1. Local analysis

Problem 1.1 (solved)**.** Prove that the index of a critical point does not depend on the change of basis.

Problem 1.2 (solved)**.** Draw level sets of a Morse function in two and three variables near a critical point of a given index.

Problem 1.3 (??)**.** Suppose f is a smooth function, $f(0,\dots,0)=0$, $Df(0,\dots,0)=0$ and $D^2f(0,\dots,0)$ has non-zero element in the bottom-right corner. Prove that f can be written as $g(x_1,\dots,x_{n-1})\pm x_n^2$.

Problem 1.4 (??)**.** Prove an analog of Morse lemma for the case, where D^2f has one dimensional kernel spanned by v, but $D^3f(v,v,v)\neq 0$.

Problem 1.5 (Problem 1.4 continued)**.** Prove an analog of Morse lemma for the case D^2f has one dimensional kernel spanned by v. Weaken the assumption on the derivatives in the v variable as much as possible.

Problem 1.6 (Problem 1.4 continued)**.** Suppose D^2f has two dimensional kernel. Assuming as much non-degeneracy of D^3f as needed, prove an analog of Morse lemma.

Problem 1.7 (solved)**.** Suppose $f\colon(\mathbb{R}^{n+m},0)\to(\mathbb{R},0)$ has non-zero derivative at 0, but $f|_{\mathbb{R}^n\times\{0\}}$ is Morse. Find a normal form in this case.

Problem 1.8 (??)**.** Study bifurcation diagrams of singularity x^3+y^2, x^4+y^2;

Problem 1.9 (solved)**.** Prove that if $f=-x_1^2-\dots-x_k^2+x_{k+1}^2+\dots+x_n^2$, then for $c>0$, $N=f^{-1}[-c,c]\cap B(0,R)$ is a disk D^n for $c\ll R$. Set $N_-=f^{-1}(-c)\cap B(0,R)$, $N_+=f^{-1}(c)\cap B(0,R)$. Prove that $N_-=S^{k-1}\times D^{n-k}$ and $N_+=D^k\times S^{n-k-1}$.

Problem 1.10 (solved)**.** Let p be a critical point of a Morse function f. Let ξ be a gradient-like vector field and let W^s, W^u be the stable and unstable manifolds of ξ. Show that if c is a level set immediately below (or above) $f(p)$, then $W^s\cap f^{-1}(c)$ (respectively $W^u\cap f^{-1}(c)$) is a sphere.

2. Generalities on Morse functions

Problem 2.1 (??)**.** Prove that on a closed manifold Morse functions are open and dense.

Problem 2.2 (??)**.** Prove the isotopy injection lemma. That is, if N is a closed manifold and $H\colon N\times[0,1]\to N$ is an isotopy (that is $H|_{N\times\{0\}}$ is the identity and $H|_{N\times\{t\}}$ is a diffeomorphism for all t), then there exist a vector field v, gradient-like for the projection on the second factor $F\colon N\times[0,1]\to[0,1]$, such that the flow ϕ_v induces the map H.

Problem 2.3 (home)**.** Prove that there exists a polynomial function on $\mathbb{R}^2$, bounded below, but without global minimum.

Problem 2.4 (solved)**.** Suppose M is a closed manifold that admits a Morse function f with precisely two critical points of index 0 and no critical points of index 1. Show that M is disconnected.

Problem 2.5 (solved)**.** Suppose M is a closed manifold that admits a Morse function f with precisely one critical points of index 0 and no critical points of index 1. Show that M is simply connected.

Problem 2.6 (home)**.** Generalizing, show that if M admits a Morse function with one local minimum, k critical points of index 1 and ℓ critical points of index 2, then $\pi_1(M)$ can be presented as a group with k generators and ℓ relations.

3. Morse functions on concrete manifolds

In Problems 3.1 through 3.4 the task is to find a Morse function with as little critical points as possible and as explicit as possible.

Problem 3.1 (almost solved)**.** Find a Morse function on a genus g surface.

Problem 3.2 (solved)**.** Find a Morse function on a sphere of dimension n.

Problem 3.3 (??)**.** Find a Morse function on a lens space. Consider both cases of dimension 3 and of general dimension.

Problem 3.4 (??)**.** Find a Morse function on a Seifert fibered space.

Problem 3.5 (home)**.** Show that there exists a Morse function on $\mathbb{R}^2$ with two local minima, without a saddle.

Problem 3.6 (solved)**.** Prove that on a closed connected manifold there always exists a Morse function with precisely one minimum and precisely one maximum.

Problem 3.7 (partially solved)**.** Prove that the embedded version of Problem 3.6 dramatically fails. Namely, if K is a knot in $\mathbb{R}^3$ and the projection $F\colon \mathbb{R}^3 \to \mathbb{R}$ is such that $F|_K$ has one minimum and one maximum, then K is an unknot. Show that for any $n > 0$ there exists a knot K_n such that for any embedding of K_n into $\mathbb{R}^3$, the restriction $F|_{K_n}$ has at least n local minima and local maxima.

Problem 3.8 (??)**.** Find a Morse function on a $S^2 \times S^2$.

Problem 3.9 (almost solved)**.** Let U be a unitary matrix of size $n \times n$. Show that the function $x \mapsto (x^T Ux)/||x||^2$ descends to a function on $\mathbb{C}P^{n-1}$ and find its critical points.

Problem 3.10 (??)**.** Is there a matrix group $G \subset GL(n, \mathbb{R})$ such that the function $A \mapsto \operatorname{Tr} A$ is Morse?

Problem 3.11 (??)**.** Is there a matrix group $G \subset U(n)$ such that the function $U \to x^T Ux$ for $x = (1, 0, \ldots, 0)$ is Morse?

Problem 3.12 (??)**.** Generalize Problem 3.7 to the following question. Suppose that K is an n-dimensional submanifold of $\mathbb{R}^m$ (with $m > n$) and the projection of $\mathbb{R}^m$ to $\mathbb{R}$ restricts to a Morse function on K having two critical points. Check, for which n and m, this condition implies that there exists a disk D^{n+1} embedded in $\mathbb{R}^m$, whose boundary is K.

Problem 3.13 (home)**.** Let $f\colon M \to \mathbb{R}$ be a self-indexing Morse function. Define $M_k = f^{-1}(-\infty, k + 1/2)$, $Y_k = f^{-1}[k - 1/2, k + 1/2]$. Show that $H_*(M_1)$ can be computed via long exact sequence of homology of the pair M_1, M_0. Draw a method of showing that singular homology is homology of the complex $\cdots \to H_k(M_k, M_{k-1}) \to H_{k-1}(M_{k-1}, M_{k-2}) \to \ldots$ without referring to the spectral sequence.

Problem 3.14 (home)**.** Let Ω be a closed $(n + k)$-dimensional manifold, $k > 1$ and M is a closed submanifold. Show that there exists a Morse function $F\colon \Omega \to \mathbb{R}$ whose all critical points of index j are on the level set $F^{-1}(j)$ and all critical points of index j of $F|_M$ are at the level set $F^{-1}(j + 1/2)$. Where is the assumption on $k > 1$ used? What if this assumption is not satisfied?

Problem 3.15 (home)**.** Suppose F as in Problem 3.14 Show that there is a map p_j between Morse chain complexes $C_j(M) \to C_j(\Omega)$ that counts trajectories from critical points of $F|_M$ to critical points of F. Check for $j = 0, 1$ that this map induces the inclusion-induced map on $H_j(M) \to H_j(\Omega)$.

Problem 3.16 (home)**.** Shpw that the map constructed in Problem 3.15 induces the map $H_j(M) \to H_j(\Omega)$ induced by inclusion.

4. Gradient-like vector fields

Problem 4.1 (??)**.** Prove that if v is a gradient-like vector field for $F\colon M \to \mathbb{R}$, then there exists a Riemannian metric on M such that $v = \nabla F$.

Problem 4.2 (solved)**.** Show the lack of functoriality of gradient. That is, if $\phi\colon M \to N$ is a smooth map between two Riemannian manifolds, $F\colon N \to \mathbb{R}$ is a smooth function and $G = F \circ \phi\colon M \to N$, then $D\phi(\nabla G) = \nabla F$ for all such F if and only if ϕ is a local isometry.

Problem 4.3 (solved)**.** Suppose v is a gradient-like vector field for two Morse functions F_0 and F_1. Is it a gradient-like vector field for a convex combination of F_0 and F_1?

5. Embedded Morse theory

Problem 5.1 (solved)**.** Suppose $M \subset \Omega$ is a smoothly embedded manifold. Let $F\colon \Omega \to \mathbb{R}$ be a Morse function whose critical points belong to $\Omega \setminus M$ and such that $F|_M$ is Morse. Let ξ be a vector field on Ω satisfying the following conditions:

- it is tangent to M;
- it is gradient-like for F;
- it is gradient-like for $F|_M$.

Show that these conditions are mutually exclusive in general.

Problem 5.2 (solved, Embedded Morse Lemma)**.** Suppose $M \subset \Omega$, $F\colon \Omega \to \mathbb{R}$ is Morse and $F|_M$ is Morse. Suppose all critical points of F are away from M (so a critical point of $F|_M$ is not a critical point of F). Consider $p \in M$. Prove that there exist local coordinates $(x_1, \dots, x_n, y_1, \dots, y_k)$ such that M is given by $y_1 = \dots = y_k = 0$ and $F(x_1, \dots, y_k) = c - x_1^2 - \dots - x_h^2 + x_{h+1}^2 + \dots + x_n^2 + y_1$.

Problem 5.3 (solved)**.** Suppose $(x_1, \dots, y_k)$ are like in Problem 5.2. Assume ξ is a vector field in these coordinates given by:

$$\xi = (-x_1, \dots, -x_h, x_{h+1}, \dots, x_n, \sum_{j=1}^{k} y_j^2, 0, \dots, 0). \tag{1}$$

Prove that ξ is tangent to M, $\partial_\xi F \geq 0$ with equality only at the critical point.

Problem 5.4 (solved)**.** Study the stable and ustable set of ξ. What is its intersection with $F^{-1}(\alpha)$ for $\alpha < c$ and $\alpha > c$? What the intersection of these level sets with M?

Problem 5.5 (??)**.** Suppose K is a knot in S^3 that bounds a disk D in B^4 such that the function 'distance to origin' on B^4 restricts to a Morse function on D with one critical point (this is a minimum). Show that K is trivial. Deduce that for some concrete examples cancellation is impossible in codimension 2.

Problem 5.6. Let K be a knot in S^3 and Σ_0, Σ_1 are closed oriented surfaces such that $\partial\Sigma_0 = \partial\Sigma_1 = K$. Prove that Σ_1 can be obtained by Σ_0 by a sequence of zero-surgeries and 1-surgeries.

Hint. Show that there exists a three-manifold $\Omega \subset S^3 \times [0,1]$ such that $\partial\Omega = \Sigma_0 \times \{0\} \cup K \times [0,1] \cup \Sigma_1 \times \{1\}$. Study the projection $\Omega \to [0,1]$ and its critical points.

6. Morse theory for manifolds with boundary

Suppose M is a manifold with boundary and F is a Morse function. A gradient-like vector field ξ for F is assumed to be tangent to M. For point $z \in \partial M$, which is a critical point of F, we say that z is boundary unstable, if the tangent space to $W^s(z)$ is a subspace of $T_z\partial M$, and boundary stable otherwise.

Problem 6.1 (important)**.** Study the behavior of level sets of F after crossing a boundary stable and boundary unstable critical points.