COMPLEX ALGEBRAIC PLANE CURVES VIA POINCARÉ–HOPF FORMULA. III. CODIMENSION BOUNDS

MACIEJ BORODZIK AND HENRYK ŻOŁĄDEK

Abstract. This work is a continuation of the papers [BZ1] and [BZ2]. Here we prove some estimates for the sum of codimensions of singularities of affine planar rational curves.

1. Introduction

In [BZ1] and [BZ2] we classified complex planar affine curves C with $b^1 = 1$, i.e. the rational curves with one place at infinity and one self-intersection and the rational curves with two places at infinity and without self-intersections. There we used essentially the inequality $\mu \leq n\nu$ for the Milnor number μ of a cuspidal singularity

$$x = \tau^n, y = c_1\tau + c_2\tau^2 + \dots, \tag{1.1}$$

where the (intrinsic) codimension ν is the number of vanishing essential Puiseux coefficients c_i (see [BZ1]). Analogous bounds are used for other degenerations (at the infinity and at the self-intersection). The sum of the Milnor numbers, or of the δ-numbers, is calculated via the Poincaré–Hopf formula applied to a suitable Hamiltonian vector field. The orders n are estimated by the degree of the curve. The problem is to estimate the intrinsic codimension ν.

We introduced the so-called external codimension, which for the cuspidal singularity equals

$$ext\nu = n + \nu - 2; \tag{1.2}$$

in the next section we define the external codimension for other singularities. We conjectured in [BZ1] (Conjecture 3.7) and in [BZ2] (Conjecture 2.40) that the sum of external codimensions does not exceed the dimension of some naturally defined space of curves modulo equivalences. For instance we claimed that $\sum ext\nu \leq p + q - 4 - \left\lfloor \frac{q}{p} \right\rfloor$ in the case of polynomial lines $x = \varphi(t)$, $y = \psi(t)$, $\deg \varphi = p < \deg \psi = q$; here $p + q - 4 - \left\lfloor \frac{q}{p} \right\rfloor$ is the dimension of the space of such curves modulo some natural equivalences.

The problem of estimating the sum of codimensions of singularities of projective rational curves was considered also by other authors. In the works of S. Orevkov and M. Zaidenberg [OZ1], [OZ2], [Or] a notion of a rough M-number of singularity $\overline{M}$ was introduced via intersection numbers of some divisors in the resolution of the singularity. For the cuspidal singularity (1.1), when n is the multiplicity, the rough

---

Date: April 20, 2007.

1991 *Mathematics Subject Classification.* Primary 14H50; Secondary 32S05, 14H15, 14H20.

Key words and phrases. Affine algebraic curve, index of vector field, Puiseux expansion.

Supported by Polish MNiSz Grant 1 P03A 015 29.

M-number coincides with $ext\nu$. In Section 2 we generalize the Orevkov's definition to the case of reducible singularities. Using the BMY inequality one can prove the inequality (see [Or])

$$\sum \overline{M}_P \leq 3d-4, \tag{1.3}$$

where the sum runs over the singular points of a rational cuspidal projective curve $C \subset \mathbb{C}P^2$ of degree d. Since the dimension of the space of such curves (modulo automorphisms of $\mathbb{CP}^2$) is $3d-9$, the bound (1.3) is presumably not optimal.

In this paper we generalize the bound (1.3) to the cases of parametric lines and parametric annuli. In particular, we prove the bounds

$$ext\nu_{\text{inf}} + \sum \overline{M}_P \leq p+q-1-\left\lfloor\frac{q}{p}\right\rfloor + \#(\text{double points}) \tag{1.4}$$

for polynomial lines with the bi-degree (p,q) (Theorem 4.25), and

$$ext\nu_{\text{inf}} + \sum \overline{M}_P \leq p+q+r+s+1+\#(\text{double points}) \tag{1.5}$$

for curves of the form $x = t^p + a_1 t^{p-1} + \ldots + a_{p+r} t^{-r}$, $y = t^q + \ldots + b_{q+s} t^{-s}$ (with some restrictions, see Theorem 4.28). The above $ext\nu_{\text{inf}}$ is the codimension of a degeneration of the curve at infinity, defined in the next section.

Our results concern only rational curves. But in the case of curves with positive genus the codimensions of singularities behave very improperly. Namely, A. Hirano [Hir] constructed a series of curves C_n of degree $d = 2\cdot 3^n$ and with $s = \frac{9}{8}(9^n-1)$ simple cusps. Therefore the genus of C_n satisfies $g \leq \frac{1}{2}(d-1)(d-2) - s = \frac{7}{8}\cdot 9^n - 3^{n+1} + \frac{17}{8}$. On the other hand, the dimension of the space of curves $\mathcal{M}_{d,g}$ of degree d and genus g (modulo $Aut(\mathbb{CP}^2)$) is $\frac{1}{2}(d+1)(d+2) - \#$(double points) $- \dim GL(3,\mathbb{C}) = 3d-9+g$. For the curves C_n it equals $\dim \mathcal{M}_{d,g} = \frac{7}{8}\cdot 9^n + 3^{n+1} - \frac{55}{8}$ which is much smaller than the sum of codimensions $\sum ext\nu_z = s$.

We spent a lot of time trying to estimate $\sum ext\nu_P$ by the (essential) dimension of the corresponding space of curves using a kind of induction argument. However, the problem turned out very rigid; it can be reduced to showing that infinite number of some determinants do not vanish. Calculation of examples (see [BZ1], [BZ2] and Section 3) suggest that the sum of external codimensions of singularities of a rational curve is bounded as expected.

There exist other, sheaf theoretical, approaches to the problem of moduli of spaces of curves with given degree, genus, and types of singularities. There notions like logarithmic deformations and 0-dimensional schemes are used. We refer an interested reader to the works [FZ1], [FZ2], [GLS], [KlPi], [FLMN]. We tried to use the latter methods to our problem, but without a visible success.

The paper is organized as follows. In the next section we introduce definitions of the external codimensions and of the rough M-numbers. In Section 3 we discuss the problem of a bound for $\sum ext\nu_P$ and prove some positive results. In Section 4 we generalize the Orevkov–Zaidenberg results about the numbers $\overline{M}_P$ and prove the bounds (1.4) and (1.5). Section 5 is devoted to an application of the inequality (1.4) to a special version of the XVIth Hilbert problem about the number of limit cycles for polynomial planar vector fields..

2. The local codimension and the rough M-number of a singular point

2.1. **Cuspidal singularity.** Let $(C, 0)$ be a germ of an analytic curve in $(\mathbb{C}^2, 0)$, singular at 0. We assume firstly that the singularity is cuspidal, i.e. that the curve has one branch.

Let us fix a coordinate system (x, y) in $\mathbb{C}^2$ and assume that $C \neq \{x = 0\}$. Then the curve can be written in the form

$$x = \tau^n, \quad y = c_1\tau + c_2\tau^2 + \dots, \quad \tau \in (\mathbb{C}, 0). \tag{2.1}$$

The form (2.1) is called the *standard Puiseux expansion of* C. We rewrite (2.1) in the following *topologically arranged Puiseux expansion*

$$\begin{aligned} y &= x^{m_0}(d_0 + \dots) + x^{\tilde{m}_1/n_1}\left(d_1 + \dots + x^{\tilde{m}_r/n_1\dots n_r}\left(d_r + \dots\right)\dots\right) \\ &= x^{m_0}(d_0 + \dots) + x^{m_1/n_1}(d_1 + \dots) + \dots + x^{m_r/n_1\dots n_r}(d_r + \dots) \end{aligned} \tag{2.2}$$

where $\tilde{m}_j \geq 1$ and $n_j \geq 2$ are integers such that $\gcd(\tilde{m}_j, n_j) = 1$ for $j \geq 1$ and the coefficients $d_j \neq 0$ for $j \geq 1$. The first polynomial term $x^{m_0}(d_0 + \dots)$ may be absent (it is inessential). The dots in the $j-$th summand mean terms $x^{k/n_1\dots n_j}$. We have $n = n_1 \dots n_r$. The coefficients $d_1, \dots, d_r$ indicated above are called the *essential Puiseux quantities.* The coefficient d_0 and those in the dots are non-essential (provided $d_1 \dots d_r \neq 0$).

The topological type of the singularity is uniquely determined by the *characteristic pairs* (m_j, n_j). In particular, the Milnor number equals

$$\mu = \sum_{j=1}^{r}(m_j n_{j+1}\dots n_r - 1)(n_j - 1)n_{j+1}\dots n_r$$

(see [BZ1]).

If we fix the $x-$order $n = \operatorname{ord}_x C > 1$ and consider the space $\mathcal{H}$ of germs (2.1) then the corresponding equisingularity stratum $\mathcal{H}_i(\mu) \subset \mathcal{H}$ (stratum with $\mu = \text{const}$ containing C) is defined by a series of equalities of the form $C_j = 0$ and equations $C_k \neq 0$. The number ν of equalities is called the *$y-$codimension* of the stratum $\mathcal{H}_i(\mu)$ and of the singularity $(C, 0)$.

Lemma 2.1. *(*[BZ1], [Or]*) We have*

$$\nu = \sum_{j=1}^{r}\left(m_j - 1 - \left\lfloor\frac{m_j - 1}{n_j}\right\rfloor\right) = \sum_{j=1}^{r}\left(\tilde{m}_j n_{j+1}\dots n_r - 1 - \left\lfloor\frac{\tilde{m}_j}{n_j}\right\rfloor\right),$$

where $\lfloor a \rfloor$ denotes the integer part of the number a. Note that $\tilde{m}_j/n_j$ are not integers.

Proof. We have $m_1 - 1$ terms x^{j/n_1} before x^{m_1/n_1} and $\lfloor m_1/n_1 \rfloor$ of them are non-essential (integer exponents). Next, we have $m_2 - 1 = m_1 n_1 + m_2 - 1$ terms $x^{j/n_1 n_2}$ before $x^{m_2/n_1 n_2}$, where $\lfloor m_2/n_2 \rfloor$ of them are of the form x^{j/n_1}. Similarly we count the terms $x^{j/n_1\dots n_k}$ for $k > 2$. □

Definition 2.2. The *external codimension of the singularity* $(C, 0)$ associated with the coordinate system (x, y) is

$$ext\nu = (n - 2) + \nu.$$

Here $n - 1$ is the number of vanishing derivatives of $x(\tau)$ and we extract 1 because the position τ_0 of the singularity may vary.

Example 2.3. For the curve $x=\tau^4$, $y=\tau^8+\tau^{10}+\tau^{11}$ the $y-$codimension is $\nu=7$. Indeed, we require $c_1=c_2=c_3=c_5=c_6=c_7=c_9=0$. The external codimension equals $ext\nu=2+7=9$.

Let us now forget about the fixed coordinate system. If the singular germ $(C,0)$ is cuspidal then there exists a local holomorphic coordinate system $\tilde{x},\tilde{y}$ such that

$$\tilde{x}=\tau^n,\quad \tilde{y}=\tau^m+\dots \tag{2.3}$$

where $1<n<m$, $m\neq 0 \pmod n$ and $n=\mathrm{mult}_0 C$ is called the *multiplicity of* C at 0; if C is defined by an equation $F(x,y)=0$ then $\mathrm{mult}_0 C$ is the degree of the first term in the Taylor expansion of F at 0. We have an expansion like in (2.2), i.e.

$$\tilde{y}=\tilde{x}^{\tilde{m}_1/n_1}\left(d_1+\dots+\tilde{x}^{\tilde{m}_2/n_1n_2}\left(d_2+\dots+\tilde{x}^{\tilde{m}_r/n_1\dots n_r}\left(d_r+\dots\right)\dots\right)\right), \tag{2.4}$$

where $1<n_1<m_1=\tilde{m}_1$.

Definition 2.4 ([Or])**.** The *rough M-number* of the singularity $(C,0)$ equals

$$\overline{M}=(\mathrm{mult}_0 C-2)+\sum_{j=1}^{r}\left(\tilde{m}_j n_{j+1}\dots n_r-1-\left\lfloor\frac{\tilde{m}_j}{n_j}\right\rfloor\right).$$

Lemma 2.5. *If (x,y) is a fixed coordinate system then for a singular curve of the form* (2.2) *we have $\overline{M}\leq ext\nu$. The equality holds only when $n\leq m=m_1n_2\dots n_r$.*

Proof. If $n=\mathrm{ord}_x C\leq m$ then clearly $\overline{M}=ext\nu$. Assume that $1<m<n$ and denote $y_1=y-x^{m_0}(d_0+\dots)$. Inverting the expansion (2.2) we get $x=y_1^{n_1/m_1}\left(d_1'+\dots y_1^{\tilde{m}_2/m_1n_2}\left(d_2'+\dots+y_1^{\tilde{m}_r/m_1n_2\dots n_r}\left(d_r'+\dots\right)\dots\right)\right)$.

Let $m_1>1$. Lemma 2.1 gives $\overline{M}=(m-2)+(n_1\dots n_r-1-\lfloor n_1/m_1\rfloor)+\sum_{j\geq 2}(\tilde{m}_j n_{j+1}\dots n_r-1-\lfloor\tilde{m}_j/n_j\rfloor)=ext\nu-\lfloor n_1/m_1\rfloor$.

If $m_1=1$ then $\overline{M}=ext\nu-(n-m)$. □

We see that $\overline{M}<ext\nu$ always when $m<n$. For example, for the curve $x=\tau^4$, $y=\tau^2+\tau^5$ we have $ext\nu=(4-2)+(5-1-\lfloor 5/4\rfloor)=5$, and after the change $\tilde{x}=y$, $\tilde{y}=y^2-x=2\tau^7+\dots$, we find $\overline{M}=(2-2)+(7-1-\lfloor 7/3\rfloor)=3$.

2.2. **Two branches.** Let the germ $(C,0)$ consists of two branches, $C=A+B$.

Let us fix the coordinate system, and let $n(A)$ and $n(B)$ be the $x-$orders of A and B respectively, i.e.

$$\begin{array}{l} A: x=\tau^{n(A)},\quad y=d_1\tau+d_2\tau^2+\dots \\ B: x=\iota^{n(B)},\quad y=e_1\iota+e_2\iota^2+\dots \end{array} \tag{2.5}$$

Definition 2.6. The *$y-$codimension* $\nu=\nu(A+B)$ of the singularity $(A+B,0)$ is the number of conditions of the form $d_i=0$, $e_j=0$ or $d_i=e_j$ that appear in the definition of the equisingularity stratum (containing $A+B$) in the space of germs of the form (2.5). The *external codimension* of this singularity is

$$ext\nu=(n(A)+n(B)-2)+\nu(A+B).$$

Remark 2.7. We can write

$$\nu(A+B)=\nu(A)+\nu(B)+\nu_{\tan}(A,B),$$

where $\nu(A)$ and $\nu(B)$ are the $y-$codimensions of A and B, and the *tangency codimension* $\nu_{\tan}(A,B)$ is the number of conditions $d_i = e_j$ that do not result from $d_i = 0$, $e_j = 0$.

Note also that on writing the equations $d_i = e_j$, we must properly choose the branches of the rational powers x^{α}; it is done in a way that the common part of the Puiseux series for the two branches is the longest possible.

Example 2.8. If $A : x = \tau^4, y = \tau^6 + \tau^7$ and $B : x = \iota^6, y = 2\iota^9 + \iota^{11}$ then $\nu(A) = 4$ (as $d_1 = d_2 = d_3 = d_5 = 0$), $\nu(B) = 8$ (as $e_1 = e_2 = e_3 = e_4 = e_5 = e_7 = e_8 = e_{10} = 0$) and $\nu_{\tan}(A,B) = 1$ (as $e_4 = d_6$).

If A is as before and $B : x = \iota^6, y = \iota^9 + \iota^{11}$ we have $\nu_{\tan}(A,B) = 2$.

Lemma 2.9. *Consider the longest possible common part of the Puiseux expansions of the branches A and B represented in the topologically arranged form*

$$y = x^{l_1/k_1}\left(f_1 + \ldots x^{\tilde{l}_2/k_1k_2}\left(f_2 + \ldots + x^{\tilde{l}_{s-1}/k_1\ldots k_{s-1}}\left(f_{s-1} + \ldots\right)\ldots\right)\right), \tag{2.6}$$

$\gcd(\tilde{l}_j, k_j) = 1$, *and let the next terms be* $C_{A,B}x^{l_s/k_1\ldots k_s}$, $l_s = \tilde{l}_1k_2\ldots k_s + \ldots + \tilde{l}_s$, $C_A \neq C_B$. *Then we have*

$$\nu_{\tan}(A,B) = \left(\sum_{i=1}^{s}\left\lfloor\frac{\tilde{l}_i - 1}{k_i}\right\rfloor\right) + s - 1.$$

Proof. Firstly we note that above it is possible that $k_1 = 1$ or $k_s = 1$. The vanishing essential coefficients in (2.6), i.e. those before x^{l/k_1}, $l < l_1$, or before x^{l/k_1k_2}, $l < l_1k_1$, etc., are not counted. The non-essential coefficients (vanishing and non-vanishing) are taken into account. There are $\left\lfloor(\tilde{l}_1 - 1)/k_1\right\rfloor$ of them before x^{l_1/k_1}, $\left\lfloor\tilde{l}_2/k_2\right\rfloor = \left\lfloor(\tilde{l}_2 - 1)/k_2\right\rfloor$ of them between f_1 and $x^{\tilde{l}_2/k_1k_2}$, etc. Finally we have $s-1$ essential coefficients $f_1, \ldots, f_{s-1}$. □

For now we leave a fixed coordinate system. We define the *multiplicity* $n = \text{mult}_0 C$ of a germ $C = A + B$ as the order of the first nonzero term in the Taylor expansion at 0 of the function F defining C. Choose a local coordinate system $\tilde{x}, \tilde{y}$ such that $\text{ord}_{\tilde{x}} A = \text{mult}_0 A$, $\text{ord}_{\tilde{x}} B = \text{mult}_0 B$, thus $n(A) + n(B) = \text{mult}_0 C$.

Definition 2.10. The *rough M-number* of the singularity $(A + B, 0)$ is defined by the formula

$$\overline{M} = (\text{ord}_0 C - 2) + \nu(A) + \nu(B) + \nu_{\tan}(A,B),$$

where $\nu(A)$ and $\nu(B)$ are the corresponding $\tilde{y}-$codimensions.

2.3. **Several branches.** Let the curve $(C,0)$ consist of k branches, $C = C_1 + \ldots + C_k$. Denote $C' = C_1 + \ldots + C_{k-1}$.

Definition 2.11. If the coordinate system (x,y) is fixed, the *$y-$codimension* and the *external codimension* of the singularity $(C,0)$ (with respect to this system) are defined by

$$\nu(C) = \nu(C') + \nu(C_k) + \max_{1\le j\le k-1}\nu_{\tan}(C_j, C_k), \tag{2.7}$$

$$ext\nu(C) = \left(\sum n(C_i) - 2\right) + \nu(C), \tag{2.8}$$

where $n(C_i)$ are the $x-$orders of C_i. We observe the recurrent relation

$$ext\nu(C) = ext\nu(C') + ext\nu(C_k) + \max_{1\leq j\leq k-1} \nu_{\tan}(C_j, C_k) + 2. \tag{2.9}$$

The *rough M-number* of the singularity $(C, 0)$ is defined as

$$\overline{M} = (\mathrm{mult}_0 C - 2) + \nu(C),$$

where $\mathrm{mult}_0 C$ is the multiplicity of C and $\nu(C)$ is the $\tilde{y}-$codimension of C and $\tilde{x}, \tilde{y}$ is the coordinate system such that $\mathrm{ord}_{\tilde{x}} C_j = \mathrm{mult}_0 C_j$. (This definition of the rough M-number, as well as that from Definition 2.6, differs slightly from a definition suggested by Orevkov in [Or]; see also Section 4.)

Proposition 2.12. *$ext\nu(C)$ does not depend on the ordering of the branches C_1, $\ldots, C_k$.*

Proof. It is sufficient to show that if we switch C_{k-1} with C_k, the codimension $ext\nu(C)$ does not change. We will use the following lemma, which trivially results from Lemma 2.9.

Lemma 2.13. *If A, B and C are three branches of one singular point and we have $\nu_{tan}(A, C) < \nu_{tan}(A, B)$ then $\nu_{tan}(A, C) = \nu_{tan}(B, C)$.*

Denote $\nu_{rs} = \nu_{tan}(C_r, C_s)$. It is sufficient to prove the formula

$$\max_{\substack{j\in\{1,\ldots,k-2\}\\ l\in\{1,\ldots,k-1\}}} \nu_{j,k-1} + \nu_{l,k} = \max_{\substack{j\in\{1,\ldots,k-2\}\\ l\in\{1,\ldots,k-2,k\}}} \nu_{j,k} + \nu_{l,k-1},$$

which corresponds to the transposition $(k-1, k)$. If $\nu_{k,k-1}$ is smaller or equal to $\max \nu_{j,k}$ and $\max \nu_{l,k-1}$, for $j, l \leq k-2$, we are clearly done. So assume $\nu_{k,k-1} > \nu_{k,j}$ for all $j \leq k-2$. Then, by Lemma 2.13, $\nu_{k,j} = \nu_{k-1,j}$. This proves the proposition. □

Example 2.14. If the branches C_j are smooth and pairwise transversal then there are $k-2$ conditions that $C_3, \ldots, C_k$ pass through the intersection $C_1 \cap C_2$.

Remark 2.15. Formula (2.9) deserves special attention if C_k is a smooth branch tangent to other branches. By (2.7), in turns out that (2.9) is still valid, provided we define the external codimension of the smooth branch (at a singular point) to be -1.

In [Or] Orevkov proposed the following

Conjecture 2.16. *The sum of rough M-numbers of a rational curve C in $\mathbb{CP}^2$ does not exceed the dimension of the space of such curves (modulo $Aut(\mathbb{CP}^2)$), i.e. $3\deg C - 9$.*

Example 2.17. Consider the *quasi-homogeneous curve*

$$C_0 : x^q = y^p,$$

where $1 < p < |q|$ and $\gcd(p, q) = 1$. If $q > p + 1$ then this curve has two singular points, denoted by 0 and ∞, with the rough M-numbers $\overline{M}_0 = p+q-3-\lfloor q/p \rfloor$ and $\overline{M}_\infty = 2q-p-3-\lfloor q/(q-p) \rfloor$; thus $\overline{M}_0+\overline{M}_\infty = 3\deg C_0 - 6 - \lfloor q/p \rfloor - \lfloor q/(q-p) \rfloor \leq 3\deg C_0 - 9$. If $q < 0$, the curve has two singularities with the sum of the rough M-numbers equal $3\deg C_0 - 8 - \lfloor q/p \rfloor$.

2.4. **Subtle codimensions.** The notion of the subtle codimension is very useful when we have a singularity given in a parametric form, with fixed orders of branches. This happens, for instance, when we are dealing with degeneracies at infinity. In fact, assume a curve is given by a pair of polynomials $x(t), y(t)$ of bidegree (p, q), ($q > p$, $q \neq kp$ for $k \in \mathbb{Z}$). Then at infinity not only the order of $u(t) = x/y$, but also of $w(t) = 1/y$ is determined by (p, q).

Definition 2.18. Let us fix two positive integers n and m, not necessarely distinct. Consider the space $\mathcal{H}_{n,m}$ of germs of parametric curves of type

$$x = \tau^n \qquad y = \tau^m + c_1\tau^{m+1} + \dots, \qquad \tau \in (\mathbb{C}, 0). \tag{2.10}$$

Then, if a given unibranched singularity C can be written in the form (2.10), we can consider the equisingularity stratum $\mathcal{H}_{n,m}(C) \subset \mathcal{H}_{n,m}$ containing C. By a *subtle codimension* ν' (with respect to (n, m)) we mean $\operatorname{codim} \mathcal{H}_{n,m}(C) \subset \mathcal{H}_{n,m}$.

Remark 2.19. The subtle codimension for one branch can be expressed by the codimension by the obvious formula

$$\nu' = \nu - \left(m - 1 - \left\lfloor \frac{m-1}{n} \right\rfloor\right). \tag{2.11}$$

Remark 2.20. If C is presented in the form

$$y = x^{m/n} + c_1 x^{(m+1)/n} + \dots$$

then ν' counts the vanishing essential Puiseux term in this expansion.

Now let us try to extend the definition of the subtle codimension to the case of singularities with more branches. Similarly as in previous subsections, we have first to define the subtle tangency codimension.

Let A and B be two branches of a singularity parametrised similarly to (2.5):

$$\begin{aligned} A:\quad & x = \tau^{n(A)}, \qquad y = d_0\tau^{m(A)} + d_1\tau^{m(A)+1} + d_2\tau^{m(A)+2} + \dots \\ B:\quad & x = \iota^{n(B)}, \qquad y = e_0\iota^{m(B)} + e_1\iota^{m(B)+1} + e_2\iota^{m(B)+2} + \dots, \end{aligned} \tag{2.12}$$

where $e_0 d_0 \neq 0$.

Definition 2.21. The *subtle codimension* $\nu' = \nu'(A + B)$ (with respect to $n(A)$, $n(B)$, $m(A)$ and $m(B)$) of the singularity $(A + B, 0)$ is the number of conditions $d_i = 0$, $e_j = 0$ $(i, j \geq 1)$ and $d_i = e_j$ $(i, j \geq 0)$ that appear in the definition of the equisingularity stratum of $A + B$ in the space of germs (2.12). The *subtle tangency codimension* is the number of conditions of the form $d_i = e_j$ that do not result from $d_i = 0$ and $e_j = 0$. In other words

$$\nu'_{tan}(A, B) = \nu'(A + B) - \nu'(A) - \nu'(B) \tag{2.13}$$

The subtle tangency codimension influences the intersection index of branches A and B as it has already been shown in [BZ1].

Example 2.22. If $n(B)m(A) - n(A)m(B) \neq 0$ the intersection index of the branches A and B does not depend on e's and d's, provided $d_0 e_0 \neq 0$. The subtle tangency codimension is then equal to 0.

The following lemma is a direct consequence of Definition 2.21

Lemma 2.23. *If $n(B)m(A) - n(A)m(B) = 0$ and we consider the common part of the Puiseux expansions of A and B*

$$y = c_0 x^{\frac{m(A)}{n(A)}} + c_1 x^{\frac{m(A)+1}{n(A)}} + \cdots + c_s x^{\frac{m(A)+s}{n(A)}} \tag{2.14}$$

then the subtle tangency codimension is the number of essential terms in (2.14)*.*

Now we are ready to define the subtle codimension for singularities of arbitrary number of branches. The formula is recursive as in Definition 2.11.

Definition 2.24. Let $C = C_1 + \cdots + C_k$ be a singular point with k branches and $C' = C_1 + \cdots + C_{k-1}$. The subtle codimension of C is

$$\nu'(C) = \nu'(C') + \nu'(C_k) + \max_{1 \le j \le k-1} \nu'_{\tan}(C_j, C_k).$$

The arguments of the proof of Proposition 2.12 are valid also in the subtle case. Hence the subtle codimension is well–defined.

Remark 2.25. The notion of the subtle codimension of multiple branches is, at the first insight, quite artificial. However it turns out to be very useful in the estimates. One can compare for example Proposition 2.11, and 2.17 from [BZ1] in which the subtle codimension plays a crucial role.

2.5. **Parametric lines.** A general rational curve C in the affine plane can be written in the form $x = \varphi(t)$, $y = \psi(t)$ with rational functions φ, ψ. Let $s_1, \ldots, s_M$ be the poles of the vector–valued function $\xi(t) = (\varphi, \psi)(t)$ and let $\left(p^{(1)}, q^{(1)}\right), \ldots,$ $\left(p^{(M)}, q^{(M)}\right)$ be the corresponding orders of poles, i.e. $\max\left(p^{(j)}, q^{(j)}\right) > 0$ for each point s_j. Usually, we consider a whole space *Curv* of such curves with fixed positions and order of poles.

The curves can be transformed using:

- changes of the parameter t,
- Cremona transformations of the plane.

Therefore some restrictions onto the above data $\left(s_j, p^{(j)}, q^{(j)}\right)$ are imposed. We describe them in two cases, considered in [BZ1] and [BZ2].

In this subsection we consider (topological) immersions of $\mathbb{C}$ (or the parametric lines), thus

$$M = 1.$$

So we set $s_1 = \infty$ and hence φ and ψ are polynomials of degree p and q, respectively. Applying *elementary transformations* of the form $(x, y + P(x))$ or $(x + Q(y), y)$ we can assume that either $\psi(t) \equiv 0$ (further we do not consider this case), or

$$0 < p < q, \quad q/p \notin \mathbb{Z}. \tag{2.15}$$

Such curves form an affine space $Curv = Curv_{p;q}$. The changes $t \to \alpha t + \beta$, $x \to \gamma x + \delta$, $y \to \epsilon y + P(x)$, $\deg P \le \lfloor q/p \rfloor$, generate the group of equivalences $Eq = Eq_{p;q}$ which acts on *Curv*. The dimension of the space $Curv/Eq$ is

$$\sigma := \dim Curv - \dim Eq = p + q - 4 - \lfloor q/p \rfloor\,. \tag{2.16}$$

(We do not consider the problem of existence and of structure of this quotient).

Note that, because of the choice (2.15), we distinguished one special coordinate system (x, y).

A curve $\xi \in Curv$, $\xi(t) = (t^p + \ldots, t^q + \ldots)$, has its Puiseux expansion at infinity

$$y = x^{q/p} + c_1^{(\infty)} x^{(q-1)/p} + \ldots$$

Definition 2.26. The external codimension $ext\nu_\infty = ext\nu_\infty(C)$ of the degeneration at $t=\infty$ is the number of vanishing essential Puiseux coefficients $c_j = c_j^{(\infty)}$ in the latter expansion. We shall also use the notation $ext\nu_{\inf} = ext\nu_\infty$. If C has one branch at infinity, this is the subtle codimension of the singularity of C at infinity (see Remark 2.20).

Note that the finite dimensional space $Curv$ contains *non-primitive curves* (or *multiply covered curves*), i.e. the curves ξ of the form $\xi(t) = \tilde{\xi}\circ\omega(t)$, where $\tilde{\xi}$ is a polynomial immersion of $\mathbb{C}$ into the plane and $\omega : \mathbb{C}\to\mathbb{C}$ is a polynomial of degree > 1. Such curves have singularities of infinite codimension. We denote by *Mult* the subspace of non-primitive curves (in [BZ1] it was denoted by $\Sigma_\infty^{\text{sin}}$).

We have the following

Conjecture 2.27. *For any non-primitive curve from $Curv_{p;q}$ the sum of external codimensions of its singularities does not exceed $\sigma+1$.*

The equality can hold only for curves of the form $x=\prod(t-t_j)^{n_j}$, $y=\prod(t-t_j)^{m_j}\tilde{\psi}(t)$, $m_j,n_j>0$ after putting the self-intersection point to $x=y=0$.

Example 2.28. (a) The space $Curv_{p;q}$, $\gcd(p,q)=1$, contains the quasi-homogeneous curve $\xi_0 : x=t^p, \quad y=t^q$, and the curves equivalent to it. For this curve we have $ext\nu_\infty=0$ and $ext\nu_0 = p+q-3-\lfloor q/p\rfloor$ and that is larger than σ. The latter fact can be explained by the property that $C_0=\xi_0(\mathbb{C})$ is invariant with respect to a one parameter subgroup of the group of automorphisms of $\mathbb{C}^2$.

(b) Consider the curve $x=t^2(1-t)^6$, $y=t^2(1-t)^8(1+2t)$ from $Curv_{8;11}$. Near $t=0$ we have $x=t^2(1-6t+\ldots)$ and $y=t^2(1-6t+\ldots)$, so $c_1=c_3=0$. Near $s=1-t=0$ we have $x=s^6(1-2s+\ldots)$ and $y=3s^8(1-\frac{8}{3}s+\ldots)$ and hence $c_1=c_2=c_3=c_4=c_5=c_7=c_9=0$. It follows that $ext\nu_0=(2+6-2)+2+7=15$, whereas $\sigma=14$.

Remark 2.29. In [BZ1] we proposed a stronger conjecture: $\sum ext\nu_z\le\sigma$ (see Conjecture 3.7 in [BZ1]). We classified the parametric lines with $b^1=1$ under the latter hypothesis. Example 2.28 shows that the latter conjecture is not true. But it turns out that no new curves obeying Conjecture 2.27 arise in this classification.

Namely, the case with $x=t^\alpha(1-t)^\beta$, $y=t^\gamma(1-t)^\delta\tilde{\psi}(t)$ is treated in [BZ1] separately; especially when $\deg\tilde{\psi}=1$. Some slight improvement in that analysis shows that there are no new cases of lines with one self-intersection.

2.6. **Parametric annuli** : $M=2$. (We follow [BZ2].) Assuming the poles to be at $t=0$ and $t=\infty$ the components φ,ψ are Laurent polynomials

$$\varphi = t^p+\alpha_1 t^{p-1}+\ldots+\alpha_{p+r}t^{-r},\quad \psi = t^q+\beta_1 t^{q-1}+\ldots+\beta_{q+s}t^{-s}. \tag{2.17}$$

If we apply a suitable Cremona transformation and, possibly, change $t\to 1/t$, we can assume that the curve is of one of the following four types.

Definition 2.30. A curve given by (2.17) is of *type* $\binom{+}{+}$ if

$$0<p<q,\ \ 0<r<s\quad \gcd(r,s)\le\gcd(p,q)\quad \min\left(\frac{q}{p},\frac{s}{r}\right)\notin\mathbb{Z};$$

the curve is of *type* $\binom{-+}{+-}$ if

$$0<q<p,\ \ 0<r<s,\quad \text{and } p+r\le q+s;$$

it is *of type* $\binom{-}{+}$ if

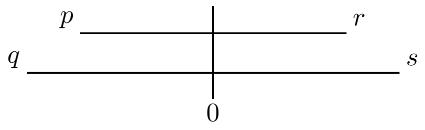
$$0 < -r \leq p, \;\; q > 0, \;\; s > 0, \text{ and } \frac{q}{p} \notin \mathbb{Z};$$

it is *of type* $\binom{-}{-}$ if

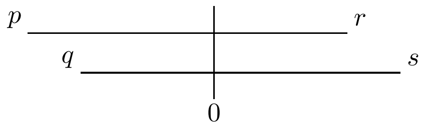
$$0 < -r \leq p, \quad 0 < -q \leq s, \text{ and } p + r \leq q + s.$$

Graphically we can present these types like that

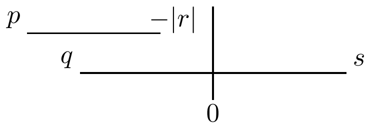
Type $\binom{+}{+}$:



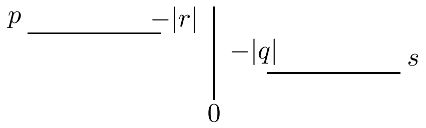
Type $\binom{+-}{-+}$:



Type $\binom{+}{-}$:



Type $\binom{+-}{-+}$:



The dimension of the space $Curv/Eq$ equals

$$\sigma := \dim Curv/Eq = p + r + q + s - 1 - \varepsilon - k, \tag{2.18}$$

where $\varepsilon = 2$ for type $\binom{+}{+}$ and type $\binom{-+}{+-}$, $\varepsilon = 1$ for type $\binom{-}{+}$ and $\varepsilon = 0$ for type $\binom{-}{-}$ and $k := \min\left(\left\lfloor\frac{q}{p}\right\rfloor, \left\lfloor\frac{s}{r}\right\rfloor\right)$ for type $\binom{+}{+}$, $k := \left\lfloor\frac{q}{p}\right\rfloor$ for type $\binom{-}{+}$ and $k = 0$ for types $\binom{-+}{+-}$ and $\binom{-}{-}$.

Definition 2.31. We define $ext\nu_0$ and $ext\nu_\infty$ exactly like in Definition 2.26, i.e. via the Puiseux expansions $y = y(x)$ at $t \to 0$ and at $t \to \infty$. We define the tangency codimension $\nu_{\tan}$ as the corresponding number of equal initial terms in these two Puiseux expansions, analogously like in Remark 2.7; in particular, $\nu_{\tan} = 0$ when $ps \neq rq$. Finally we put

$$ext\nu_{\inf} = ext\nu_0 + ext\nu_\infty + \nu_{\tan}.$$

as the *external codimension of* C *at infinity.*

In [BZ2, Conjecture 2.40] we stated the following

Conjecture 2.32. *For any non-primitive algebraic annulus of one of the types described in Definition 2.30 the sum of external codimensions of its local degenerations does not exceed* $\sigma = \dim Curv/Eq$.

3. BOUNDS FOR THE EXTERNAL CODIMENSIONS

3.1. **Regularity of sequences of Puiseux.** The problem of estimating the sum of external codimensions of several singular points of an affine rational curve can be reduced to the problem of regularity of some sequences of regular functions on suitably defined spaces of curves.

Definition 3.1. Let Z be a normal quasi-projective complex variety and let f_1, f_2, …, $f_k \in \mathbb{C}[Z]$ be a sequence of regular functions on Z. We say that this sequence is *regular at* $x_0 \in Z$ if any f_j, $j \leq k$, is not a zero divisor in the ring $\mathcal{O}_{x_0}/(f_1, \cdots, f_{j-1})$. (Here $\mathcal{O}_{x_0}$ is the local ring of germs at x_0 of holomorphic functions on Z.)

Therefore each variety $V_j = \{f_1 = \cdots = f_j = 0\}$ has codimension exactly j (if it is not empty). In particular, $V_{n+1} = \varnothing$, $n = \dim Z$, and we can assume that $k \leq \dim Z + 1$.

In the standard definition of regular sequence, see [GrHa], one requires that the number of functions equals $\dim Z$ and that the f_j belong to the maximal ideal of $\mathcal{O}_{x_0}$. In the sequel we shall assume that either all f_j vanish at x_0 or that $f_1(x_0) = \cdots = f_{k-1}(x_0) = 0 \neq f_k(x_0)$.

The role of the space Z in Definition 3.1 will be played by several spaces of the form

$$Z = Curv \setminus Mult \ , \tag{3.1}$$

where $Curv$ is a space of curves $\xi = (\varphi(t), \psi(t))$ of given form and $Mult$ denotes the subspace of $Curv$ consisting of non-primitive curves.

For example, when we want to estimate $ext\nu_0$ for a cuspidal singularity at $t = 0$ of a parametric line then we take

$$Curv = \{\varphi = a_n t^n + \ldots + a_p t^p, \psi = b_1 t + \ldots + b_q t^q : a_n a_p b_q \neq 0\} \ , \tag{3.2}$$

i.e. with fixed the $x-$order at $t = t_0 = 0$. When estimating the external codimensions of a collection of cuspidal singularities, we take

$$Curv = \left\{ \varphi = \int_0^t \prod (\tau - t_i)^{n_i - 1} d\tau, \psi = b_1 t + \ldots + b_q t^q \right\} , \tag{3.3}$$

where $t_i \neq t_j$ for $i \neq j$ and $b_q \neq 0$. To deal with a self-intersection of several local branches we use the space

$$Curv = \left\{ \varphi = \prod (t - t_j)^{n_j} \cdot \tilde{\varphi}(t), \psi = t \prod (t - t_j) \cdot \tilde{\psi}(t) \right\} , \tag{3.4}$$

where $t_i \neq t_j$ for $i \neq j$, $\tilde{\varphi}$, $\tilde{\psi}$ are polynomials and $\tilde{\varphi}(t_i) \neq 0$.

It is easy to generalize the definition of the space $Curv$ in the cases of parametric annuli and/or with several simultaneous cuspidal and self-intersection singularities. Note that when generalizing the space (3.3) to the case with Laurent polynomials φ and ψ, i.e. when $t_0 = 0$ and $n_0 < 0$, we must ensure the vanishing of the residuum at $\tau = 0$ of the subintegral form in the formula for φ in (3.3).

The space $Curv$ is acted on by a suitable group Eq of equivalences, generated by rescalings of x, y, t and by corresponding elementary transformations, like in Subsections 2.5 and 2.6. The subspace $Mult$ is invariant with respect to this action, so Eq acts on Z.

The role of functions $f_j : X \to \mathbb{C}$ in Definition 3.1 is played by functions obtained from the Puiseux coefficients $c_i^{(j)}$ in the Puiseux expansions

$$y = y_j = c_1^{(j)} (x - x_j)^{1/n_j} + c_2^{(j)} (x - x_j)^{2/n_j} + \ldots$$

of local branches of the curve C at points $(x_j, y_j) = \xi(t_j)$ (also for $t_j = \infty$).

For cuspidal singularities we consider so-called *admissible sequences of essential Puiseux coefficients* $c_i = c_i^{(j)}$, which obey the following rule:

Condition 3.2. If $c_{j_0 n'}$, $n' = n/n'' < n$, $j_0 \neq 0 \pmod{n''}$, belongs to this sequence then also all the coefficients c_i, $i < j_0 n'$, $i \neq 0 \pmod{n''}$ or $i = jn'$, $j \neq 0 \pmod{n''}$, stay in the sequence before $c_{j_0 n'}$.

For example, if $n = 6$ then the sequence $(c_1, c_2, c_3, c_5, c_7)$ is admissible, but the sequence $(c_1, c_2, c_5, c_7, c_9)$ is not admissible.

The tangency quantities $c_i^{(A)} - c_j^{(B)}$ for an intersection of two local branches A and B are ordered in natural way, by the degree of the corresponding Puiseux monomials.

It is easy to see that the coefficients $c_i^{(j)}$, treated as functions of (a,b), $a = (a_n, \dots, a_p)$, $b = (b_1, \dots, b_q)$ in (3.2) are bi–homogeneous with respect to the changes $(a,b) \to (\lambda a, \mu b)$, $\lambda, \mu \in \mathbb{C}^*$ and take the form

$$c_i^{(j)} = \hat{c}_i^{(j)} \cdot \alpha_j^{-\kappa_{ij}},$$

where α_j is the leading coefficient in the Taylor (or Laurent) expansion of φ at t_j, $\varphi = x_j + \alpha_j (t - t_j)^{n_j} + h.o.t.$, κ_{ij} are positive rational exponents and $\hat{c}_i^{(j)} = \hat{c}_i^{(j)}(a,b)$ are polynomials, linear in b and homogeneous in a. Namely, the *modified Puiseux quantities* $\hat{c}_i^{(j)}$ are the functions $f_m \in \mathbb{C}[Z]$ from Definition 3.1. Also the tangency quantities $c_i^{(A)} - c_j^{(B)}$ can be modified in a similar fashion.

We have the following interpretation of the conjectures from Section 2.

Proposition 3.3. *Conjecture 2.27 would follow from the following hypothetical properties:*

(a) *If $\xi \in Curv$ are not of the form*

$$\left(\prod (t - t_j)^{n_j}, \prod (t - t_j) \cdot \tilde{\psi}(t)\right) \tag{3.5}$$

then any admissible sequence $f_1, \dots, f_k$, which consists of modified essential Puiseux quantities of local branches at t_i and/or modifies tangency quantities, is regular at points of a suitable space $Z = Curv \setminus Mult$ of parametric lines.

(b) *If ξ's are of the form (3.5) then for any sequence $f_1, \dots, f_{\sigma+2}$ as above, $\sigma = \dim Z/Eq$, and for any $z_0 \in Z$, either the subsequence $f_1, \dots, f_{\sigma+1}$ is regular at z_0 or $f_1(z_0) = \ldots = f_{\sigma+1}(z_0) = 0$ but $f_{\sigma+2}(z_0) \neq 0$.*

Proposition 3.4. *Conjecture 2.32 would follow from the following hypothetical property:*

Any admissible sequence $f_1, \dots, f_k$, as above is regular at points a of suitable space $Z = Curv \setminus Mult$ of parametric annuli.

If a sequence $f_1, \dots, f_k$ is regular at points of Z then the maximal codimension of the varietes $V_j = \{f_1 = \ldots = f_j = 0\}$ does not exceed $\sigma = \dim Z/Eq$. Sometimes this maximal codimension is smaller than σ.

Example 3.5. Let $p = 4$, $q = 6$. Here the space of curves can be identified with $(\mathbb{C}^6 \setminus 0)/\mathbb{C}^*$ via the representation $\varphi = t^4 + a_3 t^3 + a_2 t^2 + a_1 t$, $\psi = t^6 + b_3 t^3 + b_2 t^2 + b_1 t$ and a suitable action of $\mathbb{C}^*$ stemming from the dilations of t. Thus $\sigma = \dim Curv/Eq = 5$. The subspace *Mult* consists of primitive curves of the form $\varphi = \omega^2 + a_2 \omega$, $\psi = \omega^3 + b_2 \omega$, $\omega = t^2$, and has codimension 4.

One can calculate the first topologically essential Puiseux quantities at infinity: $c_1^{(\infty)} = c_1 = -\frac{3}{2}a_3$, $c_3 = b_3 - \frac{3}{2}a_1$, $c_5 = b_1 + \frac{3}{24}a_1 a_2$, $c_7 = a_1(b_2 - \frac{3}{4}a_2^2)$, $c_9 = -\frac{1}{16}a_1^3$. We see that the equalities $c_1 = c_3 = \ldots = c_9 = 0$ lead to $a_1 = b_1 = b_3 = a_3 = 0$, i.e. we land in the subspace *Mult* of non-primitive curves. The variety $\{c_1 = c_3 = c_5 = c_7 = 0\}$ consists of two components: *Mult* and a subvariety V (of codimension 4) such that $c_9|_V \not\equiv 0$.

We have not found any example with similar behavior of the Puiseux quantities associated with finite singularities.

3.2. **Conjectures 2.27 and 2.32.** Let us present our heuristic arguments behind Conjectures 2.27 and 2.32. We begin with the case of parametric lines with cuspidal singularities.

Our initial idea was to use induction with respect to the number of critical points of φ. The case with one critical point corresponds to $\varphi = t^p$. Then the coefficients b_i, $i \neq 0 \pmod{p}$, in $\psi = b_1 t + \ldots$ play the role of the Puiseux coefficients $c_i^{(0)}$. The maximal intrinsic codimension of this singularity is $\nu_{\max} = q - 1 - \lfloor q/p \rfloor$, i.e. when $\gcd(p,q) = 1$. It corresponds to $ext\nu = p + q - 3 - \lfloor q/p \rfloor = \sigma + 1$, $\sigma = \dim Curv_{p,q}/Eq_{p,q}$.

Suppose that $\varphi = \int_0^t \tau^{n-1}(\tau - t_1)^{m-1} d\tau$, $n + m = p + 1$, i.e. with two critical points. Let us look what happens in the limit $t_1 \to 0$. One can expect that $f_i \sim t_1^{\theta_i} \cdot f_i|_{t_1=0}$, where $f_i|_{t_1=0}$ are some modified Puiseux quantities of the limiting curves; unfortunately, we do not have any rigorous proof of this statement. The codimensions ν_0 and ν_1 should then satisfy $\nu_0 + \nu_1 \leq \nu_{\max}$. Therefore, before the limit we should have $ext\nu_0 + ext\nu_1 = (n + \nu_0 - 2) + (m + \nu_1 - 2) \leq (p+1) - 4 + (q - 1 - \lfloor q/p \rfloor) = \sigma$. It is smaller than $ext\nu$ in the limit.

However, when we try the same with $\varphi = t^n(t - t_1)^m$, $n + m = p$, where the parameters $t_0 = 0$ and t_1 correspond to a double point of C and $t_1 \to 0$, then the same counting of codimensions gives $ext\nu_0 = (n + m - 2) + (\nu_0 + \nu_1 + \nu_{\tan}) \leq (p-2) + (q - 1 - \lfloor q/p \rfloor) = \sigma + 1$ before the limit. It is the same as $ext\nu$ in the limit.

These examples suggest that collapsing of several critical points of φ (some of which may be not singular for ξ) results in increasing of the sum of external codimensions by 1, while collapsing of several branches of self-intersection of C to a cuspidal singularity does not change the sum of external codimensions.

If one can apply several times the procedure of collapsing of critical points then the sum of external codimensions should be even smaller that σ. For example, elementary calculations show that, if a polynomial curve of the bi-degree $(p,q) = (5,6)$ has four cuspidal singularities then $\sum ext\nu_j = 4$, while $\sigma = 6$.

In the case of parametric annuli it looks as if any collapsing of a self-intersection can be preceded by a collapse of some critical points (maybe to $t = \infty$). Also in the case of several self-intersection points it seems that the collapsing of some such self-intersection to a cuspidal singularity can be preceded by a collapse of critical points.

3.3. **Determinants and rigidity.** Consider a cuspidal singularity at $t = 0$. For simplicity assume that $n = \operatorname{ord}_0 \varphi$ is prime. We have

$$\varphi = t^n(\alpha_0 + \ldots + \alpha_{p-n} t^{p-n})$$

and the essential Puiseux coefficients are $c_i = c_i^{(0)}$, $i \not\equiv \pmod{n}$. If the initial $\nu = l(n-1) + \rho$, $0 \leq \rho \leq n-2$, of these coefficients vanish then we have the representation

$$\psi = d_1\varphi + \ldots + d_l\varphi^l + O(t^{q_0+1}),\ q_0 = nl + \rho, \tag{3.6}$$

near $t = 0$. If ψ is a polynomial of degree q, which we assume $\leq q_0$, then we get $q_0 - q$ conditions for vanishing of the coefficients

$$b_{q+1}, \ldots, b_{q_0}$$

in the Taylor series $\sum b_j t^j$ of the polynomial $d_1\varphi(t) + \ldots + d_l\varphi^l(t)$. Then ψ equals the part of degree $\leq q$ of the latter polynomial. The coefficients b_j are functions of the coefficients $\alpha = (\alpha_0, \ldots, \alpha_{p-n})$ and $d_1, \ldots, d_l$, moreover, they are linear in d_j's. The distinguished coefficients b_i do not depend on d_j for $j \leq l_0 = \lfloor q/p \rfloor$; we denote $d = (d_{l_0+1}, \ldots, d_l)$.

We get a system of linear equations

$$A(\alpha)d = 0, \tag{3.7}$$

where $A(\alpha)$ is the matrix of coefficients $a_{ij}(\alpha)$ before d_j in the expression for b_i. The system (3.7) has an obvious solution $d = 0$, but this corresponds to a multiply covered curve $\xi = (\varphi, 0)$. We are interested in the solutions such that $d \neq 0$ and we arrive to the condition

$$\operatorname{rank} A(\alpha) < l - l_0. \tag{3.8}$$

This condition defines a system of algebraic equations on α. If $l - l_0 \leq q_0 - q$ (which usually occurs) then (3.8) is equivalent to the vanishing of $(q_0 - q) - (l - l_0)$ minors of the matrix $A(\alpha)$. The conditions (3.7) and (3.8) for $q_0 - q$ not too small constitute very rigid conditions onto the curves; usually their solution consists of isolated points in the space Z/Eq. They do not allow deformation of curves with given codimension ν.

Since we consider only non-primitive curves, we should avoid solutions α to (3.8) which correspond to composed polynomials φ, $\varphi = \tilde{\varphi} \circ \omega$ for $\omega = t^n + \ldots$., and such that the kernel of $A(\alpha)$ consists of d's which define composed polynomials, $\psi = \psi \circ \omega$.

Example 3.6. Let $n = 2$ and $p = 3$, i.e. $\varphi = t^2 + t^3$ (after normalization). Assuming $q_0 = 9$, i.e. $c_1 = c_3 = \ldots = c_9 = 0$ and $\nu = 5$, and $q = 8$, we get one equation $b_9 = 0$ for the coefficient before t^9 in $d_3\varphi^3(t) + d_4\varphi^4(t)$.

If we assume $q = 7$ then we get two conditions $b_8 = b_9 = 0$. For $q = 5$ we get four conditions $b_6 = \ldots = b_9 = 0$ for $d_2\varphi^2 + d_3\varphi^3 + d_4\varphi^4$. It is easy to check that in the latter two cases the only solution is $d = 0$.

We have the following observation.

Lemma 3.7. *Let $\varphi = t^2 + t^3$. Then the problem of regularity of the sequences $c_1, c_3, \ldots, c_{2\nu+1}$ for complex polynomial lines can be reduced to the same problem in the class of polynomial curves (φ, ψ) with real coefficients.*

The same holds true when $\varphi = \int \tau^{n-1}(\tau - 1)^{m-1} d\tau$ with two critical points or $\varphi = t^m(t-1)^n$.

Moreover, the statement holds also when $\varphi = t^2 + t^3$ and ψ is a Laurent polynomial in t.

Proof. The first two statements follow from the reality of the matrix $A(\alpha)$.

When $\psi = b_{-2s}t^{-2s} + \ldots + b_q t^q$, $s > 0$, the function $\tilde{\psi} = \psi\varphi^s$ is a polynomial and the essential Puiseux coefficients for $(\varphi, \tilde{\psi})$ correspond to the essential Puiseux coefficients for (φ, ψ). (Note that the case with odd $\operatorname{ord}_0 \psi$ is trivial). We consider polynomials $\chi(t) = d_0 + d_1\varphi + \ldots + d_l\varphi^l \pmod{t^{q_0+1}}$ and add the conditions $\chi(-1) = \chi'(-1) = \ldots = \chi^{(s-1)}(-1) = 0$ to the system of $b_i = 0$. The reality of this new system is preserved. □

In the representation (3.6) we assumed that n is prime. If n is not prime we can use an analogue of the representation (3.6) with rational powers of φ.

Also an analogous expansion can be used to study the Puiseux and tangency quantities at a self-intersection, e.g. when $\varphi = t^n(t - t_1)^m(\alpha_0 + \ldots + \alpha_u t^u)$.

When we consider sequences consisting of several singular points the situation becomes more complex and we omit its discussion.

3.4. **The argument principle.** It is easy to check the validity of Conjectures 2.27 and 2.32 for curves with low degree (Laurent) polynomials φ, ψ. But when at least one of these degrees is unbounded the problem becomes very difficult. Therefore the following result should be interesting.

We consider curves ξ with $\varphi = 3t^2 - 2t^3$, which has two critical points $t = 0$ and $t = 1$ with the critical values $\varphi = 0$ and $\varphi = 1$ respectively. Let us define the algebraic function $t(x)$ by

$$2t^3 - 3t^2 + x = 0.$$

It has three branches $t_1(x)$, $t_2(x)$ and $t_3(x)$. Assume that $t_1 < t_2 < t_3$ when $0 < x < 1$. As x tends to the critical value $x = 0$ the branches $t_1(x)$ and $t_2(x)$ tend to the critical point $t = 0$; as x tends to the critical value $x = 1$ the branches $t_2(x)$ and $t_3(x)$ tend to the second critical point $t = 1$. As x moves along a small loop around $x = 0$ (in the complex $x-$plane) the points $t_1(x)$ and $t_2(x)$ turn around $t = 0$ (two times slower) and finally exchange their positions. Analogously, as x moves along a small loop around $x = 1$ the points $t_2(x)$ and $t_3(x)$ turn around $t = 1$ and finally exchange their positions. Therefore the functions $t_1(x) + t_2(x)$, $t_1(x)t_2(x)$ and $t_3(x)$ are analytic near $x = 0$ and the functions $t_2(x) + t_3(x)$, $t_2(x)t_3(x)$ and $t_1(x)$ are analytic near $x = 1$.

We note the following relations between the codimensions of singularities and certain invariants of some algebraic functions:

(i) the codimension ν_0 of the cuspidal singularity at $t = 0$ equals $\operatorname{ord}_{x=0} \chi_{12}(x)$, where
$$\chi_{ij}(x) = \frac{\psi(t_i) - \psi(t_j)}{t_i - t_j};$$
(ii) the codimension ν_1 at $t = 1$ equals $\operatorname{ord}_{x=1} \chi_{23}$;
(iii) the tangency codimension $\nu_{\tan}$ of a self-intersection $\xi(t_i(x_*)) = \xi(t_j(x_*))$ equals $\operatorname{ord}_{x=x_*} \chi_{ij} - 1$;
(iv) sometimes we shall use interpretation of ν_0 as $\frac{1}{2}(\operatorname{ord}_{z=0} \eta_{12}(z) - 1)$, where
$$\eta_{ij}(z) = (\psi \circ t_i - \psi \circ t_j)\,|_{x=z^2},$$
and analogously we shall interpret other invariants.

We distinguish the following cases:

(1) $\psi \in \mathbb{C}[t]$ and we estimate ν_0;
(in the sequel cases we assume $\psi \in \mathbb{C}[t, t^{-1}]$)
(2) estimation of ν_0 and of ν_1 for $\psi \in \mathbb{C}[t, t^{-1}]$;
(3) estimation of $\nu_1 + \nu_2$;

(4) estimation of $\nu_{\tan}+\nu_0$ where $\nu_{\tan}$ is the codimension of the self-intersection $\xi(t_1)=\xi(t_2)$ of two smooth branches;
(5) estimation of $\nu_{\tan}$ for the self-intersection $\xi(t_1)=\xi(t_3)$;
(6) estimation of the sum of $\nu_{\tan}$ for two self-intersections $\xi(t_1)=\xi(t_2)$ and $\xi(t_2)=\xi(t_3)$ and for a triple self-intersection (here we can add $\nu_0+\nu_1$ to this sum);
(7) remaining cases.

Theorem 3.8. *Let $\varphi=3t^2-2t^3$. Then Conjectures 2.27 and 2.32 hold true in the cases 1–6 above for the class of curves where ψ is a real Laurent polynomial with fixed orders at $t=0$ and $t=\infty$.*

Remark 3.9. If $\varphi\in\mathbb{C}[t]$ has degree $p=1$, the curve is smooth. If $p=3$ then an analogue of Theorem 3.8 is elementary. Also the case with $\varphi=(t-t_0)^3$ is trivial.

Remark that, by Lemma 3.6, the restriction of reality of $\psi(t)$ can often be skipped.

In the next section we prove some general bounds for the codimensions. For the polynomial curves they are of the form $\leq p+q+R$ (see Theorem 4.25), where R is the number of double points of the curve. So for fixed p and q and large R (note that R can be quadratic in p and q) they are far from being effective, whereas Theorem 3.8 is very effective (but restricted).

Proof of Theorem 3.8. In the proof we shall use the argument principle to estimate multiplicity of a zero w_0 of certain holomorphic function f by the increment of $\arg f$ along a contour Γ which surrounds w_0. This idea was successfully used by G. Petrov [Pet] in estimating zeroes of Abelian integrals and its subsequent application to the weakened XVIth Hilbert problem. Also C. Christopher and S. Lynch [ChLy] used it to solve the case 1 from the above list (below we repeat their arguments); they applied this bound to the problem of limit cycles for the Liénard equation (see also Section 5).

Consider the case 1. The polynomial ψ, of degree $q\neq 0 \pmod 3$ has the representation

$$\psi(t)=\psi_0(x)+t\psi_1(x)+t^2\psi_2(x), \tag{3.9}$$

where $\deg\psi_1\leq\left\lfloor\frac{q-1}{3}\right\rfloor$ and $\deg\psi_2\leq\left\lfloor\frac{q-2}{3}\right\rfloor$. We consider the function $\chi_{12}=\psi_1(x)+(t_1+t_2)\psi_2(x)$. It is an algebraic function of x, holomorphic near $x=0$. In fact, χ_{12} is single valued in the domain

$$D=\mathbb{C}\setminus\{x\geq 1\}\,.$$

We estimate the $\operatorname{ord}_0\chi_{12}$ by the number of zeroes of χ_{12} in the domain D. Like in [Pet] we consider the increment of the argument of $\chi_{12}(x)$ as x varies along the following contour Γ in D: Γ consists of a large circle $\{|x|=R\}$ (in the positive anticlockwise direction), of a small circle $\{|x-1|=r\}$ (in the opposite direction) and of two segments of the cut $\{x\geq 1\}$ (from $x=1+r$ to $x=R$).

The increment of $\arg\chi_{12}$ along the small circle tends to zero with $r\to 0$, when $\chi_{12}(1)\neq 0$, and is negative otherwise. The increment of $\arg\chi_{12}$ along the large circle is bounded by

$$2\pi\cdot\max\left(\left\lfloor\frac{q-1}{3}\right\rfloor,\left\lfloor\frac{q-2}{3}\right\rfloor+\frac{1}{3}\right). \tag{3.10}$$

Using the reality of $\chi_{12}(x)$ for $0<x<1$, we find that the values of χ_{12} at the upper and at the lower ridges of the cut $\{x\geq 1\}$ are conjugate one to another. It implies that the increase of $\arg\chi_{12}$ along the two straight segments is bounded by 2π times the number of zeroes of the imaginary part of χ_{12} plus 1. But

$$2i\operatorname{Im}\chi_{12}(x)=(t_2-t_3)\psi_3(x)$$

where $t_3(x)=\bar{t}_2(x)\neq t_2(x)$ for $x>1$. So the corresponding $\Delta\arg\chi_{12}$ is bounded by

$$2\pi\cdot\left(\left\lfloor\frac{q-2}{3}\right\rfloor+1\right).$$

Summing up the above we get $\operatorname{ord}_0\chi_{12}\leq 2k$ if $q=3k+1$ and $\leq 2k+1$ if $q=3k+2$. Therefore $\nu_0\leq\sigma=p+q-4-\lfloor q/p\rfloor$, as expected.

Consider the case 2. Recall that ψ has pole at $t=0$; we can assume that its order is even, equal $2s$ (otherwise there is no degeneration).

Of course, we cannot use the representation (3.9). But we have the identity

$$t^{-3}=\frac{3}{x}t^{-1}-\frac{2}{x}.$$

It implies that $t^{-2s}=f_{-2}(\frac{1}{x})t^{-2}+f_{-1}(\frac{1}{x})t^{-1}+f_0(\frac{1}{x})$, where $\deg f_i\leq s-1$ and $\deg f_{-2}=s-1$. Representing ψ as $(g_0(x)+tg_1(x)+t^2g_2(x))\cdot t^{-2s}$, with $\deg g_0\leq\left\lfloor\frac{q+2s}{3}\right\rfloor$, $\deg g_1\leq\left\lfloor\frac{q+2s-1}{3}\right\rfloor$, $\deg g_2\leq\left\lfloor\frac{q+2s-2}{3}\right\rfloor$, we obtain

$$\psi=\psi_{-2}(x)t^{-2}+\psi_{-1}(x)t^{-1}+\psi_0(x),\quad \psi_i=\tilde{\psi}_i(x)/x^{s-1},$$

where $\tilde{\psi}_i$ are polynomials with precise bounds for their degrees and $\tilde{\psi}_{-2}(0)\neq 0$.

As in the case 1, in order to bound ν_0, we estimate the order at $x=0$ of the function

$$\tilde{\chi}_{12}(x)=t_1t_2\cdot\chi_{12}=(t_1^{-1}+t_2^{-1})\psi_{-2}+\psi_{-1}.$$

The further proof runs like in the case 1. In fact, we must more carefully control the argument of χ_{12}; the cases when $\operatorname{ord}_\infty(t_1^{-1}+t_2^{-1})\psi_{-2}$ is greater or smaller than $\operatorname{ord}_\infty\psi_{-1}$ should be considered separately.

Of course, to estimate ν_1 we use the function $\tilde{\chi}_{23}$.

Consider the case 3. If the both points $t=0$ and $t=1$ are singular then $\psi'=-6t(t-1)\tilde{\psi}$, where

$$\tilde{\psi}=\frac{d\psi}{d\varphi}$$

is a polynomial when ψ is a polynomial. The Puiseux expansions at $t=0$ and $t=1$ of the curve $(\varphi,\tilde{\psi})$ are directly related with the corresponding Puiseux expansions of the curve (φ,ψ). After applying several times this trick we reduce the problem to the case with one singular point.

But there exists another proof which works also when ψ is a Laurent polynomial. Consider the function $\eta_{12}(z)=(\psi(t_1)-\psi(t_2))(z^2)$. It is meromorphic (or holomorphic) near $z=0$ and has singularities at $z=-1$ and $z=1$. So it is meromorphic in the domain

$$E=\mathbb{C}\setminus(\{z\leq -1\}\cup\{z\geq 1\}).$$

Let Λ be the contour in E consisting of: two large half-circles in $\{|z|=R\}$ (in positive direction), two small circles around $z=-1$ and $z=1$ (in negative direction) and four straight segments along the cuts $\{z\leq -1\}$ and $\{z\geq 1\}$.

Assume that ψ is a polynomial. Let ζ_0 (respectively ζ_1) be the number of zeroes of the function $\psi(t_1)-\psi(t_2)$ (respectively $\psi(t_2)-\psi(t_3)$) in the open half-line $\{x<0\}$ (respectively $\{x>1\}$). We have $2\zeta_0+2\nu_0+1\leq\Delta_\Lambda\arg\eta_{12}$ (the increment along Λ).

The increment of $\arg\eta_{12}$ along $\{|z|=R\}$ is estimated via $\deg\psi$ and the increments along the small circles are neglected (or negative). The increment of $\arg\eta_{12}$ along each of the two cuts is bounded by 2π times 1 plus the number of zeroes of the function η_{23} in the open cut (deprived of the endpoint). Therefore

$$2\zeta_0+2\nu_0+1\leq 2\cdot(q/3)+2(\zeta_1+1) \tag{3.11}$$

(see (3.10)). The same analysis applied to η_{23} gives

$$2\zeta_1+2\nu_1+1\leq 2\cdot(q/3)+2(\zeta_0+1).$$

The both inequalities yield $\nu_0+\nu_1\leq 2\cdot(q/3)+1$. Since $\lfloor 2q/3+1\rfloor=\sigma+1$, we must estimate more carefully the increment of the argument along the cuts (like in the end of the case 2); for example, if $q=3k+1$ then the inequality (3.11) is replaced with $2\zeta_0+2\nu_0+1\leq 2(k+1/3)+2(\zeta_1+1/3)$.

In the case of Laurent polynomial ψ we have to replace $2\nu_0+1$ with $2\nu_0+1-2s$, where $2s=-\operatorname{ord}_0\psi$.

Consider the case 4. Recall that the tangency codimension $\nu_{\tan}$ of the self-intersection $\xi(t_1(x_*))=\xi(t_2(x_*))$ equals the order at x_* of the function $\chi_{12}(x)$. Recall that $x_*\neq 0,1$. If $x_*\notin\{x>1\}$ then we estimate $\operatorname{ord}_{x_*}\chi_{12}$ like in the case 1. Moreover, the same proof allows to estimate the sum $\operatorname{ord}_{x_*}\chi_{12}+\operatorname{ord}_0\chi_{12}$. If $x_*>1$ then we modify the contour Γ from the case 1 by adding two small half-circles in $\{|x-x_*|=r\}$ in the opposite direction. The increment of the argument of χ_{12} along these half-circles equals $-\operatorname{ord}_{x_*}\chi_{12}$.

Note that when x_* is not real we cannot use Lemma 3.7 to guarantee that $\psi(t)$ has real coefficients; here the assumption of reality of ψ in Theorem 3.8 is essential.

Consider the case 5. Of course, we use the function χ_{13}. It is singular at $x=0$ and $x=1$. So the domain D should be replaced with

$$D'=\mathbb{C}\setminus(\{x\leq 0\}\cup\{x\geq 1\})$$

and the contour Γ should be suitably modified.

Consider the case 6. Assume that $\xi(t_1(x_*))=\xi(t_2(x_*))$ and $\xi(t_2(x_{**}))=\xi(t_3(x_{**}))$. If $x_*\neq x_{**}$ then we have two double points of the curve C, otherwise we have a triple self-intersection. We estimate $\operatorname{ord}_{z_*}\eta_{12}+\operatorname{ord}_{-z_*}\eta_{12}+\operatorname{ord}_{z_{**}}\eta_{23}+\operatorname{ord}_{-z_{**}}\eta_{23}$, $z_*^2=x_*$, $z_{**}^2=x_{**}$, like in the case 3.

Of course the same arguments allow to estimate $\nu_0+\nu_1+\sum\nu_{\tan 12}+\sum\nu_{\tan 23}$, where we sum over self-intersections $\xi(t_1)=\xi(t_2)$ and $\xi(t_2)=\xi(t_3)$. □

Remark 3.10. This method does not allow to get a good estimate for the sum of $\nu_{\tan}$ for the simultaneous self-intersections $\xi(t_1)=\xi(t_2)$, $\xi(t_1)=\xi(t_3)$ and maybe $\xi(t_2)-\xi(t_3)$. Note that for a generic such curve the sum of zeroes of all the functions $\psi(t_i)-\psi(t_j)$ is about the total number of double points, i.e. $\sim q$ in the polynomial case.

It seems that there exists a whole class of problems, like 1–6 above, which can be solved using the argument principle. Below we present one such generalization.

Theorem 3.11. *Let* $\varphi = 12\int_0^t \tau(1-\tau)^2\, d\tau$ *and* ψ *be a polynomial of degree* $q \neq 0 \pmod 4$*. Then* $\nu_0 \leq q - \lfloor q/4 \rfloor$*.*

Proof. The function φ has two critical points $t = 0$ and $t = 1$ (of multiplicity 3) with the corresponding critical values $x = 0$ and $x = 1$. The equation $\varphi(t) = x$ has four solutions $t_1(x), \dots, t_4(x)$, where $t_1(x) < 0 < t_2(x) < 1$ for $0 < x < 1$. $t_1(x)$ and $t_2(x)$ collapse to $t = 0$ as $x \to 0$ and $t_2(x), t_3(x)$ and $t_4(x)$ collapse to $t = 1$ as $x \to 1$.

By Lemma 3.7 we can assume that the polynomial ψ is real. We write $\psi = \psi_0(x) + t\psi_1(x) + t^2\psi_2(x) + t^3\psi_3(x)$. We have $\nu_0 = \operatorname{ord}_0 \chi_{12} = \psi_1 + (t_1 + t_2)\psi_2 + (t_1^2 + t_1t_2 + t_2^2)\psi_3$.

As in the case 1 of the previous proof we reduce the problem to calculation of the number of zeroes of the function $\operatorname{Im}\chi_{12}(x)$ at he cut $\{x > 1\}$. From the monodromy properties of the algebraic branches $t_j(x)$ near $x = 1$ we find $2\operatorname{Im}\chi_{12} = (t_2 - t_4)\left(\psi_2 + (t_1 + t_2 + t_4)\psi_3\right)$ (here $t_3 > 1$ and t_2, $t_4 = \bar{t}_2$ are nonreal).

The function $\theta(z) = \left(\psi_2 + (t_1 + t_2 + t_4)\psi_3\right)|_{x=1+z^2}$ is holomorphic near $z = 0$, but it may have singularities at $z = -1$ and $z = e^{\pm\pi i/3}$. At each of the latter points two branches t_1 and t_j collapse. It follows that only the point $z = -1$ is singular for θ and we can repeat the argument principle argument to estimate the number of zeroes of θ. □

4. Resolution of singularities, splice diagrams and the BMY inequality

4.1. **Dual graphs.** Let $(C, 0)$ be a (singular) germ of a curve at $(\mathbb{C}^2, 0)$. Let

$$\pi : (V, D) \to (U, C),$$

$U \subset (\mathbb{C}^2, 0)$, be the minimal resolution of the singularity at 0. Here $D = \widetilde{C} + E$, where $\widetilde{C} = \pi'(C)$ is the strict transform of C and $E = E_1 + \ldots + E_u$ is the exceptional divisor (with smooth components $E_i \simeq \mathbb{CP}^1$ and normal intersections). We associate with this resolution two weighted graphs Γ_E, $\Gamma_{E,C}$, called *dual graphs.*

The graph Γ_E has u vertices corresponding to the divisors E_i and the *weight* w_i of a vertex E_i is the self-intersection index $E_i \cdot E_i = E_i^2$. The edges $[E_i, E_j]$ correspond to the intersection points $E_i \cap E_j$. It is clear that Γ_E is a tree graph. The *valence* v_i of a vertex E_i equals to the number of edges attached to E_i.

The graph $\Gamma_{E,C}$ arises from Γ_E by attaching to a vertex E_l an edge with arrowhead vertex whenever a component $\widetilde{C}_j$ of the curve $\widetilde{C}$ intersects the divisor E_l. The arrowhead vertices are labeled by C_j and the valences of vertices E_i in $\Gamma_{E,C}$ are denoted by $\bar{v}_i$. Therefore $\bar{v}_i - v_i$ is the number of components of $\widetilde{C}$ intersecting E_i. The vertices with $\bar{v}_i \geq 3$ are called *branching vertices* and those with $\bar{v}_i = 1$ are called the *ends.*

Introduce the vector space

$$\operatorname{Vect}(E) = \mathbb{Q}E_1 \oplus \ldots \oplus \mathbb{Q}E_u$$

The dual graph Γ_E encodes the *intersection matrix* A with entries $E_i \cdot E_j$. It is known that the *discriminant*

$$d(\Gamma_E) := \det(-A)$$

of Γ_E equals 1. Since the quadratic form on $\mathrm{Vect}(E)$ defined by the matrix A is non-degenerate, we can define three elements of $\mathrm{Vect}(E)$:

- the *canonical divisor* K_E, via the adjunction formula $(K_E+E_j)\cdot E_j=-2$,
- a representation of C_E as a combination $\sum a_jE_j$ such that $\sum a_jE_j\cdot E_k=C_E\cdot E_k$, $k=1,\dots,u$, and
- $D_E:=C_E+\sum E_j=C_E+E$.

Proposition 4.1. *The rough M-number of the singularity* $(C,0)$*, defined in Section 2, equals*

$$\overline{M}=K_E\cdot(K_E+D_E). \tag{4.1}$$

Remark 4.2. Orevkov in [Or] introduced the rough M-number as $(K_E+D_E)^2+\mu$, where μ is the Milnor number. As we shall see, this number agrees with $K_E(K_E+D_E)$ in the case of cuspidal singularity. But already in the case of simple double point with the dual graph $C_1\longleftarrow\overset{E}{\circ}\longrightarrow C_2$, $E^2=-1$, we find that $K=E$, $C_E=-2E$ and $\mu=1$. Thus $(K_E+D_E)^2+\mu=1$, while $\overline{M}=0$. Of course, the simple double point singularity has zero codimension.

Let us introduce also another invariant that was extensively used by [OZ1], [OZ2] and [Or]

Definition 4.3. With the notation as above, let $K_E+D_E=P_E+N_E$ be the Zariski–Fujita decomposition of K_E+D_E (see [Fuj],[Or]). Here P_E is the numerically efective part and N_E, the negative part of K_E+D_E. The quantity

$$\eta_E=-N_E^2\geq 0$$

is called the *excess* of a singular point. The *M–number* (without a bar) is defined to be $M_E=\overline{M}_E+\eta_E$.

We note also the following formulas equivalent to (4.1):

$$\begin{aligned}\overline{M}&=(K_E+D_E)^2+\mu+1-C_E\cdot E,\\ \overline{M}&=(K_E+D_E)^2+1+C_E\cdot K_E.\end{aligned}$$

They follow from the (arithmetic) genus formula $0=p_a(E)=\frac{1}{2}E(K_E+E)+1$ and from the following expression for the Milnor number

Lemma 4.4 ([Or])**.** *We have* $\mu=1-C_E(K_E+D_E)$.

We recall that for an algebraic curve C on an algebraic surface S its arithmetic genus $p_a(C):=\frac{1}{2}\chi(\mathcal{O}_C)+1=\frac{1}{2}(h^0(\mathcal{O}_C)-h^1(\mathcal{O}_C))+1$ equals $\frac{1}{2}C(K_S+C)+1$. If $\widetilde{C}$ is the normalization of C then $p_a(C)=p_a(\widetilde{C})+\sum\delta_P$, where δ_P is the number of double points at the singular point $P\in C$. In particular, if C is a connected union of m rational curves with r simple double points as the only singularities then $p_a(C)=1-m+r=1-e(\Gamma_C)$, where $e(\Gamma_C)$ is the Euler characteristic of the dual graph of C. All this can be found in [Har].

Let us pass to the proof of Proposition 4.1. The following lemma is Proposition 4.1 from [OZ1] and is proved by induction with respect to the number of blowing-ups.

Lemma 4.5 ([OZ1])**.** *We have*

$$(K_E+E)^2+2=\sum_{i=1}^{u}(-b_{ii})(v_i-2)$$

where b_{ii} *are the diagonal elements of the matrix* $B = A^{-1} = (b_{ij})_{i,j=1,\dots,n}$ *and* v_i *are the valences of vertices in* Γ_E*.*

It is easy to get the following

Lemma 4.6. *If a component* $\widetilde{C}_j$ *of* $\widetilde{C}$ *intersects a divisor* E_l*,* $l = l(j)$*, then* $\widetilde{C}_j^2 = b_{ll}$*.*

Corollary 4.7. *We have*

$$K_E(K_E + D_E) = \sum_{i=1}^{u}(-b_{ii})(\bar{v}_i - 2) - \sum_{j=1}^{k}\mu(C_j) = W(\Gamma_{E,C}) - \sum_{j=1}^{k}\mu(C_j),$$

where

$$W(\Gamma_{E,C}) = \sum_{E_i:\text{branching}}(-b_{ii})(\bar{v}_i - 2) - \sum_{E_i:\text{end}}(-b_{ii}). \tag{4.2}$$

Proof. By Lemma 4.4 we can write $\mu(C_j) = -\widetilde{C}_j(K_E + \widetilde{C}_j)$, where $\widetilde{C}_j$ are represented as combination of E_i's. Therefore

$$\begin{aligned} K_E(K_E + E + \textstyle\sum \widetilde{C}_j) &= (K_E + E)^2 - E(K_E + E) + \textstyle\sum \widetilde{C}_j(K_E + \widetilde{C}_j) - \textstyle\sum \widetilde{C}^2 \\ &= (K_E + E)^2 + 2 - \textstyle\sum \widetilde{C}^2 - \textstyle\sum \mu(C_j). \end{aligned}$$

But by Lemmas 4.5 and 4.6 we have

$$\Big((K_E + E)^2 + 2\Big) - \sum \widetilde{C}_j^2 = \sum(-b_{ii})(v_i - 2) + \sum(-b_{ii})(\bar{v}_i - v_i).$$

From this the corollary follows. □

Now our task is to calculate the entries b_{ii}.

Lemma 4.8. *We have*

$$b_{ij} = -d(\Gamma_{ij}),$$

where $d(\Gamma_{ij})$ *is the discriminant of the subgraph* Γ_{ij} *of* Γ_E *obtained by deleting the shortest path between the vertices* E_i *and* E_j E_i *and* E_j *being deleted. In particular,*

$$b_{ii} = -\prod d(\Gamma_m),$$

where Γ_m *are the connected components of* $\Gamma_E - E_i$*.*

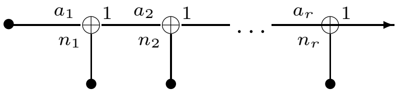
Proof. Recall that the discriminant is the determinant of the minus intersection matrix. Therefore $b_{ij} = \pm\det(-A')_{ij}/\det(-A) = \pm\det(-A')_{ij}$ where A'_{ij} is obtained from A by deleting the $i-$th row and $j-$th column. Some additional analysis gives the formula from the lemma. □

4.2. **Eisenbud–Neumann splice diagrams.** The latter lemma justifies introduction of so-called *splice diagram* Δ defined as follows:

- one replaces each linear chain in $\Gamma_{E,C}$ by an edge;
- one assigns to each end of an edge at a branching vertex in Δ a weight, equal to the discriminant of the corresponding branch of $\Gamma_{E,C}$ at this vertex.

Example 4.9 ([EiNe])**.** For a cuspidal singularity $C : y = x^{m_1/n_1}(1 + \dots + x^{m_2/n_1n_2}(1 + \dots + x^{m_r/n_1\dots n_r}(1 + \dots)\dots))$ the splice diagram is presented at the below figure with

$$a_1 = m_1, \quad a_j = a_{j-1}n_jn_{j-1} + m_j = \sum_{i=1}^{j-1}(m_in_i)(n_{i+1}\dots n_{j-1})^2 + m_j. \tag{4.3}$$



Using the splice diagrams we can express the quantities $(-b_{ii})$ in Lemma 4.7 for the branching vertices:

$$-b_{ii} = \pi_i := \prod (\text{weights of edges incident to } E_i)\,. \tag{4.4}$$

Therefore it remains to calculate the quantities $(-b_{ii})$ for the boundary vertices in $\Gamma_{E,C}$. To this aim we use the following lemma whose proof is in [OZ1, Lemmas 4.1, 4.2, 4.3, Corollaries 4.4, 4.5].

Lemma 4.10 ([OZ1])**.** *Let L be a linear extremal chain (twig) of a graph Γ with vertices $E_1, \ldots, E_m$ such that E_m is the end of Γ and E_1 is connected with $\Gamma - L$ by an edge $[E_0, E_1]$, where E_0 is a branching vertex of Γ $\ldots \overset{E_0}{\circ} - \overset{E_1}{\circ} - \ldots - \overset{E_m}{\circ}$. Then we have*

$$d(\Gamma - L - E_0) = d(\Gamma - E_m) d(L) - d(L - E_m).$$

Therefore, for $\Gamma = \Gamma_E$ with $d(\Gamma_E) = 1$, we get

$$-b_{mm} = d(\Gamma - E_m) = \frac{d(\Gamma - L - E_0)}{d(L)} + \frac{d(L - E_m)}{d(L)} = \left\lfloor \frac{d(\Gamma - L - E_0)}{d(L)} \right\rfloor + 1. \tag{4.5}$$

(Note that $0 < d(L - E_m)/d(L) < 1$ and from this the latter identity follows.)

Let e_m denote the weight of the end at E_0 of the edge in the splice diagram Δ corresponding to the twig L. Then we get $d(\Gamma - L - E_0) = \pi_0 / e_m$, $d(L) = e_m$ (see Lemma 4.8) and hence

$$-b_{mm} = \sigma_m := \left\lfloor \pi_0 / e_m^2 \right\rfloor + 1. \tag{4.6}$$

Thus (4.2) can be rewritten in the following form

$$W(\Gamma_{E,C}) = \sum_{E_i:\text{branching}} (\bar{v}_i - 2)\pi_i - \sum_{E_i:\text{end}} \sigma_i. \tag{4.7}$$

Proof of Proposition 4.1. We use induction with respect to the number k of components C_j of C. The case of irreducible curve $C = C_1$ was considered in [OZ1] and [Or].

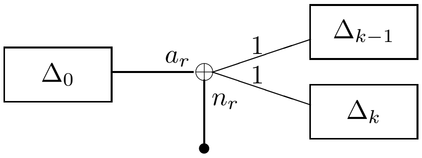
Let now $C = C_1 + \ldots + C_k$ has $k > 1$ branches. Assume that C_k has the maximal order of tangency with C_{k-1} (among all C_i's). Let $\nu_{\tan} = \nu_{\tan}(C_k, C_{k-1})$. By Definition 2.11 it suffices to prove the recursive formula

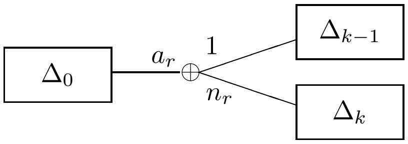
$$\overline{M}(C) = \overline{M}(C') + \overline{M}(C_k) + \nu_{\tan} + 2,$$

where $C' = C_1 + \ldots + C_{k-1}$. Let Γ, Γ', Γ_k denote the dual graphs of C, C' and C_k respectively. The recursive formula for $\overline{M}$ is equivalent to

$$W(\Gamma) - W(\Gamma') - W(\Gamma_k) = \nu_{\tan} + 2. \tag{4.8}$$

Consider the splice diagram of the curve $C_{k-1} + C_k$. It is of one of the two types presented at the below figures (with $a_1 > n_1$). In the first case the first difference between the Puiseux expansions of the two curves occurs in the term $x^{\tilde{m}_1/n_1 \ldots n_r}$, with different and nonzero coefficients. In the second case the term $x^{\tilde{m}_1/n_1 \ldots n_r}$ is present in the expansion of C_{k-1} but is absent in the expansion of C_k.





The splice diagram Δ of $C_1 + \dots C_k$ is obtained from the splice diagram of $C_{k-1} + C_k$ by replacing some boundary edges by trees. Similarly, the diagram Δ' of $C_1 + \ldots + C_{k-1}$ is obtained from the diagram for $C_{k-1} + C_k$.

Now it is not difficult to see that the left-hand side of (4.8) equals

$$\sigma_{r+1}+\ldots+\sigma_{2r+1}-\pi_1-\ldots-\pi_r = \left(1+\left\lfloor\frac{n_1}{a_1}\right\rfloor\right)+\left(1+\left\lfloor\frac{a_1}{n_1}\right\rfloor\right)+\ldots+\left(1+\left\lfloor\frac{a_r}{n_r}\right\rfloor\right)- \\ - a_1n_1 - \ldots a_{r-1}n_{r-1},$$

where the vertices with indices $r+1, \ldots, 2r+1$ are the bold vertices in the diagram from Example 4.9. Using the recursive formulas (4.3) for a_j we find that this expression equals $\sum \left\lfloor \frac{m_j}{n_j} \right\rfloor + r + 2 = \nu_{\tan} + 2$ (see Lemma 2.9). ☐

We end up this subsection by providing a way to compute η_i in terms of the Eisenbud–Neumann diagramm.

Proposition 4.11 ([OZ2]). *The quantity $\eta_E = -N_E^2$ (see Definition 4.3) equals*

$$\eta_E = \sum_{E_i:\text{end}} g_m, \tag{4.9}$$

where $g_m = \sigma_m - \pi_0/e_m^2 = \{\pi_0/e_m^2\}$ and $\{a\} = \min_{n>a, n\in\mathbb{Z}}(n-a)$ denotes the upper fractional part.

Corollary 4.12 ([OZ1],[OZ2]). *If the number of branches of the singular point is equal to one then $\eta_E > \frac{1}{2}$. Moreover, if the multiplicity of the singular point is 2 then $\eta_E \geq \frac{5}{6}$.*

Proof. If (n, m) is the first characteristic pair of the singularity then by Proposition 4.11 and formula (4.3)

$$\eta_E \geq \left\{\tfrac{n}{m}\right\} + \left\{\tfrac{m}{n}\right\}.$$

From this the first part follows (see [OZ1]). If $n = 2$, $m = 2k + 1$ then $\eta_E = \frac{2k-1}{2k+1} + \frac{1}{2} \geq \frac{5}{6}$. ☐

4.3. **Relative minimality of resolution.** The sum of $\overline{M}$ numbers of a given curve $C \subset \mathbb{C}P^2$ can be bounded using the following deep result, which is known as the BMY inequality.

Theorem 4.13 ([Miyo], [KNS]). *Assume that V_0 is an open algebraic surface and D its normal crossing completion, such that $V = V_0 \cup D$ is projective and the pair (V_0, D) is relatively minimal. Let $K = K_V$ be the canonical divisor.*

(a) *If $\bar{\kappa}(V_0) \geq 0$ then*

$$(K_V + D)^2 \leq 3\chi(V_0). \tag{4.10}$$

(b) *If* $\bar{\kappa}(V_0) = 2$ *and* $K + D = P + N$ *is the Zariski–Fujita decomposition* [Fuj] *then*

$$P^2 \leq 3\chi(V_0). \tag{4.11}$$

Here χ is the topological Euler characteristic, $\bar{\kappa}(V_0)$ is the logarithmic Kodaira dimension of V_0: $\bar{\kappa}(V_0) = \limsup \frac{1}{\log n} \log h^0(V, n(K_V + D))$. Let us recall the notion of the relative minimality of the pair (V, D). Assume that $\bar{\kappa}(V_0) \geq 0$.

Definition 4.14. The pair (V, D) is *relatively minimal* if D is minimal, i.e. it does not contain a (-1)–curve G with branching index $v(G) \leq 2$, and the negative part of $K_V + D$ (in the sense of Zariski–Fujita decomposition) is supported on D.

In the case where V_0 is the complement of an irreducible curve $C \subset \mathbb{C}P^2$ Wakabayashi [Wak] computed that $\bar{\kappa}(V_0) \geq 0$ if, for example, C is rational with at least two singular points. If C has at least three cusps, or at least two singular points and one of them has more than one branch, then $\bar{\kappa} = 2$. In our method we shall use mostly part (b) of the theorem.

Our setting is the following. Let C_0 be a degree d curve in $\mathbb{C}P^2$. Let $x_1, \ldots, x_k$ be its singular points at finite distance. Take $C = C_0 + L_\infty$, where L_∞ is the line at infinity. Denote $y_1, \ldots, y_l$ the singular points of C lying in L_∞. In all statements below we assume that $\bar{\kappa}(\mathbb{C}P^2 \setminus C)$ is either ≥ 0, or is equal to 2; this condition is relatively easy to check.

Let us resolve the singularities of C. We obtain a resolution map $X \xrightarrow{\pi} \mathbb{C}P^2$. Let D be the reduced inverse image of C. We want to apply Theorem 4.13 to the space $X \setminus D$.

The problem is that this space may be not relatively minimal. In the sequel we deal with this problem. First we cite a variant of Lemma 6.20 from [Fuj].

Lemma 4.15. *Suppose that D is connected and there does not exist a (-1)–curve F in X satisfying one of the following conditions:*

(a) *F is contained in D and the branching index of F, $v(F) = F(D - F) \leq 2$ (non-minimality of D);*

(b) *F is not contained in D and $F \cdot D \leq 1$.*

Rhen the pair (X, D) is relatively minimal.

Before discussing when curves satisfying the conditions (a) or (b) of the above Lemma may in fact occur, let us see how such appearance affects the BMY estimates (4.10). Firstly we shall deal with curves of type (a).

Lemma 4.16. *Assume we are given a reduced divisor D_0 on a surface X_0, and let $K_0 = K_{X_0}$ be the canonical divisor. Let us blow up a point $x_0 \in X_0$, $\xi : X_1 \to X_0$, and let $D_1 = \xi^*(D_0)_{red}$ be the reduced inverse image. Moreover let K_1 be the canonical divisor on X_1. If $\operatorname{mult}_{x_0} D_0 = m > 0$ then*

$$(K_1 + D_1)^2 = (K_0 + D_0)^2 - (m - 2)^2. \tag{4.12}$$

Proof. Let E be the exceptional divisor. Then $K_1 = \xi^*(K_0) + E$ and $D_1 = \xi^* D_0 - (m-1)E$ (because we take the the reduced inverse image). So $K_1 + D_1 = \xi^*(K_0 + D_0) - (m - 2)E$. But $\xi^*(A) \cdot E = 0$ for any divisor A on X_0. In fact, by the projection formula $\xi_*(\xi^* A \cdot E) = \xi_*(E) \cdot A = 0$ in the Chow group $A^0(X_0)$. Hence $(K_1+D_1)^2 = (K_0+D_0)^2+(-(m-2)E)^2$, where $E^2 = -1$. The lemma is proved. □

Corollary 4.17. *Let V be a surface and $D \subset V$ a reduced normal crossing divisor. Suppose that, in order to obtain relatively minimal model, we have to contract l_1 (-1)–curves contained in D with branch index 2 and l_2 (-1)–curves in D with branch index 1. Let Y be the resulting space and D' the image of D. Then on V one has*

$$(K + D)^2 = (K_Y + D')^2 - l_2. \tag{4.13}$$

Now let us discuss how large the numbers l_1 and l_2 appearing in Lemma 4.17 may be. In fact, we shall mostly be interested in the number l_2, since it affects the codimension bounds.

Let D contain a (-1)–curve G with branching index at most 2. From the definition of the desingularisation process, we conclude that G cannot be an exceptional curve of the map π. Hence, G is either $\pi'(C_0)$, or $\pi'(L_\infty)$ (the strict transform). But the first possibility can occur only in few cases. Namely we have

Lemma 4.18. *If C_0 satisfies at least one of the following:*

- (i) *the geometrical genus $p_g(C_0) > 0$, so C_0 is not rational;*
- (ii) *C_0 has two branches at infinity and at least one singular point at finite distance;*
- (iii) *C_0 has one place at infinity and at least two cusps or one multiple branched point at finite distance;*
- (iv) *C_0 has three branches at infinity.*

Then $\pi'(C_0)$ is not a (-1)–curve with branching index at most 2.

Proof. Condition (i) says that $\pi'(C_0)$ is not rational so it cannot be a (-1)–curve. Conditions (ii)—(iv) imply that $\pi'(C_0)$ has branching index at least three. □

Therefore $\pi'(C_0)$ can violate the relative minimality condition in few cases. The only interesting case is when C_0 is an annulus that has no singular points at finite distance.

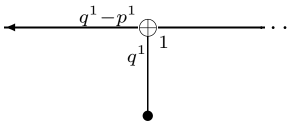
On the other hand, $L = \pi'(L_\infty)$ becomes a (-1) curve rather often. Suppose that this is indeed the case. Let A_1 be a component of D such that $L \cdot A_1 = 1$. In order to obtain a relatively minimal model we have to contract L. But then A_1 may become a (-1) curve as well (if $A_1^2 = -2$ at the beginning). If $v(L) = 2$ then contracting L does not change $(K + D)^2$. A more interesting situation occurs when L is a (-1) curve that is the end of a chain of (-2) curves. Below we study this case more carefully.

Lemma 4.19. *Assume that C_0 has one branch at infinity. Let t be a local parameter on C_0 near the point $C_0 \cap L_\infty$, so that C_0 is parametrised by $x(t) \sim t^p$, $y(t) \sim t^q$ with $t \to \infty$ and $p < q$. Then $\pi'(L_\infty)$ is a (-1)–curve if and only if $q \geq 2p$. Moreover, $\pi'(L_\infty)$ is the end of a chain of l (-2)–curves if and only if $q \geq (2 + l)p$.*

Proof. In local coordinates around infinity we have $u = x/y = s^{q-p} + \dots$, $v = 1/y = s^q + \dots$. After the first blow-up we have $u_2 = s^{q-p} + \dots$, $v_2 = s^p + \dots$ and the strict transform of the line at infinity is given by $v_2 = 0$. Now, if $q - p < p$ then after next blow–up the line at infinity will not be separated from C_0. Hence altogether points on L_∞ are blown–up at least thrice. So $\pi'(L_\infty)^2 \leq -2$. This proves the first part of the lemma.

Let $p^1 = p/\gcd(q, p)$ and $q^1 = q/\gcd(q, p)$. Then the Eisenbud–Neumann diagramm near infinity has the form

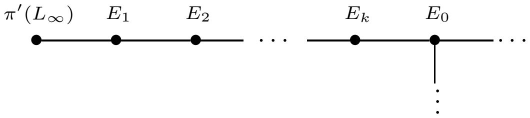
(4.14)



Here the dots denote an uninteresing for a moment part of the diagramm. The arrowhead is at the place where $\pi'(L_\infty)$ is attached.

The procedure described in [EiNe] allows to reconstruct the dual graph of a given singularity. In this particular case the corresponding part of the dual graph has the following form

(4.15)



E_0 is a branching vertex of the diagramm. The self–intersection indices $E_i^2 = -e_i$ are related to q and $q-p$ by the formula

$$\frac{q}{q-p} = 2 - \cfrac{1}{e_1 - \cfrac{1}{e_2 - \cdots - \cfrac{1}{e_k}}} \tag{4.16}$$

The first 2 comes from the assumption that $\pi'(L_\infty)$ is a (-1)–curve. By induction and monotonicity of the function $x \to 2 - \frac{1}{x}$ we obtain the following

Lemma 4.20. *$e_1 = \cdots = e_l = 2$, $l \le k$, if and only if $\frac{q^1}{q^1-p^1} \le \frac{l+2}{l+1}$.*

Therefore the condition $\frac{q}{q-p} \le \frac{l+2}{l+1}$ means that $q \ge (l+2)p$. Lemma 4.19 is proved. ☐

The case of one branch is done. If C_0 intersects L_∞ at more than one point then $\pi'(L_\infty)$ will have the branching index at least 2. So suppose C_0 has r branches $B_1, \dots, B_r$ at one point at infinity. B_i is locally parametrised by $x_i(t) \sim t^{p_i}$, $y_i(t) \sim t^{q_i}$, with $t \to \infty$. We may assume that $p_i < q_i$ (this means that $C_0 \cap L_\infty = [z_1 : z_2 : z_3] = [0 : 1 : 0] \subset \mathbb{CP}^2$ for $x = z_1/z_3$ and $y = z_2/z_3$). If for some i $p_i < 0$ then the branch B_i is not tangent to the line at infinity. It will be separated from L_∞ after the first blowing–up, so it does not influence the lenght of the (-2)–chain. Moreover if for some i we have $q_i < 2p_i$ then B_i will not be separated from L_∞ after two blowing–ups. So L_∞ will eventually have the self–intersection at most -2. Assume therefore that for $i \ge r_0 + 1$ we have $p_i < 0$ and for $i \le r_0$ we have $q_i \ge 2p_i > 0$. Without loss of generality we may suppose that $q_1/p_1 \le q_2/p_2 \le \cdots \le q_{r_0}/p_{r_0}$. Then the Eisenbud–Neumann diagramm of the singularity has the form similar to (4.14) with $q^1 = q_1/\gcd(q_1,p_1)$ and $p^1 = p_1/\gcd(q_1,p_1)$: it is exactly as in (4.14) if $q_1/p_1 < q_2/p_2$. If we have the equalities then the first branching vertex might have higher valency, but the reasoning remains unchanged. Application of Lemma 4.19 yields then the following

Corollary 4.21. *The number of succesively contracted* (-1)*–curves is equal to*

$$\min_{p_i>0}\left\lfloor\frac{q_i}{p_i}\right\rfloor-1. \tag{4.17}$$

Now we shall deal with curves of case (b) of Lemma 4.15.

Lemma 4.22. *Let* $F\not\subset D$ *be a smooth rational curve such that* $F\cdot D\le 1$*. Then* $\pi(F)$ *is isomorphic to a line.*

Proof. Obviously $F\cdot D=1$. Then $\pi(F)$ is a rational curve, possibly singular, of positive degree. Thus it intersects the line at infinity L_∞. Therefore $F\cdot\pi^*(L_\infty)>0$, so F intersects D at the preimage of L_∞. It follows that $\pi(F)$ is a rational curve with one place at infinity, smooth at finite distance. The lemma follows from the Abyankhar–Moh–Suzuki theorem. □

Assume now that there are mutually disjoint (-1)–curves $F_1,\dots,F_n$ such that $F_i\cdot D=1$. Assume also that D does not contain any (-1)–curve with branching index less or equal to 2 (as in point (a)). Let $D_1=D+F_1+\dots+F_n$. Obviously we have

$$(K+D_1)^2=(K+D)^2-n,\qquad \chi(X\setminus D_1)=\chi(X\setminus D)-n. \tag{4.18}$$

But now D_1 is not minimal. We have to contract curves $F_1,\dots,F_n$, as in point (a). Let $\xi:X\to Y$ be the contraction map. Let $D_2=\xi(D_1)$. Then by Corollary 4.17 we obtain

$$(K_Y+D_2)^2=(K+D_1)^2+n=(K+D)^2. \tag{4.19}$$

But $\chi(Y\setminus D_2)=\chi(X\setminus D)-n$. Therefore the BMY inequality gives the following bound

$$(K+D)^2\le 3\chi(X\setminus D)-3n.$$

We see that the appearance of curves satisfying point (a) of Lemma 4.15 alone leads to an improved bound for $(K+D)^2$. The presence of curves satisfying (b) of this lemma improves the estimates, too. It remains to show that if (V,D) contains curves of both types (a) and (b) then the BMY estimates are improved. In fact, theoretically it might happen that $\pi'(L_\infty)$ is a (-1)–curve attached to the chain of (-2)–curves $A_1,\dots,A_m$, but some of above F_i's intersects A_j or $\pi'(L_\infty)$. Then we cannot contract both A_j and F_i. The following lemma shows that such F_i cannot exists.

Lemma 4.23. *Let* $\pi'(L_\infty)$ *be a* (-1) *curve attached to the chain of* m (-2)*–curves* $A_1,\dots,A_m$ *as in Lemma 4.19 (*m *may be zero as well). Let* F *be a rational curve not contained in* D *such that* $F\cdot D=1$ *and* $F\cdot(\pi'(L_\infty)+A_1+\dots+A_m)=1$*. Then* $F^2>0$*.*

Proof. Let G denote $\pi(F)$. By assumption C_0 intersects L_∞ at one point, possibly with many branches. If $F\cdot\pi'(L_\infty)=1$ then G does not intersect the closure of C_0. This contradicts the Bezout theorem. Therefore F must intersect A_r for some $r\le m$.

Let, locally near the point at infinity, G be given by $u = s^a + \dots$, $w = s^b + \dots$, $b > a$ where dots denote terms of higher order and the line at infinity is given by $w = 0$. We shall argue that $a = r$, $b = r + 1$.

The first time we blow up the point $C_0 \cap L_\infty$ we use the map $\pi_0 : (u_1, w_1) \to (u, w) = (u_1, u_1 w_1)$. Here the exceptional divisor is given by $A_0 = \{u_1 = 0\}$. Then come precisely m blow–ups of the form $\pi_{k-1} : (u_k, w_k) \to (u_{k-1}, w_{k-1}) = (u_k w_k, w_k)$. The exceptional divisor of π_{k-1} is $\{w_k = 0\}$ and this is precisely A_{k-1} (by abuse of notation, we denote the exceptional divisor of π_{k-1} with the strict transform under all the remaining blow–ups by the same symbol).

The strict transform of G under the map $\xi_k = \pi_{k-1} \circ \dots \circ \pi_1 \circ \pi_0$ is easy to see to be given by $u_k = s^{a-k(b-a)} + \dots$, $w_k = s^{b-a} + \dots$, provided $k(b-a) \leq a$. Suppose that F intersects A_r. It follows that the strict transform $\xi'_r(G)$ intersects A_r away from the divisor A_0 and A_{r-1}. Therefore $u_r = \text{const} + O(s)$, so $a - r(b-a) = 0$. Moreover, as the intersection index $F \cdot A_r = 1$, we infer that $b - a = 1$, otherwise $\xi'_r(G)$ is tangent to A_r. Therefore $a = r$, $b = r + 1$ as claimed.

Now the self–intersection index of G is equal to $(r+1)^2$. G at the point of multiplicity r at the center of π_0, so $\pi'_0(G)$ has the self–intersection index $(r+1)^2 - r^2$. Then $\pi'_0(G)$ has multiplicity 1 at the center of π_1, and $\xi'_1(G)$ has the multiplicity 1 at the center of π_2 and so on. Therefore $\xi'_r(G)^2 = (r+1) - r^2 - r = r + 1$. As all subsequent blow ups constituting π have centers away from $\xi'_r(G)$, the self–intersection index does not change. Thus $F^2 = r + 1 > 0$. ☐

Remark 4.24. The arguments with degrees and explicit writing of blowing–ups described above could be used to give a more down-to-earth proof of Lemma 4.19.

So now let us consider $D_1 = D + F_1 + \dots + F_n$, where F_i are (-1)–curves not contained in D and $F_i \cdot D = 1$, but D is not necessarely minimal. We have to contract n (-1)–curves F_i and, possibly, the chain of $m + 1$ curves starting from $\pi'(L_\infty)$. Note that if D contains a (-1)–curve L'_∞ with a chain of m (-2)–curves then the same holds for D_1 by virtue of Lemma 4.23, since no F_i intersect this chain. Let $\xi : X \to Y$ be the contraction map such that $D_2 = \xi(D_1)$. Then the BMY inequality for (Y, D_2) yields

$$(K_Y + D_2)^2 - N_Y^2 \leq 3\chi(Y \setminus D_2).$$

which gives

$$(K + D_2)^2 - N_Y^2 \leq 3\chi(X \setminus D) - 3n - (m+1).$$

4.4. **Application of BMY inequality.**

Theorem 4.25. *Assume that C_0 is a rational curve with one place at infinity (and $x(t) = t^p + \dots$, $y(t) = t^q + \dots$, $p < q$) and with precisely R self–intersection at finite distance (more precisely, with arithmetic genus equal to R).*

If $\bar{\kappa}(\mathbb{C}^2 \setminus C_0) = 2$ then

$$\sum \overline{M}_i + ext\nu_\infty \leq p + q - 2 - \left\lfloor \frac{q}{p} \right\rfloor - \sum_i \eta_i + R. \tag{4.20a}$$

If $\bar{\kappa}(\mathbb{C}^2 \setminus C_0) = 0$ then

$$\sum \overline{M}_i + ext\nu_\infty \leq p + q - 2 - \left\lfloor \frac{q}{p} \right\rfloor + R. \tag{4.20b}$$

Proof. Let $x_1, \dots, x_l$ be singular points of C_0 and x_∞ be the point at infinity. Let each point $x_1, \dots, x_k$ have more than one branch and $x_{k+1}, \dots, x_l$ be cuspidal. Let $\pi : X \to \mathbb{C}P^2$ be the resolution of singularities. By $E_{i1}, \dots, E_{ik_i}$, $i \in \{1, 2, \dots, l, \infty\}$ we will denote the exceptional divisors lying over the point x_i and V_i is the subspace of $\operatorname{Pic} X \otimes \mathbb{Q}$ spanned by $E_{i1}, \dots, E_{ik_i}$. Let $\pi'(H)$ be the strict transform of the generic line on $\mathbb{C}P^2$, which, by abuse of notation, we will denote also by H. Denote by V_0 the subspace of $\operatorname{Pic} X \otimes \mathbb{Q}$ spanned by $\mathbb{Q} \cdot [H]$.

The splitting $\operatorname{Pic} X \otimes \mathbb{Q} = \bigoplus_{i=1}^{l} V_i \oplus V_0 \oplus V_\infty$ is orthogonal with respect to the intersection form.

Let us now contract all possible (-1)–curves and let $\xi : X \to Y$ be the contraction. We obtain a divisor $D' \subset Y$, $D' = \xi_*(D)$. By construction ξ is an isomorphism on preimages $\pi^{-1}\{x_{k+1}, \dots, x_l\}$. So it preserves the spaces V_i for $i = k+1, \dots, l$. Let N_i be the negative part of the divisor $K_Y + D'$ projected on V_i. Then, by definition, $-N_i^2 = \eta_i$ is the excess of the singular point x_i. Moreover, $-N^2 \geq \sum_{i=k+1}^{l} \eta_i$. By Theorem 4.13 we get

$$
\begin{aligned}
(K_Y + D')^2 &\leq 3\chi(\mathbb{C} \setminus C_0) && \text{if } \bar{\kappa}(\mathbb{C} \setminus C_0) \geq 0,\\
(K_Y + D')^2 &\leq 3\chi(\mathbb{C} \setminus C_0) - \sum_{i=k+1}^{l} \eta_i && \text{if } \bar{\kappa}(\mathbb{C} \setminus C_0) = 2.
\end{aligned}
\tag{4.21}
$$

(In the presence of (-1)–curves F_i of type (b) we could improve the bound; nevertheless we do not have a satisfactory criterion for the existence of such curves.) Let us assume that $\bar{\kappa}(\mathbb{C} \setminus C_0) = 2$, the other case being treated identically. By Corollary 4.17 and Lemma 4.19 we obtain from (4.21)

$$
(K + D)^2 \leq 3\chi(\mathbb{C} \setminus C_0) - \sum_{i=k+1}^{l} \eta_i + 1 - \left\lfloor \frac{q}{p} \right\rfloor. \tag{4.22}
$$

Now we will examine the left hand side of (4.22).

Firstly observe that $(K+D)^2 = K(K+D) + D(K+D) = K(K+D) + 2p_a(D) - 2$, where p_a is the arithmetic genus. Moreover $K(K+D) = \sum_{i=0}^{l} K_i(K_i + D_i) + K_\infty(K_\infty + D_\infty)$, where K_s and D_s are projections of K and D onto the space V_s. Then $K_0 = -3H$ and $D_0 = (q+1)H$. Moreover, for $i \neq \infty$ the number $K_i(K_i + D_i) = \overline{M}_i$ is the rough $\overline{M}$ number of the i–th singular point (by Proposition 4.1). Only $K_\infty(K_\infty + D_\infty)$ remains to be computed.

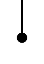
Lemma 4.26. *We have*

$$K_\infty(K_\infty + D_\infty) = q + (q-p) - 2 + ext\nu_\infty.$$

Proof. We can prove this lemma in two ways. Either we compute $K(K+D)$ by method described in Corollary 4.7 or we use the fact that $K_\infty(K_\infty + D_\infty)$ is the codimension of a two–branched singularity at infinity.

<u>First method.</u> The two–branched singularity has the splice diagram at infinity Γ_0 as in (4.14). The singularity of C_0 at this point has the diagramm Γ_1

(4.23)



which differs from Γ_0 only by the end vertex standing in place of the arrowhead. Here $q^1 = q/\gcd(p,q)$, $p^1 = p/\gcd(q,p)$. Then, by (4.7)

$$W(\Gamma_0) = W(\Gamma_1) + \left\lfloor \frac{q^1}{q^1 - p^1} \right\rfloor + 1.$$

As the Milnor number of the smooth branch is zero, we get

$$\overline{M}_1 = \overline{M}_0 + \left\lfloor \frac{q^1}{q^1 - p^1} \right\rfloor + 1,$$

where $\overline{M}_1$ is the $\overline{M}$ number of the singularity $C_0 \cap L_\infty$, and $\overline{M}_0$ of C_0. But, by (2.11),

$$\overline{M}_0 = q - 1 - \left\lfloor \frac{q}{q-p} \right\rfloor + (q - p - 2) + ext\nu_\infty.$$

Therefore $\overline{M}_1 = q + (q-p) + ext\nu_\infty - 2$.

If $p|q$ these computations have to be suitably altered, but the final formula remains unchanged.

Second method. Here we will use the inductive formula (2.9) for $ext\nu(C_0 + L_\infty)$. The tangency codimension of the two branches is $\lfloor \frac{q}{q-p} \rfloor$. So, using Remark 2.15 we get $\overline{M}_1 = \overline{M}_0 + \lfloor \frac{q}{q-p} \rfloor + 1$. □

Using Lemma 4.26 we get

$$(K+D)^2 = 2p_a(D) - 2 + \sum_{i=1}^{k} \overline{M}_i + (6 - 3q) + 2q - p + ext\nu_\infty - 2. \tag{4.24}$$

By assumption, $p_a(C) = R$. Therefore we have $p_a(D) = R$ since the arithmetic genus is a birational invariant (see [GrHa]). Hence, $\chi(C_0) = 1 - R$, so $3\chi(\mathbb{C}^2 \setminus C_0) = 3R$. Using (4.22) and (4.24) yields then the required result.

The proof of Theorem 4.25 is now complete. □

Theorem 4.25 can be extended to arbitrary cases, when the topology of underlying curve C_0 is fixed as well as the behaviour of branches at infinity. The only difficulty is Lemma 4.26 that must be generalised to the case when more branches meet at infinity.

Lemma 4.27. *Let C has n branches $C_1, \dots, C_n$ at one point at infinity with local parametrisation $x_i(t) \sim t^{p_i}$, $y_i(t) \sim t^{q_i}$ as $t \to \infty$ with $0 < p_i < q_i$ and m branches $C_{n+1}, \dots, C_{n+m}$ with parametrisation $x_j(t) \sim t^{-r_j}$, $y_j(t) \sim t^{s_j}$ with $-r_j < 0 < s_j$. Then we have*

$$K_\infty(K_\infty + D_\infty) = \sum_{i=1}^{n}(2q_i - p_i - 1) + \sum_{j=1}^{m}(2s_j - r_j - 1) - \\ - \max_{j=n+1,\dots,n+m} \left\lfloor \frac{s_j + |r_j| - 1}{s_j} \right\rfloor - 1 + ext\nu_{inf},$$

where $ext\nu_{inf}$ is the subtle codimension at infinity (like in Definition 2.26).

Proof. In local coordinates around infinity $v = 1/y$, $u = x/y$ and with $\tau = t^{-1}$ we have

$$u_i(\tau) \sim \tau^{q_i - p_i} \qquad v_i(\tau) \sim \tau^{q_i} \qquad \text{for } 1 \le i \le n$$

$$u_j(\tau) \sim \tau^{s_j + r_j} \qquad v_j(\tau) \sim \tau^{s_j} \qquad \text{for } n+1 \le j \le n+m.$$

The line at infinity can be parametrised by

$$u_0(\tau) = \tau \qquad\qquad v_0(\tau) \equiv 0.$$

The lemma is proved by induction on n and m. Assume firstly that $m = 0$. For $n = 1$ this is exactly Lemma 4.26. Suppose the lemma is proved for $n-1$. By (2.9)

$$ext\nu(L_\infty, C_1, \dots, C_n) = ext\nu(L_\infty, C_1, \dots, C_{n-1}) + ext\nu(C_n) + \nu_{tan} + 2,$$

where ν_{tan} is the tangency codimension of C_n and $L_\infty + C_1 + \dots + C_{n-1}$. But $ext\nu(C_n) = \nu'(C_n) + 2q_n - p_n - 3 - \lfloor\frac{q_n}{q_n-p_n}\rfloor$. On the other hand, $\nu_{tan} = \nu'_{tan} + \lfloor\frac{q_n}{q_n-p_n}\rfloor$. In fact, $\lfloor\frac{q_n}{q_n-p_n}\rfloor$ is the number of common initial inessential terms of the Puiseux expansion of (u_n, v_n) and (u_0, v_0). Then the formula in Definition 2.24 provides the induction step.

Increasing m is very similar. Let us order the branches in such a way that $(s_{n+1}+r_{n+1})/s_{n+1} \le \dots \le (s_{n+m}+r_{n+m})/s_{n+m}$. Then the tangency codimension ν_{tan} of the $(n+m)$–th branch C_{n+m} to $L_\infty + C_1 + \dots + C_{n+m}$ is equal to

$$\nu_{tan} = \left\lfloor \frac{s_{n+m-1} + r_{n+m-1}}{s_{n+m-1}} \right\rfloor + \nu'_{tan},$$

for $\lfloor\frac{s_{n+m-1}+r_{n+m-1}}{s_{n+m-1}}\rfloor$ is the number of common initial inessential terms of the Puiseux expansions of (u_{n+m}, v_{n+m}) and (u_{n+m-1}, v_{n+m-1}). No branch $C_{m+1}, \dots, C_{n+m-2}$ can have more common initial inessential terms because of the choice of ordering of branches (we note by the way that $\nu_{tan}(L_\infty, C_{n+m}) = \nu_{tan}(C_k, C_{n+m}) = 0$ for $k \le n$). The induction step in now routine. □

Lemma 4.27 can be applied to bound the sum of codimensions for annuli.

Theorem 4.28. *Let C_0 be an annulus as in Subsection 2.6 and $p_a(C_0 + L_\infty) = R+1$ (the notation comes from the fact, that if C_0 has no self–intersection at finite distance then $p_a = 1$). Let $K = \max(\lfloor\frac{q}{p}\rfloor, 0)$ if $r < 0$ and $K = \min(\lfloor\frac{q}{p}\rfloor, \lfloor\frac{s}{r}\rfloor)$ if $p, r > 0$. Let also $K_1 = \max(\lfloor\frac{p-q-1}{p}\rfloor, 0)$, $K_2 = \max(\lfloor\frac{s-r-1}{s}\rfloor, 0)$. Then the sum of codimension is bounded by the following formula*

$$\sum \overline{M}_i + ext\nu_{inf} \le p + q + r + s + R + 1 - K + K_1 + K_2. \tag{4.25a}$$

Moreover, if $\kappa(\mathbb{C} \setminus C_0) = 2$ then we can substract $\sum \eta_i$ from the right hand side obtaining

$$\sum \overline{M}_i + ext\nu_{inf} \le p + q + r + s + R + 1 - K + K_1 + K_2 - \sum \eta_j. \tag{4.25b}$$

So for the types $\binom{+}{+}$ and $\binom{-+}{+-}$ we get

$$\sum \overline{M}_i + ext\nu_{inf} \le p + q + r + s + R + 1 - K. \tag{4.26a}$$

For type $\binom{-}{+}$

$$\sum \overline{M}_i + ext\nu_{inf} \le p - |r| + q + s + R + 2 - K + \left\lfloor \frac{|r|-1}{s} \right\rfloor. \tag{4.26b}$$

For type $\binom{-}{-}$

$$\sum \overline{M}_i + ext\nu_{inf} \le p - |r| - |q| + s + R + 3 + \left\lfloor \frac{|r|-1}{s} \right\rfloor + \left\lfloor \frac{|q|-1}{p} \right\rfloor. \tag{4.26c}$$

Proof. As the proof in all cases is very similar, we will focus on the case of type $\binom{+}{+}$ and prove (4.26a). By Lemma 4.19, if $K \geq 2$ then $\pi'(L_\infty)$ becomes a (-1)–curve attached to a chain of $(K-2)$ curves with self–intersection -2. By the BMY inequality we have $(K+D)^2 \leq 4+3R-K$. As $D(K+D) = 2p_a - 2$ we infer that $K(K+D) = 4+R-K$. But $K(K+D) = K_0(K_0+D_0) + \sum \overline{M}_i + K_\infty(K_\infty + D_\infty)$ and $K_0(K_0+D_0) = 6 - 3(q+s)$.

By Lemma 4.27, $K_\infty(K_\infty + D_\infty) = \nu'_{inf} + (2q-p) + (2s-r) - 3$. Hence

$$\sum \overline{M}_i + \nu'_{inf} \leq p+q+r+s+1-K,$$

and the proof in this case is completed. □

Proposition 4.29. *Let C_0 be either a parametric line or an annulus. If $\bar{\kappa}(\mathbb{C}\setminus C_0) = 2$ and at least one branch of C_0 at infinity is not smooth then the inequality* (4.20a) *or* (4.25b) *is sharp.*

Proof. If a branch of C_0 at infinity is not smooth then the resolution of $C_0 \cup L_\infty$ contains a (-1) curve E' with branching index at least 3. If, as in Lemma 4.19, we start contracting the (-1)–curves then we will never contract E'. In fact, we would have to reduce its branching index by blowing down some adjacent curve, but then E' will have self–intersection zero; so it is definitely not contracted. Hence on Y (notation from the proof of Theorem 4.25) D' has some components that lie in the image $\xi_* V_\infty$. As the dual graph of D' is a tree, it follows that D' has a component E_0 in $\xi_* V_\infty$ such that $v(E_0) = 1$. Then $(K_Y + D')E_0 = -1$, so $E_0 \in \operatorname{supp} N$ by the construction of the Zariski–Fujita decomposition [Fuj]. Therefore $N - \sum N_i > 0$, so $-N^2 > \sum \eta_i$. Hence already in (4.21) the inequality is sharp. □

Remark 4.30. If there are no singular points at finite distance the both sides of inequalities (4.20a) or (4.25b) are integers. Therefore, having a sharp inequality improves bound for the sum of codimensions already by 1. Readers of [BZ1] or [BZ2] may appreciate, how important this “1” can be.

Remark 4.31. In the proof of Theorems 4.25 and 4.28 we have tacitly assumed that C_0 satisfies one of the conditions of Lemma 4.18. This guarantees that $\pi'(C_0)$ does not become a (-1) curve that must be contracted in order to obtain a relative minimal model. If we must contract $\pi'(C_0)$ then we cannot argue that the spaces V_i and V_∞ are pairwise orthogonal (see the proof of Theorem 4.25). But this orthogonality is necessary only if C_0 has at least one singular point at finite distance. If C_0 additionally does not satisfy the assumptions of Lemma 4.18 then it is a rational curve with one place at infinity and one unibranched singular point at finite distance. All such curves were classified by Zaidenberg and Lin (see [ZaLi]) so we do not have to worry about them.

5. Application to the problem of limit cycles

Consider the Liénard vector field

$$\dot{x} = y - F(x), \quad \dot{y} = -G'(x), \tag{5.1}$$

where F and G are polynomials of degree $m+1$ and $n+1$ respectively. It is related with the second order Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ via the formulas $f(x) = F'(x)$, $g(x) = G'(x)$. The principal problem concerning the system (5.1) is to find a maximal number $H(m,n)$ of its limit cycles (a special case of the Hilbert's

16th problem). We study a weaker problem, we ask about the number of small limit cycles.

We assume that the origin $x = y = 0$ is a singular point of the center or focus type. Therefore

$$F(x) = a_1 x + \ldots + a_{m+1} x^{m+1}, \quad G(x) = b_2 x^2 + \ldots + b_{n+1} x^{n+1}, \tag{5.2}$$

where $a_1^2 < 8b_2$. We can also assume that $b_2 = 1$. When we introduce the local analytic variable $u = \sqrt{G(x)} = x + \ldots$ then the system (5.1) becomes orbitally equivalent to

$$\dot{u} = y - \Phi(u), \quad \dot{y} = -2u, \quad \Phi = c_1 u + c_2 u^2 + \ldots. \tag{5.3}$$

Here the series $X = c_1 Y^{1/2} + c_2 Y + c_3 Y^{3/2} + \ldots$ is the Puiseux expansion at the point $X = Y = 0$ of the curve

$$C : X = F(x), \;\; Y = G(x). \tag{5.4}$$

It is well known, see [Che], that the system (5.1) (equivalently, (5.3)) has center at the origin if and only if $c_1 = c_3 = \ldots = 0$, i.e. $\Phi(u) = \widetilde{\Phi}(u^2)$ is an even function. From the algebraic point of view this means that the curve (5.4) is multiply covered.

The coefficients $c_1, c_3, c_5, \ldots$ are the essential Puiseux quantities of the singularity $X = Y = 0$ of the curve C. They are related with the *Poincaré–Lyapunov quantities* $g_1, g_3, \ldots$, which appear in the Taylor expansion of the Poincaré return map

$$r \to P(r) = r + g_1 r(1 + \ldots) + g_3 r^3 (1 + \ldots) + \ldots, \quad r \to 0^+, \tag{5.5}$$

from the section $\{(x, y) = (r, 0) : r \geq 0)\}$ to itself. Namely, g_j are proportional to c_j with coefficients depending only on j. We refer the reader to [ChLy] for details.

Since the fixed points of the map (5.5) correspond to the limit cycles of the Liénard vector field, the essential Puiseux quantities of the curve C become responsible for the small amplitude limit cycles of the system (5.1).

The quantities c_j and g_j depend on the coefficients a_k and b_l in the polynomials F and G (see (5.2)). In fact, they are polynomials in $a = (a_1, \ldots, a_{m+1})$ and $b = (b_3, \ldots, b_{n+1})$, e.g. for $b_2 = 1$. So the expansion (5.5) varies with varying (a, b). This variation results in bifurcation of fixed points of the map $P(r)$ from the point $r = 0$ (the generalized Hopf bifurcation). For instance, when $g_{2\nu+1} \neq 0$ and the coefficients $g_1, g_3, \ldots, g_{2\nu-1}$ vary independently then they can be chosen such that either

$$\begin{gathered} 0 < g_1 << -g_3 << g_5 << \ldots \pm g_{2\nu+1}, \qquad \text{or} \\ 0 < -g_1 << g_3 << -g_5 << \ldots \mp g_{2\nu+1}. \end{gathered}$$

Thus one finds exactly ν limit cycles of small amplitude.

Since $g_j(a, b)$ are real polynomials, one cannot ensure free choice of signs, like above (although the functions g_j may be independent).

C. Christopher and S. Lynch in [ChLy] introduced the following quantities:

$\widehat{H}(m, n)$— the maximal number of limit cycles which can bifurcate from the origin;

$H^*(m, n)$— the maximal cyclicity of the focus at $x = y = 0$, i.e. $\max\{\nu : c_1 = c_3 = \ldots = c_{2\nu-1} = 0 = c_{2\nu+1}\}$;

$\widehat{H}_{\mathbb{C}}(m, n)$— the maximal number of limit cycles bifurcating from the origin in the complex sense, i.e. $\frac{1}{2} \times$maximal number of zeroes $r_i \neq 0$ of the function $P(r) - r$ for $r \in (\mathbb{C}, 0)$ (counted with multiplicities);

$H^*_{\mathbb{C}}(m,n)$— the maximal cyclicity of $x=y=0$ in the complex sense, i.e. the codimension ν_0 of the cuspidal singularity of C at $X=Y=0$.
In the definitions of $\widehat{H}_{\mathbb{C}}(m,n)$ and $H^*_{\mathbb{C}}(m,n)$ one assumes complex coefficients a_i, b_j and considers the complex foliation defined by (1.1) in $(\mathbb{C}^2,(0,0))$.

We have the following simple relations

$$\widehat{H}(m,n)\le H^*(m,n)\le H^*_{\mathbb{C}}(m,n)=\widehat{H}_{\mathbb{C}}(m,n).$$

Christopher and Lynch stated several conjectures concerning the above quantities.

Conjecture 5.1 ([ChLy]). *(1) $\widehat{H}_{\mathbb{C}}(m,n)=\widehat{H}_{\mathbb{C}}(n,m)=m+n-2-\left\lfloor\frac{m+1}{n+1}\right\rfloor$ for $2\le n\le m$;*
(2) $\widehat{H}(m,n)=\widehat{H}(n,m)$;
(3) $H^*(m,n)=H^*(n,m)$.

Remark 5.2. Note that when $1\le n\le m$ and we denote $p=n+1$ and $q=m+1$ then $m+n-2-\left\lfloor\frac{m+1}{n+1}\right\rfloor=\sigma=p+q-4-\left\lfloor\frac{q}{p}\right\rfloor$ is the dimension of the space $Curv/Eq$.

When $n=2$ (or $m=2$ and $c_1=0$) the problem is trivial: we have $\widehat{H}(m,1)=H^*(m,1)=H^*_{\mathbb{C}}(m,1)=\left\lfloor\frac{m+1}{2}\right\rfloor$.

In [BZ3] we proved that

$$H^*_{\mathbb{C}}\le\delta_{\max}-1 \tag{5.6}$$

for $m,n\ge 2$, where

$$\delta_{\max}=\delta_{\max}(m,n)=\frac{1}{2}(mn-\gcd(m+1,n+1)+1)$$

is the maximal number of double points of a curve of the form (5.4). In the proof we used the fact that the Milnor number μ_0 of the singularity $X=Y=0$ equals $2\cdot H^*_{\mathbb{C}}$, on the one hand, and the number of double points hidden in the singularity, on the other hand. Moreover, by the Zaidenberg–Lin theorem [ZaLi], the case when all the double points become hidden in the singularity, corresponds to a quasi-homogeneous curve (after reduction) which implies either $m=1$ or $n=1$.

Here we have the following improvement of the bound (5.6).

Theorem 5.3. *If $2\le m<n$ then*

$$H^*_{\mathbb{C}}\le\frac{1}{4}mn+\frac{1}{2}\left(m+n+1-\left\lfloor\frac{n+1}{m+1}\right\rfloor\right)-\frac{1}{4}(\gcd(m+1,n+1)-1).$$

Proof. By Theorem 4.25 we have $\nu_0=\overline{M}_0\le(n+1)+(m+1)+R-1-\left\lfloor\frac{n+1}{m+1}\right\rfloor$, where R is the number of double points. On the other hand, $\nu_0+R\le\delta_{\max}$, i.e. $R\le\frac{1}{2}(mn-\gcd(m+1,n+1)+1)-\nu_0$. These two inequalities give the bound from the thesis of the theorem. □

Acknowledgements. We want to thank Mariusz Koras, Zbigniew Jelonek, Stepan Orevkov and Peter Russell for their interest in our work and interesting discussions.

References

[BZ1] M. Borodzik, H. Żołądek, *Complex algebraic curves via Poincaré–Hopf formula. I. Parametric lines,* Pacific J. Math. (in press).

[BZ2] M. Borodzik and H. Żołądek, *Complex algebraic plane curves via Poincaré–Hopf formula. II. Annuli*, Preprint, University of Warsaw, 2007.

[BZ3] M. Borodzik and H. Żołądek, *Small amplitude limit cycles for the polynomial Liénard system*, Preprint, University of Warsaw, 2007.

[Che] L. A. Cherkas, *On the conditions for a center for certain equations of the form* $yy' = P(x) + Q(x)y + R(x)y^2$, Differential Equations **8** (1974), No 8, 1104–1107 [Russian: **8** (1972), No 8, 1435–1439].

[ChLy] C. Christopher and S. Lynch, *Small–amplitude limit cycle bifurcations for Liénard systems with quadratic damping or restoring forces*, Nonlinearity **12** (1999), 1099–1112.

[EiNe] D. Eisenbud and W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Annals Math. Studies **110**, Princeton University Press, Princeton, 1985.

[FLMN] J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández and A. Némethi, *On rational cuspidal curves, open surfaces and local singularities*, arXiv:math.AG/0604421 v1 19 Apr 2006.

[FZ1] H. Flenner and M. Zaidenberg, $\mathbb{Q}-$*acyclic surfaces and their deformations*, Contemporary Math. **89** (1996), 439–459.

[FZ2] H. Flenner and M. Zaidenberg, *On a class of rational cuspidal plane curves*, Manuscripta Math. **89** (1996), 439–459.

[Fuj] T. Fujita, *On the topology of non-complete algebraic surfaces*, J. Fac. Sci. Univ. Tokyo (Ser. 1A) **29** (1982), 503–566.

[GLS] G-M. Greuel, C. Lossen and E. Shustin, *Castelnuovo function, zero–dimensional schemes and singular plane curves*, J. Algebraic Geometry **9** (2000), No 4, 663–710.

[GrHa] P. Griffiths and J. Harris, *Principles of algebraic geometry*, J. Wiley & Sons, New York, 1978.

[Har] R. Hartshorne, “Algebraic geometry”, Springer–Verlag, New York, 1993.

[Hir] A. Hirano, *Construction of plane curves with cusps*, Saitama Math. J. **10** (1992), 21–24.

[Iit] S. Iitaka, *On logarithmic Kodaira dimension of algebraic varietes*, in: ‘Complex Analysis and Algebraic Geometry’ (A collection of papers dedicated to K. Kodaira), Iwanami, 1977, pp. 175–189.

[KlPi] S. Kleiman and R. Piene, *Enumerating singular curves on surfaces*, Conemporary Math. **241** (1999), 209–239.

[KNS] R. Kobayashi, S. Nakamura and F. Sakai, *A numerical characterization of ball quotients for normal surfaces with branch loci*, Proc. Japan. Acad. Ser. A Math Sci. **65** (1989), 238–241.

[Miyo] Y. Miyaoka, *The minimal number of quotient singularities with given numerical invariants*, Math. Ann. **268** (1984), 159–171.

[Or] S. Yu. Orevkov, *On rational cuspidal curves. I. Sharp estimate for degree via multiplicities*, Math. Ann. **324** (2002), 657–673.

[OZ1] S. Yu. Orevkov and M. G. Zaidenberg, *Some estimates for plane cuspidal curves*, in: “Seminaire d’Algébre et Geometrie, Grenoble 1993”, pp. 1–13.

[OZ2] S. Yu. Orevkov and M. G. Zaidenberg, *On the number of singular points of plane curves*, in: “Algebraic Geometry, Saithana 1995”.

[Pet] G. S. Petrov, *Number of zeroes of complete elliptic integrals*, Funct. Anal. Appl. **18** (1984), 148–149 [Russian: **18** (1984), No 2, 73–74].

[Rus] P. Russell, *On affine ruled rational surfaces*, Math. Ann. **255** (1981), No 3, 287–302.

[Wak] I. Wakabayashi, *On the logarithmic Kodaira dimension of the complement of a curve in* $\mathbf{P}^2$, Proc. Japan Acad. Ser. A. Math. Sci. **54** (1978), 157–162.

[ZaLi] M. G. Zaidenberg and V. Ya. Lin, *An irreducible, simply connected algebraic curve in* $\mathbb{C}^2$ *is equivalent to a quasi-homogeneous curve*, Dokl. Akad. Nauk SSSR **271** (1983), 1048–1052 [Russian].

Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland
E-mail address: `mcboro@mimuw.edu.pl`

E-mail address: `zoladek@mimuw.edu.pl`