduction

In this paper we deal with singularities of plane curves. A plane curve singularity is meant to be a zero set of a polynomial $C = \{ f = 0 \}$, such that its gradient $\nabla f$ vanishes at some point (mostly we will assume this is $(0, 0) \in \mathbb{C}^2$) and $\nabla f \neq (0, 0)$ in some punctured neighbourhood of $(0, 0)$. This property ensures that $f$ is a reduced polynomial and $(0, 0)$ is an isolated singularity.

By the classical Puiseux theorem (see [Zol]) the set $C$ intersected with a ball centered at $(0, 0)$ is an union of sets $C_1 \cup C_2 \cup \cdots \cup C_r$, such that each $C_i$ is homeomorphic to a disk, $C_i \cap C_j = \{(0, 0)\}$ and each $C_i$ can be given a local parametrisation

\[ y = c_i^0 x^{q_i/p} + c_i^1 x^{(q_i+1)/p} + \cdots, \]

called a Puiseux expansion.

In this paper we will mostly deal with singularities with one branch, i.e. when $r = 1$.

In one branch case, the Puiseux expansion determines uniquely the topological type of singularity $C_i$. This type can be viewed as the homotopy type of the pair $(S^3_\varepsilon, S^2_\varepsilon \cap C_i)$, where $S^2_\varepsilon$ is a sufficiently small sphere around zero in $\mathbb{C}^2$. Since Puiseux expansions are of vital interest to us in this paper, we recall a following definition.

Definition 1.1. Let us consider a Puiseux expansion of the singularity such that $p$ is the multiplicity.

\[ y = c_q x^{q/p} + c_{q+1} x^{(q+1)/p} + c_{q+2} x^{(q+2)/p} + \ldots. \]

The sequence $(p, q_1, \ldots, q_n)$ is called a characteristic sequence of the singularity if the following conditions are satisfied

1. $q_i$ is an increasing sequence. $q_i = q$ if and only if $q/p$ is not integer.
2. $c_{q_i} \neq 0$ for $i = 1, \ldots, n$;

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Remark 1.3. In algebraic geometry the notion of degeneration is called a singularity of a fiber. Given a deformation of a plane curve singularity we define its degeneration.

Problem 1.4. What characteristic sequences of a plane curve singularity with a characteristic sequence \((p; q_1, \ldots, q_n)\) is a polynomial map
\[
F : D^2 \times B \to \mathbb{C},
\]
where \(D^2\) is a ball in \(\mathbb{C}^2\) around zero and \(B = \{\varepsilon \in \mathbb{C}, |\varepsilon| < 1\}\) such that
1. For any \(\varepsilon\) the map \(f_\varepsilon : D^2 \to \mathbb{C}\), \(f_\varepsilon(x, y) = F(x, y, \varepsilon)\) defines a singularity \(C_\varepsilon\) at \((0, 0)\), i.e. \(C_\varepsilon = f_\varepsilon^{-1}(0)\) has singularity at \((0, 0)\).
2. For \(\varepsilon \neq 0\) the singularity of \(C_\varepsilon\) has characteristic sequence of \((p; q_1, \ldots, q_n)\).
3. The gradient of \(F\) does not vanish away from \((0, 0) \times B\).

Given a deformation of a plane curve singularity we define its degeneration as the singularity of a fiber \(C_0\). The singularity at \(\varepsilon \neq 0\) will be called generic.

Remark 1.3. In algebraic geometry the notion of degeneration is called rather specialisation. We prefer the first term since our methods are mostly analytic.

The main question we put is

Problem 1.4. What characteristic sequences \((r; s_1, \ldots, s_m)\) may arise as degenerations of a singularity with a characteristic sequence \((p; q_1, \ldots, q_n)\)?
From now on we shall always assume that the deformation can be given in a parametric form (1.2)

Remark 1.6. In the paper we will sometimes abuse the language to make statements shorter and more concise, namely

(i) a singularity with a characteristic sequence \((p; q_1, \ldots, q_n)\) will sometimes be called simply a \((p; q_1, \ldots, q_n)\) singularity;
(ii) a singularity with multiplicity \(p\) will sometimes be called a \(p\)-singularity;
(iii) we will say that the limit singularity is at least \((r; s_1', \ldots, s_m')\) if it is a \((r; s_1', \ldots, s_m')\) singularity with \(s_1' \geq s_1\);
(iv) all limits in the article are taken with \(\varepsilon \to 0\);
(v) unless specified otherwise, bounded will mean bounded from above when \(\varepsilon \to 0\), unbounded — going to infinity for some subsequence \(\varepsilon_n \to 0\).

2. Classical facts

The Milnor number of a \((p; q_1, \ldots, q_n)\) singularity can be computed by the Milnor formula

\[
\mu = (p - 1)(q_1 - 1) + \sum_{i=2}^{n} (p_i - 1)(q_i - q_{i-1}),
\]
where \(p_i = \gcd(p_{i-1}, q_{i-1})\) and \(p_1 = p\) (see Definition 1.1).

We have the following

Proposition 2.1. The Milnor number is upper semicontinuous with respect to deformations, namely if \(\mu\) is the Milnor number in a non-degenerate case and \(\mu_0\) is the Milnor number of the special fiber then

\[
\mu_0 \geq \mu
\]
with an equality if and only if the family over the whole disk is topologically equisingular.

The first part is classical (see [AVG] or [Zol]), while the second part is due to Teissier [Tes], [LR].

Apart of the semicontinuity of Milnor number we have also the semicontinuity of multiplicity, which can be rephrased as

Lemma 2.2. If a \(p\)-singularity specialises to \(r\)-singularity then \(r \geq p\).

The lemma follows from the definition of the multiplicity as the intersection index of a germ of a singular curve with a generic line passing through the singular point.

When studying the case \(p = r\), so if the multiplicity is preserved, we can always assume that \(x_\varepsilon(t) = t^p\), which solves completely the problem of degenerations in this case. This statement follows from the following

Lemma 2.3. If \(p = r\) then \(x_\varepsilon(t)^{1/p}\) is analytic with respect to \(\varepsilon\) around \(\varepsilon = 0\).

Proof. As \(a_0(0) \neq 0\), it is bounded away from zero for small \(\varepsilon\). We can then write

\[
x_\varepsilon(t)^{1/p} = a_0(\varepsilon)^{1/p} t \left(1 + \frac{a_1(\varepsilon)}{a_0(\varepsilon)} t + \cdots + \frac{a_N(\varepsilon)}{a_0(\varepsilon)} t^N \right)^{1/p},
\]
and if we apply the expansion \((1 + z)^{1/p} = 1 + \frac{1}{p} z + \ldots\), the resulting power series in \(\varepsilon\) and \(t\) will be convergent in a small polydisk around \(|t| < \delta, |\varepsilon| < \delta\). \(\square\)
3. Spectral numbers

One of the tools in studying the deformation of singular points comes from theory mixed Hodge structures. The result of Varchenko [Var] shows that the spectrum of planar singular points is semicontinuous under deformations. This can give some partial answer to the Problem 1.4.

We shall not give the precise and rather complicated definition of spectral numbers of a singular point referring the curious reader either to [Sai] or to a wonderful book by Żołdek [Zol]. Two things we need are first, how to compute spectral numbers of a plane curve singularity, second: how these numbers behave under specialisation.

Let \((C, 0) \subset (\mathbb{C}^2, 0)\) be a germ of a plane curve singularity of one branch and with a characteristic sequence \((p, q_1, \ldots, q_n)\) as above. Let us define the Eisenbud–Neumann diagram of the singularity

\[
\begin{array}{cccccc}
& \ldots & & \vdots & & \\
& & & 1 & & \\
& & & k_1 & & w_1 \\
& & & & & \\
\end{array}
\]

Here \(w_i\)’s and \(k_i\)’s are related to \((p, q_1, \ldots, q_n)\) as follows: we write the Puiseux expansion (1.1) in the topologically arranged form

\[
y = x^{m_1/k_1}(d_1 + \ldots + x^{m_2/k_1}k_2(d_2 + \ldots + x^{m_3/k_1}k_2k_3(d_3 + \ldots + x^{m_n/k_1\ldots k_n}d_n + \ldots))).
\]

where \(p = k_1 \ldots k_n\) and the fact that the expansion is topologically arranged means \(d_i \neq 0\) and between each \(d_i\) and \(x^{m_{i+1}/k_1\ldots k_i}k_{i+1}\) there are only terms with \(x^{\text{fractional power}}\) \(x^{j/k_1\ldots k_i}\). In other words \(k_1 = p/\gcd(p, q)\), \(m_1 = q/\gcd(p, q)\) and, more generally, \(k_i = p_{i-1}/\gcd(p_{i-1}, q_{i-1})\) and \(m_i = (q_i - q_{i-1})/\gcd(p_{i-1}, q_{i-1})\).

Proposition 3.1. [Sai] The spectral numbers less then 1 of the singularity are given by the set

\[
\left\{ \frac{1}{k_{\nu+1}\ldots k_n} \left( \frac{i}{k_\nu} + \frac{j}{w_\nu} \right) + \frac{r}{k_{\nu+2}\ldots k_n} \right\}.
\]

where \(\nu\) goes through all numbers 1 to \(n\) and \(1 < i < k_\nu\), \(1 < j < w_\nu\), \(0 \leq r \leq k_{\nu+1}\ldots k_n\) with the additional condition that \(\frac{i}{k_\nu} + \frac{j}{w_\nu} < 1\).

The remaining half of spectral numbers (those in the interval \((1, 2)\)) arise as the symmetrical reflection of the above set with respect to point 1. The total number of spectral numbers of a singular point is equal to its Milnor number.

Example 3.2. For the singularity \((4; 6, 9)\) we have \(k_1 = k_2 = 2\), \(w_1 = 3\) and \(w_2 = 15\). We have 18 spectral numbers:

\[
(3.1) \quad 5, 17, 7, 19, 5, 23, 5, 9, 29, 31, 11, 7, 37, 13, 41, 17, 43, 19
\]

\[
12, 30, 12, 30, 10, 30, 6, 10, 30, 10, 6, 30, 10, 6, 30, 12, 30, 12.
\]
Example 3.3. For the singularity (5, 6) the spectral numbers are

\[
\begin{align*}
11 & \quad 8 & \quad 17 & \quad 11 & \quad 23 & \quad 13 & \quad 9 & \quad 14 & \quad 29 \\
30' & \quad 15' & \quad 30' & \quad 10' & \quad 15' & \quad 30' & \quad 10' & \quad 15' & \quad 30'
\end{align*}
\]

(3.2)

Example 3.4. Let us consider a deformation of a plane curve singularity as in Definition 1.2. Assume that the spectral numbers for \( \varepsilon \neq 0 \) are

\[\{a_1, a_2, \ldots, a_{\mu'}\}\]

whereas for the degenerate fiber \( \varepsilon = 0 \) we have

\[\{b_1, \ldots, b_{\mu}\}\]

Then for any \( \alpha \in \mathbb{R} \) we have

\[
\# \{i : a_i < \alpha\} \leq \# \{i : b_i < \alpha\}
\]

(3.3)

Example 3.5. Consider a singularity (32; 48, 56, 60, 62, 63) with a following Puiseux diagramm

\[
\begin{array}{cccccccc}
3 & 2 & 1 & 13 & 1 & 53 & 1 & 213 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}
\]

Its Milnor number is 1612. The smallest spectral number is

\[
\left( \frac{1}{3} + \frac{1}{2} \right) \cdot \frac{1}{16} = \frac{5}{96}
\]

On the other hand consider a singularity with a diagram

\[
\begin{array}{cc}
& \beta_1 \\
\beta_2 & \beta_1
\end{array}
\]

where \( \beta_1 \) and \( \beta_2 \) are not fixed. This is a \((33; 3 \cdot \beta_1, \beta_2 - 30/\beta_1)\) singularity.

Its smallest spectral number is

\[
\left( \frac{1}{\beta_1} + \frac{1}{11} \right) \cdot \frac{1}{3}
\]

it does not depend on \( \beta_2 \) and for

\[
\beta_1 < \frac{352}{23} = 15 \frac{8}{23}
\]

it is strictly larger than \( \frac{5}{36} \). It follows, that the singularity (32; 48, 56, 60, 62, 63) can never specialise to a singularity \((33; \beta, \ldots)\) with \( b = 36, 39, 42, 45 \), independently on the Milnor number and even the codimension (see Conjecture 6.2 and above) of a degenerate singular point: we could put \( \beta_2 = 10000 \) and this will not help.
4. Case \( p = 2 \) and \( r = 3 \)

This case is one of the most elaborated and is very close to be completed.

The motivation for this case is the following result proved originally by Petrov [Pet] in a different context.

**Proposition 4.1.** Let
\[
x(t) = a_2 t^2 + t^3
\]
\[
y(t) = b_2 t^2 + b_3 t^3 + \cdots + b_{w-1} t^{w-1} + t^w
\]
be a polynomial curve in \( \mathbb{C}^2 \) with 3 \( w \). Assume that the coefficients \( a_2, b_2, \ldots, b_{w-1} \) are real and the curve (4.1) has an \( A_{2k} \) singularity at \( t = 0 \).

Then \( k \leq w - \left\lfloor \frac{w}{3} \right\rfloor - 1 \).

The proof, based on argument principle, can be found also in [BZ3].

Now assume that there exists a curve of the form (4.1) having the A\( 2k \) singularity. Then we apply a change of singularities of multiplicity 2 to multiplicity 3. If a family of A\( 2k \) singularities specialises to a singularity (3; \( b \)) we would have found a complete criterion for the specialisation of singularities of multiplicity 2 to multiplicity 3.

**Conjecture 4.2.** If a family of A\( 2k \) singularities specialises to a (3; \( b \)) singularity then \( k \leq b - \left\lfloor \frac{b}{4} \right\rfloor - 1 \).

There is a very elementary approach to this conjecture, which could also lead to a complete answer in the case when a singularity with multiplicity \( p \) specialises to a singularity of multiplicity \( p + 1 \). Let us start with a following example.

**Example 4.3.** Let us consider a deformation
\[
x(t) = \varepsilon t^2 + t^3
\]
\[
y(t) = b_2(\varepsilon)t^2 + b_3(\varepsilon)t^3 + \ldots,
\]
Here \( x \) and \( y \) depend implicitly on \( \varepsilon \). Assume that \( b_i \)'s are chosen in such a way that for \( \varepsilon \neq 0 \) the resulting singularity is an \( A_{10} \) singularity. This amounts to the fact that we have
\[
y = c_2(\varepsilon)x + c_4(\varepsilon)x^2 + c_6(\varepsilon)x^3 + c_8(\varepsilon)x^4 + c_{10}(\varepsilon)x^5 + c_{11}(\varepsilon)x^{11/2} + \ldots.
\]
Substituting \( x \) from (4.2) into (4.3) yields
\[
y = c_2(\varepsilon^2 c_2 t^3) + \\
+ c_4(\varepsilon^2 t^4 + 2\varepsilon t^5 + t^6) + \\
+ c_6(\varepsilon^3 t^6 + 3\varepsilon^2 t^7 + 3\varepsilon t^8 + t^9) + \\
+ c_8(\varepsilon^4 t^8 + 4\varepsilon^3 t^9 + 6\varepsilon t^{10} + \ldots) + \\
+ c_{10}\varepsilon^5 t^{10} + \ldots
\]
Corollary 4.5. If a singularity

Remark 4.6. Some $c_i$’s can diverge infinity. Yet, for example, if $c_4(\varepsilon)$ is unbounded, then, since $b_9(\varepsilon)$ is bounded (otherwise $y$ would diverge), $\varepsilon^2 c_9$ must be unbounded so as to cancel the term $c_4(\varepsilon)$ at $t^6$. But then $\varepsilon c_9$ is unbounded, too. This contradicts the boundedness of $b_7$. So $c_4$ must be bounded. Using similar argument we can prove the following

Lemma 4.4. $\varepsilon c_k(\varepsilon)$ is bounded as $\varepsilon \to 0$.

Proof. Assume contrary. Let us take a subsequence $\varepsilon_n \to 0$ such that $\varepsilon c_9(\varepsilon_n) \to \infty$. To shorten the notation, we will call $c_k(\varepsilon_n)$ as $c_k^n$.

Let us pick $n$ sufficiently large and consider the terms $b_8(\varepsilon), b_9(\varepsilon)$ and $b_{10}(\varepsilon)$ written in the following way

$$
\begin{pmatrix}
3\varepsilon_n & 1 & 0 \\
\varepsilon_n^4 & 4\varepsilon_n^3 & 6\varepsilon_n^2 \\
0 & 0 & \varepsilon_n^6
\end{pmatrix}
\begin{pmatrix}
c_9^n \\
c_9^n \\
c_{10}^n
\end{pmatrix}
= 
\begin{pmatrix}
b_8^n \\
b_9^n \\
b_{10}^n
\end{pmatrix}
$$

We claim that the above $3 \times 3$ matrix must have non–trivial kernel. In fact, the leading terms in $\varepsilon c_9^n, \varepsilon^4 c_9^n$ and $\varepsilon^7 c_{10}^n$ are unbounded and must mutually cancel so that $b_8, b_9$ and $b_{10}$ stay bounded as $\varepsilon \to 0$.

The desired contradiction comes from the fact that

$$
D_{10} := 
\begin{pmatrix}
3 & 1 & 0 \\
1 & 4 & 6 \\
0 & 0 & 1
\end{pmatrix}
= 11 \neq 0.
$$

From this it follows that in the limit expansion $b_7(\varepsilon) \to 0$.

Corollary 4.5. If a singularity $A_{10}$ specialises to a singularity (3; $b$) then $b \geq 8$.

Remark 4.6. For $n = 2k$ let $l = \left\lfloor \frac{n+3}{2} \right\rfloor$. Consider the determinant

$$
D_n :=
\begin{vmatrix}
(k-l-1) & (k-l-1) & \ldots & (k-l-1) \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1
\end{vmatrix}
$$

Conjecture 4.2 will follow once we have proved that for all even positive $n$ we have $D_n \neq 0$. Using computer we were able to check this up to $n = 400$, but no general formula for $D_n$ has been found.

Example 4.7. The curve

$$
x = t^2 + t^3 \\
y = \frac{3}{4}t^6 + \frac{9}{4}t^7 + t^8
$$

Has an $A_{10}$ singularity at $t = 0$, so by rescaling we obtain a deformation $(2; 11) \to (3; 8)$. 

Now if all $c_2, c_4, c_8, c_{10}$ and $c_{10}$ stay bounded (from above) while $\varepsilon \to 0$ then, on passing to the limit $\varepsilon = 0$, all terms with $\varepsilon$ in a positive power will vanish. Then, the resulting singularity would be at least $(3;11)$. 
5. Analytic theory

In this section we will assume that for \( \varepsilon \neq 0 \) the Puiseux expansion of \( y_\varepsilon \) in powers of \( x_\varepsilon \) (see (1.2)) has the form

\[
y = c_1(\varepsilon)x^{q_1/p} + c_2(\varepsilon)x^{q_2/p} + c_3(\varepsilon)x^{q_3/p} + \ldots.
\]

Here \( q = q_1 < q_2 < q_3 < \ldots \) and the Puiseux coefficients between \( x^{q_i/p} \) and \( x^{q_{i+1}/p} \) are supposed to vanish. Note that Puiseux terms \( x^{q_i/p} \) are not necessarily essential, which is contrary to the convention we have been accepting in previous sections. The reason for this will clarify in the following.

Let us divide (5.1) by \( x^{q_1/p} \), differentiate both parts with respect to \( t \) and multiply it again by \( x^{1+q_1/p} \) (cf. [BZ2], proof of Lemma 3.2). We obtain

\[
\dot{y}x - \frac{q_1}{p}y\dot{x} = \frac{q_2 - q_1}{p}c_2(\varepsilon)x^{q_2/p} + \frac{q_3 - q_1}{p}c_3(\varepsilon)x^{q_3/p} + \ldots.
\]

Let us denote

\[
P_2(\varepsilon, t) = \dot{y}_\varepsilon x_\varepsilon - \frac{q_1}{p}y_\varepsilon \dot{x}_\varepsilon.
\]

As \( x_\varepsilon \sim t^p \) and so \( x^{q_2/p} \sim t^{q_2} \) we get that

\[
\text{ord}_{t=0} P_2(\varepsilon, t) \geq q_2 + p - 1,
\]

for \( \varepsilon \neq 0 \). The equality here holds for those \( \varepsilon \neq 0 \) such that \( c_2(\varepsilon) = 0 \).

But \( P_2(\varepsilon, t) \to P_2(0, t) \) for \( \varepsilon \to 0 \) uniformly in \( t \) (in some neighbourhood of 0). Therefore

\[
\text{ord}_{t=0} P_2(0, t) \geq q_2 + p - 1.
\]

Now let us put \( \varepsilon = 0 \) and regard the equation (5.3) as an ordinary differential equation for \( y_0 \). Solving it we get

\[
y_0 = x_0^{q_1/p} \left( \int_0^t P_2(0, s)x^{-q_1/p-1}ds + D \right),
\]

with \( D \) an integration constant.

**Lemma 5.1.** If \( q_1r/p \) is not an integer then \( D = 0 \).

**Proof.** In this case the r.h.s. of (5.5) is analytic in \( t = 0 \) iff \( D = 0 \). \( \square \)

**Lemma 5.2.** If \( q_1/p \) is integer, we can assume that \( D = 0 \).

**Proof.** If \( q_1/p = n \) we can apply the global change of coordinates \( y \to y - Dx^n \). \( \square \)

**Corollary 5.3.** If either \( q_1r/p \notin \mathbb{Z} \) or \( q_1/p \in \mathbb{Z} \) then

\[
\text{ord}_{t=0} y_0 \geq q_2 - (r - p).
\]

This gives some restrictions for possible Puiseux terms in the limit. We illustrate them in the following

**Example 5.4.** Assume that the multiplicity sequence of the singularity at \( \varepsilon \neq 0 \) is \((9; 17)\), so that the Puiseux expansion is

\[
y = c_1x^{9/9} + c_2x^{17/9} + \ldots.
\]

Assume that at \( \varepsilon = 0 \) the order of \( x \) is 10. Then, by Corollary 5.3, the order of \( y_0 \) at \( t = 0 \) is at least 16. It follows that the characteristic sequence of the specialised singularity is at least \((10; 16)\).
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The Milnor number of singularity (9; 17) is 128. The Milnor number of singularity (10; 16, 17) is 136. But the Milnor number of (10; 15, 16) is 130 > 128. Therefore the semicontinuity of the Milnor numbers does not exclude the possibility of degeneration of (9; 17) to (10; 15, 16).

Remark 5.5. One could be tempted to do the following trick in this case. If $y_\varepsilon = c_1(\varepsilon)x^{9/9} + c_2(\varepsilon)x^{17/9} + \ldots$, we apply changes $y \rightarrow y - c_1(\varepsilon)x$ so that the resulting new $y_\varepsilon$ has the order 17 as $t = 0$. What is wrong is that nothing can prevent $c_1$ from escaping to infinity as $\varepsilon \rightarrow 0$ (see Example 4.3). Then the new $y_\varepsilon$ cease to converge uniformly to $y_0$. There are examples (see Proposition 5.12) that the order of $y_0$ is precisely 16.

The above method admits further improvements. Let us take the equation (5.2). Let us divide both sides by $x^{q_2/p+1}\varepsilon^2$. We obtain

$$P_3(\varepsilon, t) = \frac{q_3 - q_1}{p} \frac{q_3 - q_2}{p} x^{q_3/p} + \ldots,$$

where

$$P_3(\varepsilon, t) = x\dot{x}P'_2 - \left(\frac{q_2}{p} \frac{q_3 - q_2}{p} x^2 + \varepsilon x\right)P_2.$$

Here and in the following $P'_2$ means $\frac{\partial}{\partial t}P_2(\varepsilon, t)$: we never differentiate w.r.t. $\varepsilon$.

We can repeat this procedure of dividing, differentiating and multiplying several times. The reader may easily verify the following formula valid for $n > 2$

$$P_n(\varepsilon, t) = \prod_{k=1}^{n-1} q_n - q_k \frac{c_n(\varepsilon)}{p} x^{2n-3}x^{q_n/p} + \ldots,$$

where $P_n$ are defined inductively by the formula

$$P_{n+1}(\varepsilon, t) = x\dot{x}P'_n - \left(\frac{q_n}{p} \frac{x^2}{p} + (2n-3)x\right)P_n.$$

An analogue of equation (5.4) is

$$\text{ord}_{t=0} P_n(\varepsilon, t) \geq q_n + (2n-3)(p-1).$$

The inequality for orders is valid for $\varepsilon \neq 0$ by virtue of (5.8) (since $\text{ord}_{t=0} x^{2n-3} = (2n-3)(p-1)$ and $\text{ord}_{t=0} x^{q_n/p} = q_n$). It holds also for $\varepsilon = 0$ because $P_n(\varepsilon, t) \rightarrow P_n(0, t)$ uniformly in $t$ (in some neighbourhood of $t = 0$) if $\varepsilon \rightarrow 0$.

As before we can treat the equation (5.9) as the ordinary differential equation with the known function $P_{n+1}(0, t)$ and unknown $P_n(0, t)$. We get the following solution

$$P_n(0, t) = x^{q_n/p} x^{2n-3} \left(\int_0^t P_{n+1}(0, s)\dot{x}(s)^{2n-2}x(s)^{-q_n/p-1}ds + D\right).$$

The condition that $P_n(0, t)$ is analytic near $t = 0$ implies the following

Lemma 5.6. If $q_nr/p$ is not an integer then $D = 0$. Moreover, if $q_n/p$ is integer, we can still perform a change of coordinates so that $D = 0$.

Proof. Only the second part of the proof requires some comments. If $q_n/p = k \in \mathbb{Z}$, we apply the change $y \rightarrow y - \dot{D}x^k$. Such a change induces, by virtue of formulae (5.3) and (5.9) the change $P_1 \rightarrow P_1 - D_kx^k\varepsilon \cdot x^{2l-3}$, where $D_k$ depends linearly on $\dot{D}$. It is now clear that picking a suitable $\dot{D}$ we can ensure that $D = 0$. □
**Proposition 5.7.** Assume that for all $i = 1, \ldots, n$ either $q_i/p \in \mathbb{Z}$ or $q_ir/p \not\in \mathbb{Z}$. Then we have

$$\text{ord}_{t=0} y_0 \geq \max_{i=2, \ldots, n} q_i - (2i - 3)(r - p).$$

**Proof.** If $D = 0$ in (5.11) then

$$\text{ord}_{t=0} P_n(t, 0) \geq \text{ord}_{t=0} P_{n+1}(t, 0) - 2(r - p).$$

The statement follows now from an easy induction on $n$. □

**Remark 5.8.** These (non)integrality assumptions are automatically satisfied if $r$ and $p$ are coprime.

In some cases Proposition 5.7 gives better estimates than the semicontinuity of Milnor number.

**Example 5.9.** In this example the main issue is that $r - p > 1$. The numbers occurring here are quite large, but the bound is about 20% better than the bound that could be obtained from semicontinuity of Milnor numbers.

Let the singularity at $ε \neq 0$ be $(32; 112, 5196, 5201)$. Its Milnor number is 39364. Moreover in the expansion (5.1) we have

$$q_1 = 32, q_2 = 64, \ldots, q_8 = 152, \ldots, q_{638} = 5192, q_{639} = 5196, q_{640} = 5200, q_{641} = 5201.$$ 

Suppose, that the multiplicity of singularity increases to $r = 35$. Therefore, by Proposition 5.7, the order of $y_0$ at $t = 0$ is greater or equal $q_{638} - 3 \cdot (2 \cdot 638 - 3) = 1373$. Note, in passing, that $q_{639} - 3 \cdot 1275 < 1373$. Hence the characteristic pair of the singularity at $ε = 0$ is $(35; b, \ldots)$ with $b \geq 1373$.

The Milnor number of the singularity $(35; 1373)$ is equal to 46648. Yet even the singularity $(35; 1158)$ has Milnor number greater than 39364. Hence the bound here is stronger than the Milnor bound.

**Remark 5.10.** The semicontinuity of spectral numbers gives here weaker result, namely $b \geq 1177$. This result is close to the Milnor number bound.

**Remark 5.11.** In Example 5.9 we had that $q_{639} - 3 \cdot 1275 \leq q_{639} - 3 \cdot 1273$. This means that our method provides the same bound for the order of $y_0$ in the case of singularity $(32; 112, 152, 5196, 5201)$ as in the case of $(32; 112, 152, 5196, 5197)$. In general, the method gives the better results, the larger is the difference $q_{i+1} - q_i$ compared to $2(r - p)$. In particular in case of a $2$–singularity degenerating to any $r$–singularity with $r \geq 3$, this method does not give anything interesting.

We shall prove one more proposition and give some example that will shed light on some phenomena occurring here.

**Proposition 5.12.** Assume that $q_2 - q_1 > r - p$. Then there exists a deformation as in (1.2) such that the order of $y_0$ is precisely $q_2 - (r - p)$.

**Proof.** Let $d = q_2 - q_1$ and $q = q_1$. Consider a vector space $V = V_x \oplus V_y$ of pairs of polynomials

$$x(t) = a_0 t^p + a_1 t^{p+1} + a_2 t^{p+2} + \cdots + a_{d+1} t^{p+d}$$

$$y(t) = b_0 t^q + b_1 t^{q+1} + b_2 t^{q+2} + \cdots + b_{d+1} t^{q+d}.$$
To simplify notation let us denote
\[ h_{m,n} = m - n \frac{q}{p}. \]
The conditions that \( \text{ord}_t = 0 \) \( P_i(t) = q_2 + p - 1 \) translate into a set of equations
\[ F_i(a, b) = \cdots = F_{d-1}(a, b) = 0 \neq F_d(a, b), \]
where
\[ F_i(a, b) = \sum_{i+j=l} h_{ij}a_ib_j. \]
For \( k \leq d \) define
\[ \Sigma_k = \{(a, b) \in V : F_1(a, b) = \cdots = F_k(a, b) = 0\}. \]

**Lemma 5.13.** Both \( \Sigma_k \)'s are smooth away from the set \( \{a_0 = 0\} \).

**Proof of Lemma 5.13.** The matrix of partial derivatives of function \( F = (F_1, \ldots, F_k) \) with respect to \( b \) variables is
\[
\begin{pmatrix}
  h_{10}a_1 & h_{01}a_0 & 0 & \cdots & \\
  h_{20}a_2 & h_{11}a_1 & h_{02}a_0 & 0 & \cdots \\
  h_{30}a_3 & h_{21}a_2 & h_{12}a_1 & h_{03}a_0 & 0 & \cdots \\
  \vdots & \vdots & \ddots & \ddots & \\
  h_{k,0}a_k & h_{k-1,1}a_{k-1} & \cdots & h_{0,k}a_0
\end{pmatrix}
\]
This is a \( k \times (k+1) \) matrix. If \( a_0 \neq 0 \) it is obvious that its rows are linearly independent. The lemma follows from the implicit function theorem.

The next thing we need is the structure of the set \( N_{d-1} = \Sigma_{d-1} \cap \{a_0 = 0\} \).

**Lemma 5.14.** The set \( N_{d-1} \) is an union of sets \( N_{d-1,0}, \ldots, N_{d-1,d-1} \), where
\[ N_{d-1,k} = \{a_0 = a_1 = \cdots = a_k = 0 = b_0 = \cdots = b_{d-2-k}\}. \]
In particular set \( N_{d-1} \) is a codimension one subset in \( \Sigma_{d-1} \).

**Proof.** We shall prove slightly more. Namely let
\[ N_{l,k} = \{a_0 = \cdots = a_k = 0 = b_0 = \cdots = b_{l-1-k}\}. \]
Then the set \( N_l = \Sigma_l \cap \{a_0 = 0\} \) is a sum of \( N_{l,k} \)'s for \( k = 0, \ldots, l \).

For \( l = 1 \) the statement is trivial. Assume it has been proved for \( l-1 \). Consider the equation \( F_1 = 0 \) on \( N_{l-1,k} \). From the definition of this space we infer (see (5.13)) that the only monomial in \( F_l \) that does not vanish identically is
\[ h_{k+1,0}a_kb_{l-k-1}. \]
In fact in all other \( a_ib_j \)'s either \( i \leq k \) or \( j \leq l - k - 2 \). It follows that
\[ N_{l-1,k} \cap \{F_l = 0\} = N_{l,k} \cap N_{l,k+1}, \]
and the induction step is proved.

**Corollary 5.15.** The set \( N_{d-1} = \Sigma_{d-1} \cap \{a_0 = 0\} \) is not contained in \( \Sigma_d \).

**Proof.** From (5.14) we infer that \( F_d \) does not vanish identically on any subset \( N_{d-1,k} \).\]
The rest of the proof is easy. Consider the set $N_{d-1,r-p-1}$ in $\Sigma_{d-1}$. There exists a small topological disk $D$ in $\Sigma_{d-1}$ omitting $\Sigma_d$ and intersecting $N_{d-1,r-p-1}$ in one point disjoint from $\Sigma_d$. Hence $a_{r-p} \neq 0$ at this point (otherwise $F_d = 0$).

Then this disk represents a specialisation of a singularity

$$y = c_1 x^{q_1/p} + c_2 x^{q_2/p} + \ldots$$

to the singularity with order of $x$ equal to $r$ and order of $y$ equal to $q_2 - (r - p)$ (because if $(a, b) \in N_{d-1,r-p-1}$ then $b_0 = b_1 = \cdots = b_{q_2 - q_1 - (r - p) - 1} = 0$ so the order of $y$ jumps to $q_2 - (r - p)$).

The above result answers one questions and makes us ask another one: when is the bound in Proposition 5.7 sharp? The example below will hopefully shed some light on the phenomena occuring in the problem.

Let $g$ be some unknown constant (indeedent of $\varepsilon$). Consider a deformation

$$x = \varepsilon^2 t^p + \varepsilon gt^{p+1} + t^{p+2}$$

$$y = b_0 t^q + b_1 t^{q+2} + \ldots$$

such that

$$P_2(\varepsilon, t) = h_{22} t^{q+p+3}.$$ 

According to Corollary 5.3 we should have $\text{ord}_{t=0} y_0 \geq q + 2$.

Let us write down equations for $y$ and see what happens

$$h_{01} \varepsilon^2 b_1 + h_{10} \varepsilon g b_0 = 0$$

$$h_{02} \varepsilon^2 b_2 + h_{11} \varepsilon g b_1 + h_{20} b_0 = 0$$

$$h_{03} \varepsilon^2 b_3 + h_{12} \varepsilon g b_2 + h_{21} b_1 = 0$$

$$h_{04} \varepsilon^2 b_4 + h_{13} \varepsilon g b_3 + h_{22} b_2 = h_{22}$$

(5.15)

Since all functions $b_i(\varepsilon)$ are supposed to be bounded as $\varepsilon \to 0$, from the last equation we deduce that $b_2 = 1 + O(\varepsilon)$.

The third equation implies that

$$b_1 = -g \frac{h_{12}}{h_{21}} \varepsilon + O(\varepsilon^2).$$

Next, from the second equation we infer that

$$b_0 = \frac{h_{11} h_{12} g^2 - h_{02} h_{21}}{h_{20} h_{21}} \varepsilon^2 + O(\varepsilon^3).$$

But now, unless

$$h_{10} h_{11} h_{12} g^3 - (h_{10} h_{02} h_{21} + h_{01} h_{20} h_{12}) g = O(\varepsilon).$$

(5.16)

the first equation cannot hold.

If it does not, we get a contradiction. The only assumption we have made is that $b_i(\varepsilon)$ remain bounded. So, in general, this assumption is false. It means that whenever we consider the $y_\varepsilon$ defined by

$$y_\varepsilon = x^{q_1/p} \int_0^t P_2(\varepsilon, s) x^{-q_1/p-1} ds,$$

the function $x_\varepsilon(t)$ and $P_2(\varepsilon, t)$ must be chosen so that $y_\varepsilon \to y_0$ uniformly in some neighbourhood of $t = 0$. Without additional assumptions this turns out not to be true. These additional assumption could, for example, be that the order of $P_2(\varepsilon, t)$ increases as $\varepsilon \to 0$. This would lead to the increment of the order of $y_0$ at $t = 0$. We
were not able to find in literature satisfactory criteria for the analytic dependence of solutions to (5.2) or (5.9) at \( \varepsilon = 0 \).

**Remark 5.16.** There is no contradiction between this example and Proposition 5.7. In fact, if \( g \) does not satisfy (5.16) and for a given \( x_\varepsilon \) we want to find \( y_\varepsilon \) such that \((x_\varepsilon, y_\varepsilon) \in N_3\) then the analysis of (5.15) implies that \( y_\varepsilon \) will pass to infinity, \((x_\varepsilon, y_\varepsilon)\) will approach a point in \( N_4 \).

6. **Conjectures**

In papers [Or] and [BZ3] a new invariant of the singularity has been introduced, namely the codimension of the singular point. Let us recall briefly its definition.

**Definition 6.1.** Let \((C, 0) \subset \mathbb{C}\) be a germ of a plane curve singularity. Let \( \pi : X \to \mathbb{C}^2 \) be the minimal resolution of the singularity of the curve \( C \). We denote by \( E \) the reduced exceptional locus of \( \pi \); by \( \tilde{C} \) the strict transform of \( C \) and by \( K \) the canonical divisor, i.e. the divisor associated to the holomorphic form \( \pi^* dx \wedge dy \), where \( x \) and \( y \) are the coordinates on \( \mathbb{C}^2 \).

The codimension of the singular point is the quantity

\[
\nu = K \cdot (K + \tilde{C} + E).
\]

It has been shown in [Or] for unibranched singularity and in [BZ3] in general case, that the codimension is the number of conditions for the given singularity in a suitably defined parameter space of curves. Given this interpretation we state the following

**Conjecture 6.2.** The codimension is semi-continuous with respect to deformation of singular points. More precisely, given a deformation like in Definition 1.2 if \( \nu_\varepsilon \) denotes the codimension of the singularity \( C_\varepsilon \) and \( \nu_0 \) the codimension of the singularity \( C_0 \) then

\[
\nu_\varepsilon \leq \nu_0,
\]

with the equality if and only if the deformation is equisingular on the whole disc.

This conjecture is motivated by the definition of the invariant \( \nu \): if we are given parameter space \( V_d \) of curves of degree \( d \) in \( \mathbb{CP}^2 \), its dimension is equal to \( \binom{d+2}{2} \).

For a singularity with codimension \( \nu \), with \( \nu \) small compared to \( d \), the space of degree \( d \) curves having among their singularities the given one, has codimension \( \nu \) in \( V_d \). The "number of conditions" should increase during the specialisation. But we have by now no idea of a possible proof of this conjecture.

**Remark 6.3.** A singularity \( A_{2k} \) has codimension \( k \). The singularity \( (t^3, t^b) \) with \( 3 \not| b \) has codimension \( b - \lfloor \frac{b}{3} \rfloor \). The results and conjectures in Section 4 stay in agreement with Conjecture 6.2.

Conjecture 6.2 apart of its own interest could have many applications. First of all, in case of \( p = 2 \) it would allow a linear bound for limit cycles bifurcating from origin in case of Liénard system, the one precisely conjectured by Christopher and Lynch [ChLy].

On the other hand, if we consider a polynomial curve in \( \mathbb{C}^2 \) with bidegree \((m, n)\) with \((m, n)\) coprime then such a curve, by a parameter change \( t \to \lambda, x \to \lambda^{-m}x, y \to \lambda^{-n}y \) (see discussion after Proposition 4.1), deforms to the curve \((t^m, t^n)\). Now the codimension of the \((m; n)\) singularity is equal to \( m + n - \left\lfloor \frac{m}{m} \right\rfloor - 3 \). The semicontinuity of codimensions would lead to a very strong bound for the sum of
codimensions of polynomial curves, similar to the conjectured one in [BZ1]. Similar bound is known to hold (see [BZ3]), but the sum of codimensions is bounded by the sum of degrees plus the number of self-intersections at finite distance, which itself can be proportional to the product \( mn \).

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References


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