Heegaard Floer homologies and rational cuspidal curves

Maciej Borodzik
www.mimuw.edu.pl/~mcboro

Institute of Mathematics, Polish Academy of Science

Warsaw, February 2016
Heegaard decomposition of 3-manifolds

Definition

Let $Y$ be a closed oriented and connected 3–manifold. A Heegaard decomposition is the presentation of $Y$ as a union $Y = H_1 \cup \Sigma \cup H_2$, where $H_1$ and $H_2$ are handlebodies and $\Sigma$ is a closed connected surface.
Heegaard decomposition of 3–manifolds

Definition
Let $Y$ be a closed oriented and connected 3–manifold. A Heegaard decomposition is the presentation of $Y$ as a union $Y = H_1 \cup \Sigma \cup H_2$, where $H_1$ and $H_2$ are handlebodies and $\Sigma$ is a closed connected surface.

Theorem
Each three manifold admits a Heegaard decomposition. Any two Heegaard decompositions are related by stabilizations and destabilizations.
Heegaard decomposition of 3-manifolds

Definition
Let $Y$ be a closed oriented and connected 3–manifold. A Heegaard decomposition is the presentation of $Y$ as a union $Y = H_1 \cup \Sigma \cup H_2$, where $H_1$ and $H_2$ are handlebodies and $\Sigma$ is a closed connected surface.

Theorem
Each three manifold admits a Heegaard decomposition. Any two Heegaard decompositions are related by stabilizations and destabilizations.

Sketch of proof.
Use Morse theory.
Example

The only manifold admitting a Heegaard decomposition of genus 0 is $S^3$. 
The only manifold admitting a Heegaard decomposition of genus 0 is $S^3$.

For genus 1 $H_1$ and $H_2$ are two solid tori glued along their boundary. Then $Y$ is either $S^3$ or a lens space.
Example

- The only manifold admitting a Heegaard decomposition of genus 0 is $S^3$.
- For genus 1 $H_1$ and $H_2$ are two solid tori glued along their boundary. Then $Y$ is either $S^3$ or a lens space.

**Problem**

*Prove the last statement.*
Heegaard diagrams

Definition

Let $H_1 \cup \Sigma \cup H_2$ be a Heegaard decomposition. Let $g = \text{genus}(\Sigma)$. A **Heegaard diagram** is a triple $(\Sigma, \alpha, \beta)$, where $\alpha$ and $\beta$ are unordered collections of $g$ simple closed curves, $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$, such that
Heegaard diagrams

Definition

Let $H_1 \cup \Sigma \cup H_2$ be a Heegaard decomposition. Let $g = \text{genus}(\Sigma)$. A Heegaard diagram is a triple $(\Sigma, \alpha, \beta)$, where $\alpha$ and $\beta$ are unordered collections of $g$ simple closed curves, $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$, such that

- $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$ if $i \neq j$;
Heegaard diagrams

Definition

Let $H_1 \cup \Sigma \cup H_2$ be a Heegaard decomposition. Let $g = \text{genus}(\Sigma)$. A Heegaard diagram is a triple $(\Sigma, \alpha, \beta)$, where $\alpha$ and $\beta$ are unordered collections of $g$ simple closed curves, $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$, such that

- $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$ if $i \neq j$;
- The curves $\{\alpha_1, \ldots, \alpha_g\}$ form a basis of ker $H_1(\Sigma) \to H_1(H_1)$ and $\{\beta_1, \ldots, \beta_g\}$ form a basis of ker $H_1(\Sigma) \to H_1(H_2)$.
Definition

Let $H_1 \cup \Sigma \cup H_2$ be a Heegaard decomposition. Let $g = \text{genus}(\Sigma)$. A Heegaard diagram is a triple $(\Sigma, \alpha, \beta)$, where $\alpha$ and $\beta$ are unordered collections of $g$ simple closed curves, $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$, such that

1. $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$ if $i \neq j$;
2. The curves $\{\alpha_1, \ldots, \alpha_g\}$ form a basis of $\ker H_1(\Sigma) \to H_1(H_1)$ and $\{\beta_1, \ldots, \beta_g\}$ form a basis of $\ker H_1(\Sigma) \to H_1(H_2)$.

Another way: $\alpha_1, \ldots, \alpha_g$ bound disjoint disks in $H_1$ such that the complement of these disks in $H_1$ is a 3–ball.
Heegaard diagrams

Definition

Let $H_1 \cup \Sigma \cup H_2$ be a Heegaard decomposition. Let $g = \text{genus}(\Sigma)$. A Heegaard diagram is a triple $(\Sigma, \alpha, \beta)$, where $\alpha$ and $\beta$ are unordered collections of $g$ simple closed curves, $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$, such that

- $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$ if $i \neq j$;
- The curves $\{\alpha_1, \ldots, \alpha_g\}$ form a basis of $\ker H_1(\Sigma) \to H_1(H_1)$ and $\{\beta_1, \ldots, \beta_g\}$ form a basis of $\ker H_1(\Sigma) \to H_1(H_2)$.

Another way: $\alpha_1, \ldots, \alpha_g$ bound disjoint disks in $H_1$ such that the complement of these disks in $H_1$ is a 3–ball.

A pointed Heegaard diagram is a quadruple $(\Sigma, \alpha, \beta, z)$, where $z \in \Sigma \setminus (\alpha \cup \beta)$.
Let \((\Sigma, \alpha, \beta, z)\) be a pointed Heegaard diagram. Let \(g = \text{genus}(\Sigma)\).
Let \((\Sigma, \alpha, \beta, z)\) be a pointed Heegaard diagram. Let \(g = \text{genus}(\Sigma)\). For technical reasons assume that \(g > 2\).
Let \((\Sigma, \alpha, \beta, z)\) be a pointed Heegaard diagram. Let \(g = \text{genus}(\Sigma)\). For technical reasons assume that \(g > 2\). There will be some technicalities now:
Symmetric products

Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram. Let $g = \text{genus}(\Sigma)$. For technical reasons assume that $g > 2$. There will be some technicalities now: Choose a complex structure on $\Sigma$. 
Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram. Let $g = \text{genus}(\Sigma)$. For technical reasons assume that $g > 2$. There will be some technicalities now: Choose a complex structure on $\Sigma$. Let $\text{Sym}^g(\Sigma) = \Sigma \times \ldots \Sigma / S_g$, where $S_g$ is the symmetric group.
Let \((\Sigma, \alpha, \beta, z)\) be a pointed Heegaard diagram. Let \(g = \text{genus}(\Sigma)\). For technical reasons assume that \(g > 2\). There will be some technicalities now: Choose a complex structure on \(\Sigma\). Let \(\text{Sym}^g(\Sigma) = \Sigma \times \ldots \Sigma / S_g\), where \(S_g\) is the symmetric group. The product of \(\alpha\)-curves is a torus \(T_\alpha\) and the product of \(\beta\)-curves is a torus \(T_\beta\).
Let \((\Sigma, \alpha, \beta, z)\) be a pointed Heegaard diagram. Let \(g = \text{genus}(\Sigma)\). For technical reasons assume that \(g > 2\). There will be some technicalities now: Choose a complex structure on \(\Sigma\). Let \(\text{Sym}^g(\Sigma) = \Sigma \times \ldots \Sigma / S_g\), where \(S_g\) is the symmetric group. The product of \(\alpha\)-curves is a torus \(T_\alpha\) and the product of \(\beta\)-curves is a torus \(T_\beta\). These tori are lagrangian.
Let \((\Sigma, \alpha, \beta, z)\) be a pointed Heegaard diagram. Let \(g = \text{genus}(\Sigma)\). For technical reasons assume that \(g > 2\). There will be some technicalities now: Choose a complex structure on \(\Sigma\). Let \(\text{Sym}^g(\Sigma) = \Sigma \times \ldots \Sigma / S_g\), where \(S_g\) is the symmetric group. The product of \(\alpha\)-curves is a torus \(T_{\alpha}\) and the product of \(\beta\)-curves is a torus \(T_{\beta}\). These tori are lagrangian. The main object is the set of intersection points \(T_{\alpha} \cap T_{\beta}\).
Let \((\Sigma, \alpha, \beta, z)\) be a pointed Heegaard diagram. Let \(g = \text{genus}(\Sigma)\). For technical reasons assume that \(g > 2\). There will be some technicalities now: Choose a complex structure on \(\Sigma\). Let \(\text{Sym}^g(\Sigma) = \Sigma \times \ldots \Sigma / S_g\), where \(S_g\) is the symmetric group. The product of \(\alpha\)-curves is a torus \(T_\alpha\) and the product of \(\beta\)-curves is a torus \(T_\beta\). These tori are lagrangian. The main object is the set of intersection points \(T_\alpha \cap T_\beta\).

**Problem**

Show that there is a 1–1 correspondence between points \(x \in T_\alpha \cap T_\beta\) and \(g\)–tuples of points \((x_1, \ldots, x_g) \in \Sigma\) such that there exists a permutation \(\sigma : \{1, \ldots, g\} \to \{1, \ldots, g\}\) and \(x_i \in \alpha_i \cap \beta_{\sigma(i)}\).
Let \((\Sigma, \alpha, \beta, z)\) be a pointed Heegaard diagram. Let \(g = \text{genus}(\Sigma)\). For technical reasons assume that \(g > 2\). There will be some technicalities now: Choose a complex structure on \(\Sigma\). Let \(\text{Sym}^g(\Sigma) = \Sigma \times \ldots \Sigma / S_g\), where \(S_g\) is the symmetric group. The product of \(\alpha\)-curves is a torus \(T_\alpha\) and the product of \(\beta\)-curves is a torus \(T_\beta\). These tori are lagrangian. The main object is the set of intersection points \(T_\alpha \cap T_\beta\).

**Problem**

*Show that there is a 1–1 correspondence between points \(x \in T_\alpha \cap T_\beta\) and \(g\)-tuples of points \((x_1, \ldots, x_g) \in \Sigma\) such that there exists a permutation \(\sigma: \{1, \ldots, g\} \to \{1, \ldots, g\}\) and \(x_i \in \alpha_i \cap \beta_{\sigma(i)}\).*

Let \(D_z = \{z\} \times \Sigma \times \ldots \times \Sigma\). Then \(D_z\) is a divisor in \(\text{Sym}^g(\Sigma)\).
The chain complex $\widehat{CF}$ and $CF^-$

The chain complex $\widehat{CF}$ is defined (over $\mathbb{Z}_2$) by intersection points $T_\alpha \cap T_\beta$. 
The chain complex $\widehat{CF}$ and $CF^-$

- The chain complex $\widehat{CF}$ is defined (over $\mathbb{Z}_2$) by intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.
- The chain complex $CF^-$ is defined over $\mathbb{Z}_2[U]$ by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. 
The chain complex $\widehat{CF}$ and $CF^-$

- The chain complex $\widehat{CF}$ is defined (over $\mathbb{Z}_2$) by intersection points $T_\alpha \cap T_\beta$.
- The chain complex $CF^-$ is defined over $\mathbb{Z}_2[U]$ by $T_\alpha \cap T_\beta$.
- The differential counts holomorphic maps $\phi: D \to \text{Sym}^g$ such that $\phi(-1) = x$, $\phi(1) = y$, $\phi_{\partial_- D} \subset T_\alpha$ and $\phi_{\partial_+ D} \subset T_\beta$. We make it precise.
The chain complex $\hat{CF}$ and $CF^-$

- The chain complex $\hat{CF}$ is defined (over $\mathbb{Z}_2$) by intersection points $T_\alpha \cap T_\beta$.
- The chain complex $CF^-$ is defined over $\mathbb{Z}_2[U]$ by $T_\alpha \cap T_\beta$.
- The differential counts holomorphic maps $\phi: D \to \text{Sym}^g$ such that $\phi(-1) = x$, $\phi(1) = y$, $\phi_{\partial_-}D \subset T_\alpha$ and $\phi_{\partial_+}D \subset T_\beta$. We make it precise.
- Define $\pi_2(x, y)$ a set of homotopy classes of continuous maps as above.
The chain complex \( \hat{CF} \) and \( CF^- \)

- The chain complex \( \hat{CF} \) is defined (over \( \mathbb{Z}_2 \)) by intersection points \( T_\alpha \cap T_\beta \).
- The chain complex \( CF^- \) is defined over \( \mathbb{Z}_2[U] \) by \( T_\alpha \cap T_\beta \).
- The differential counts holomorphic maps \( \phi: D \to \text{Sym}^g \) such that \( \phi(-1) = x, \phi(1) = y, \phi_{\partial_- D} \subset T_\alpha \) and \( \phi_{\partial_+ D} \subset T_\beta \). We make it precise.
- Define \( \pi_2(x, y) \) a set of homotopy classes of continuous maps as above.
- For each \( \phi \in \pi_2(x, y) \) there is a uniquely defined integer, the Maslov class, \( \mu(\phi) \). This is the dimension of the moduli space of holomorphic maps.
The chain complex $\hat{CF}$ and $CF^-$

- The chain complex $\hat{CF}$ is defined (over $\mathbb{Z}_2$) by intersection points $T_\alpha \cap T_\beta$.
- The chain complex $CF^-$ is defined over $\mathbb{Z}_2[U]$ by $T_\alpha \cap T_\beta$.
- The differential counts holomorphic maps $\phi: D \to \text{Sym}^g$ such that $\phi(-1) = x$, $\phi(1) = y$, $\phi_{\partial_-}D \subset T_\alpha$ and $\phi_{\partial_+}D \subset T_\beta$. We make it precise.
- Define $\pi_2(x, y)$ a set of homotopy classes of continuous maps as above.
- For each $\phi \in \pi_2(x, y)$ there is a uniquely defined integer, the Maslov class, $\mu(\phi)$. This is the dimension of the moduli space of of holomorphic maps.
- The differential for $\hat{CF}$ is

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y): \mu(\phi)=1, n_z(\phi)=0} \# \mathcal{M}(\phi)y.$$
The differential for $CF^-$ (and later for $CF^\infty$) is

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y) : \mu(\phi) = 1} \# M(\phi) U_n^\tau(\phi) y.$$
Complexes $CF^+$ and $CF^\infty$

The differential for $CF^-$ (and later for $CF^\infty$) is

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x,y): \mu(\phi)=1} \# M(\phi) U^{n_2(\phi)} y.$$

The complex $CF^\infty$ arises from $CF^-$ by replacing $\mathbb{Z}_2[U]$ by $\mathbb{Z}_2[U, U^{-1}]$. 
Complexes $CF^+$ and $CF^\infty$

The differential for $CF^-$ (and later for $CF^\infty$) is

$$\partial x = \sum_{y \in T_{\alpha} \cap T_{\beta}} \sum_{\phi \in \pi_2(x, y) : \mu(\phi) = 1} \# \mathcal{M}(\phi) U^{n_z(\phi)} y.$$  

The complex $CF^\infty$ arises from $CF^-$ by replacing $\mathbb{Z}_2[U]$ by $\mathbb{Z}_2[U, U^{-1}]$. The complex $CF^+$ is the quotient $CF^\infty / CF^-$. 

Maciej Borodzik (Institute of Mathematics, Polish Academy of Science) 
Heegaard Floer homologies and rational cuspidal curves 
Warsaw, February 2016 7 / 45
Complexes $CF^+$ and $CF^\infty$

The differential for $CF^-$ (and later for $CF^\infty$) is

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x,y): \mu(\phi) = 1} \# M(\phi) U^{n_\pi(\phi)} y.$$ 

The complex $CF^\infty$ arises from $CF^-$ by replacing $\mathbb{Z}_2[U]$ by $\mathbb{Z}_2[U, U^{-1}]$. The complex $CF^+$ is the quotient $CF^\infty / CF^-$. The short exact sequence $0 \rightarrow CF^- \rightarrow CF^\infty \rightarrow CF^+ \rightarrow 0$ gives rise to an exact triangle in homology.
The differential for $CF^-$ (and later for $CF^\infty$) is

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x,y) : \mu(\phi) = 1} \# M(\phi) U^{n_2}(\phi) y.$$

The complex $CF^\infty$ arises from $CF^-$ by replacing $\mathbb{Z}_2[U]$ by $\mathbb{Z}_2[U, U^{-1}]$. The complex $CF^+$ is the quotient $CF^\infty / CF^-$. The short exact sequence $0 \to CF^- \to CF^\infty \to CF^+ \to 0$ gives rise to an exact triangle in homology.

**Problem**

*Prove that there exists a short exact sequence*

$$0 \to \hat{CF} \to CF^+ \xrightarrow{U} CF^+ \to 0$$

*giving rise to a long exact sequence in homology.*
Complexes $CF^+$ and $CF^\infty$

The differential for $CF^-$ (and later for $CF^\infty$) is

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x, y): \mu(\phi) = 1} \#M(\phi) U^{n_z(\phi)} y.$$ 

The complex $CF^\infty$ arises from $CF^-$ by replacing $\mathbb{Z}_2[U]$ by $\mathbb{Z}_2[U, U^{-1}]$. The complex $CF^+$ is the quotient $CF^\infty / CF^-$. The short exact sequence $0 \to CF^- \to CF^\infty \to CF^+ \to 0$ gives rise to an exact triangle in homology.

Problem

Prove that there exists a short exact sequence

$$0 \to \hat{CF} \to CF^+ \xrightarrow{U} CF^+ \to 0$$

giving rise to a long exact sequence in homology.

A remark. The complexes have a relative grading, called the Maslov grading, with $M(x) - M(y) = \mu(\phi) - 2n_z(\phi)$. 
Spin-c structures

Definition

A spin-c structure on a manifold $M$ is a choice of a complex line bundle $L \to M$ together with a choice of spin structure on $TM \otimes L^{-1}$. 
Spin-c structures

Definition

A spin-c structure on a manifold \( M \) is a choice of a complex line bundle \( L \to M \) together with a choice of spin structure on \( TM \otimes L^{-1} \).

- We can think of spin-c structure as line bundles.
Spin-c structures

Definition

A spin-c structure on a manifold $M$ is a choice of a complex line bundle $L \to M$ together with a choice of spin structure on $TM \otimes L^{-1}$.

- We can think of spin-c structure as line bundles.
- Spin-c structures admit an action of $H_1(M; \mathbb{Z})$. 
Spin-c structures

Definition

A spin-c structure on a manifold $M$ is a choice of a complex line bundle $L \to M$ together with a choice of spin structure on $TM \otimes L^{-1}$.

- We can think of spin-c structure as line bundles.
- Spin-c structures admit an action of $H_1(M; \mathbb{Z})$.
- If $H_1(M; \mathbb{Z}_2) = 0$, then we have a correspondence between spin-c structures and elements in $H_1(M; \mathbb{Z})$. 
Spin-c structures

Definition

A spin-c structure on a manifold $M$ is a choice of a complex line bundle $L \rightarrow M$ together with a choice of spin structure on $TM \otimes L^{-1}$.

- We can think of spin-c structure as line bundles.
- Spin-c structures admit an action of $H_1(M; \mathbb{Z})$.
- If $H_1(M; \mathbb{Z}_2) = 0$, then we have a correspondence between spin-c structures and elements in $H_1(M; \mathbb{Z})$.

Theorem (Turaev)

Suppose $\dim M = 3$. Consider the set of non-vanishing vector fields on $M$. Consider two vector fields equivalent if they are homotopic through vector fields non-vanishing outside of a point. Then the set of abstraction classes is in a bijective correspondence with the set of spin-c structures.
Choose a generator $x \in \widehat{CF}$. 
Choose a generator $x \in \widehat{CF}$. This is the g-tuple of points $x_1, \ldots, x_g$ such that $x_j \in \alpha_j \cap \beta_{\sigma(j)}$. 

Choose a generator $x \in \widehat{CF}$. This is the g-tuple of points $x_1, \ldots, x_g$ such that $x_j \in \alpha_j \cap \beta_{\sigma(j)}$. Each point $x_j$ corresponds to a trajectory of $\nabla F$ connecting index 1 and 2 critical points.
Choose a generator $x \in \widehat{CF}$. This is the g-tuple of points $x_1, \ldots, x_g$ such that $x_j \in \alpha_j \cap \beta_{\sigma(j)}$. Each point $x_j$ corresponds to a trajectory of $\nabla F$ connecting index 1 and 2 critical points. Remove all these trajectories from $M$. 
Choose a generator $x \in \widehat{CF}$. This is the g-tuple of points $x_1, \ldots, x_g$ such that $x_j \in \alpha_j \cap \beta_{\sigma(j)}$. Each point $x_j$ corresponds to a trajectory of $\nabla F$ connecting index 1 and 2 critical points. Remove all these trajectories from $M$. Remove the trajectory of $\nabla F$ through the marked point $z$. 
Choose a generator $x \in \widehat{CF}$. This is the g-tuple of points $x_1, \ldots, x_g$ such that $x_j \in \alpha_j \cap \beta_{\sigma(j)}$. Each point $x_j$ corresponds to a trajectory of $\nabla F$ connecting index 1 and 2 critical points. Remove all these trajectories from $M$. Remove the trajectory of $\nabla F$ through the marked point $z$. Then $\nabla F$ is non-vanishing.
Choose a generator $x \in \hat{CF}$. This is the g-tuple of points $x_1, \ldots, x_g$ such that $x_j \in \alpha_j \cap \beta_{\sigma(j)}$. Each point $x_j$ corresponds to a trajectory of $\nabla F$ connecting index 1 and 2 critical points. Remove all these trajectories from $M$. Remove the trajectory of $\nabla F$ through the marked point $z$. Then $\nabla F$ is non-vanishing.

Each generator $x \in \hat{CF}$ determines a spin-c structure on $M$.
Choose a generator \( x \in \widehat{CF} \). This is the \( g \)-tuple of points \( x_1, \ldots, x_g \) such that \( x_j \in \alpha_j \cap \beta_{\sigma(j)} \). Each point \( x_j \) corresponds to a trajectory of \( \nabla F \) connecting index 1 and 2 critical points. Remove all these trajectories from \( M \). Remove the trajectory of \( \nabla F \) through the marked point \( z \). Then \( \nabla F \) is non-vanishing.

Each generator \( x \in \widehat{CF} \) determines a spin-c structure on \( M \)

**Problem**

*Show that if the differential from \( x \) to \( y \) is non-trivial, then \( x \) and \( y \) determine the same spin-c structure.*
Choose a generator \( x \in \widehat{CF} \). This is the g-tuple of points \( x_1, \ldots, x_g \) such that \( x_j \in \alpha_j \cap \beta_{\sigma(j)} \). Each point \( x_j \) corresponds to a trajectory of \( \nabla F \) connecting index 1 and 2 critical points. Remove all these trajectories from \( M \). Remove the trajectory of \( \nabla F \) through the marked point \( z \). Then \( \nabla F \) is non-vanishing.

Each generator \( x \in \widehat{CF} \) determines a spin-c structure on \( M \).

Problem

Show that if the differential from \( x \) to \( y \) is non-trivial, then \( x \) and \( y \) determine the same spin-c structure.

Given that, the chain complexes split as direct sums of subcomplexes corresponding to different spin-c structures.
The homologies $HF^+$, $HF^-$, $HF^\infty$ and $\widehat{HF}$ are independent of the choices made and are invariants of $(Y, s)$. Moreover, if $Y$ is a rational homology sphere, then $HF^\infty(Y, s) = \mathbb{Z}_2[U, U^{-1}]$. 

Maciej Borodzik (Institute of Mathematics, Polish Academy of Science)

Heegaard Floer homologies and rational cuspidal curves

Warsaw, February 2016
Independence

Theorem

The homologies $HF^+, HF^-, HF^\infty$ and $\hat{HF}$ are independent of the choices made and are invariants of $(Y, s)$. Moreover, if $Y$ is a rational homology sphere, then $HF^\infty(Y, s) = \mathbb{Z}_2[U, U^{-1}].$

The original result shows that a change of data induces an isomorphism of HF groups. Therefore the isomorphism class of groups is well defined.
Independence

Theorem

The homologies $HF^+, HF^-, HF^\infty$ and $\hat{HF}$ are independent of the choices made and are invariants of $(Y, s)$. Moreover, if $Y$ is a rational homology sphere, then $HF^\infty(Y, s) = \mathbb{Z}_2[U, U^{-1}]$.

- The original result shows that a change of data induces an isomorphism of HF groups. Therefore the isomorphism class of groups is well defined.
- It is a result of Juhász and Thurston that the homology groups are well-defined and not just their isomorphism classes.
Independence

Theorem

The homologies \( HF^+, HF^-, HF^\infty \) and \( \hat{HF} \) are independent of the choices made and are invariants of \((Y, s)\). Moreover, if \( Y \) is a rational homology sphere, then \( HF^\infty(Y, s) = \mathbb{Z}_2[U, U^{-1}] \).

- The original result shows that a change of data induces an isomorphism of HF groups. Therefore the isomorphism class of groups is well defined.
- It is a result of Juhász and Thurston that the homology groups are well-defined and not just their isomorphism classes.
- There is a subtle difference between having an isomorphism class of a group or a group.
Example. $L(p, q)$

A Heegaard diagram for $L(p, q)$ is as on the picture.
Example. $L(p, q)$

A Heegaard diagram for $L(p, q)$ is as on the picture.
Example. \( L(p, q) \)

A Heegaard diagram for \( L(p, q) \) is as on the picture.

The complex for \( L(3, 1) \) has three generators and no differentials. Each generator corresponds to another \( \text{spin-c} \) structure. We get that \( CF^{-}(L(3, 1), s) \cong \mathbb{Z}_2[U] \) for each \( s \).
Example. $L(p, q)$

A Heegaard diagram for $L(p, q)$ is as on the picture.

The complex for $L(3, 1)$ has three generators and no differentials. Each generator corresponds to another spin-c structure. We get that $CF^-(L(3, 1), s) \cong \mathbb{Z}_2[U]$ for each $s$. The same statement hold for every lens space.
Definition

A rational homology three–sphere is called an $L$–space if for every spin-c structure $s$ we have $\widehat{HF}(Y, s) = \mathbb{Z}_2$ and $HF^{-}(Y, s) = \mathbb{Z}_2[U]$ (these two are equivalent to each other and also equivalent to saying that $HF^{+} = \mathbb{Z}_2[U, U^{-1}]/\mathbb{Z}_2[U]$).
L–spaces

Definition

A rational homology three–sphere is called an \( L–space \) if for every spin-c structure \( s \) we have \( \hat{HF}(Y, s) = \mathbb{Z}_2 \) and \( HF^-(Y, s) = \mathbb{Z}_2[U] \) (these two are equivalent to each other and also equivalent to saying that \( HF^+ = \mathbb{Z}_2[U, U^{-1}]/\mathbb{Z}_2[U] \)).
Problem

Suppose \((Y_1, s_1)\) and \((Y_2, s_2)\) are two three–manifolds. Prove the following Künneth formula for \(CF^-\) and \(CF^\infty\):

\[
CF^-(Y_1 \# Y_2, s_1 \# s_2) \cong CF^-(Y_1, s_1) \otimes CF^-(Y_2, s_2).
\]
Adjunction inequality

Theorem (Ozsváth–Szabo 2003)

Suppose $Y$ has $b_1(Y) > 0$. Let $Z \subset Y$ be a closed oriented surface in $Y$. If $HF^+(Y, \xi) \neq 0$, then $|\langle c_1(\xi), Z \rangle| \leq 2g(Z) - 2$. 
Adjunction inequality

Theorem (Ozsváth–Szabo 2003)

Suppose $Y$ has $b_1(Y) > 0$. Let $Z \subset Y$ be a closed oriented surface in $Y$. If $HF^+(Y, s) \neq 0$, then $|\langle c_1(s), Z \rangle| \leq 2g(Z) - 2$.

This is the one of the two main technical tools in dealing with Heegaard Floer theory. This is also one of the sources of its power.
Theorem

Suppose $Y$ is a homology three–sphere and $K \subset Y$ is a knot. Then there exists an exact sequence

$$\ldots \rightarrow HF^+(Y) \rightarrow HF^+(Y_0) \rightarrow HF^+(Y_1) \rightarrow HF^+(Y) \rightarrow \ldots,$$

where $Y_1$ is the $+1$ surgery and $Y_0$ is the 0-surgery.

Idea of proof.

Construct a suitable triple Heegaard diagram and define various maps by counting holomorphic triangles.
Grading

Theorem (Ozsváth–Szabó 2003)
Theorem (Ozsváth–Szabó 2003)

There is an absolute $\mathbb{Q}$ grading of the homologies.
Theorem (Ozsváth–Szabó 2003)

- There is an absolute $\mathbb{Q}$ grading of the homologies.

- If $(W, t)$ is a smooth spin-c cobordism between $(Y_1, s_1)$ and $(Y_2, s_2)$, then there exists maps $F_{Wt}^\circ : HF^\circ(Y_1, s_1) \to HF^\circ(Y_2, s_2)$ with $\circ \in \{+, -, \infty\}$ making the obvious diagrams commute. The grading shift of $F$ is equal to 
  \[ \deg F_W := \frac{1}{4}(c_1(t)^2 - 2\chi(W) - 3\sigma(W)). \]
Grading

Theorem (Ozsváth–Szabó 2003)

• There is an absolute $\mathbb{Q}$ grading of the homologies.

• If $(W, t)$ is a smooth spin-c cobordism between $(Y_1, s_1)$ and $(Y_2, s_2)$, then there exists maps $F_{Wt}^\circ : HF^\circ (Y_1, s_1) \to HF^\circ (Y_2, s_2)$ with $\circ \in \{+, -, \infty\}$ making the obvious diagrams commute. The grading shift of $F$ is equal to

  $$\deg F_W := \frac{1}{4}(c_1(t)^2 - 2\chi(W) - 3\sigma(W)).$$

• If $W$ has negative definite intersection form, then $F_{WW}^{\infty}$ is an isomorphism.
$d$-invariants

Let $W$ be a spin-c cobordism between $Y_1$ and $Y_2$. We have maps
Let $W$ be a spin-c cobordism between $Y_1$ and $Y_2$. We have maps

\[ \longrightarrow HF^{-}(Y_1) \longrightarrow HF^{\infty}(Y_1) \longrightarrow HF^{+}(Y_1) \longrightarrow \]

\[ \downarrow F_{W}^{-} \quad \downarrow F_{W}^{\infty} \quad \downarrow F_{W}^{+} \]

\[ \longrightarrow HF^{-}(Y_2) \longrightarrow HF^{\infty}(Y_2) \longrightarrow HF^{+}(Y_2) \longrightarrow \]
Define $d(Y_1, s_1)$ and $d(Y_2, s_2)$ as the minimal grading of an element in $HF^+$ that is in the image of $HF^\infty$. 

Let $W$ be a spin-c cobordism between $Y_1$ and $Y_2$. We have maps

$$
\begin{align*}
\rightarrow & \quad HF^{-}(Y_1) \quad \rightarrow \quad HF^{\infty}(Y_1) \quad \rightarrow \quad HF^{+}(Y_1) \\
\downarrow & \quad F_{W}^{-} \quad \downarrow \quad F_{W}^{\infty} \quad \downarrow \quad F_{W}^{+} \\
\rightarrow & \quad HF^{-}(Y_2) \quad \rightarrow \quad HF^{\infty}(Y_2) \quad \rightarrow \quad HF^{+}(Y_2) 
\end{align*}
$$

- Define $d(Y_1, s_1)$ and $d(Y_2, s_2)$ as the minimal grading of an element in $HF^{+}$ that is in the image of $HF^{\infty}$.
- If $W$ is negative definite, then the red arrow is an isomorphism so we obtain the fundamental inequality between $d$-invariants:

$$
d(Y_1, s_1) \geq d(Y_2, s_2) + \deg F_{W}.
$$
Power of $d$-invariants.

The inequality for $d$-invariants is strong enough to:
Power of $d$-invariants.

The inequality for $d$-invariants is strong enough to:
- Reprove the Donaldson’s diagonalization theorem.
Power of $d$-invariants.

The inequality for $d$-invariants is strong enough to:

- Reprove the Donaldson’s diagonalization theorem.
- Reprove the Kronheimer–Mrowka result on the unknotting number of torus knots.
Power of $d$-invariants.

The inequality for $d$-invariants is strong enough to:

- Reprove the Donaldson’s diagonalization theorem.
- Reprove the Kronheimer–Mrowka result on the unknotting number of torus knots.
- many other things.
Corollary

If $(Y, s)$ bounds a rational homology ball $W$ (that is $H_k(W; \mathbb{Q}) = 0$ for $k \geq 1$) and the spin-c structure $s$ extends over $W$, then $d(Y, s) = 0$. 
Corollary

If \((Y, s)\) bounds a rational homology ball \(W\) (that is \(H_k(W; \mathbb{Q}) = 0\) for \(k \geq 1\)) and the spin-c structure \(s\) extends over \(W\), then \(d(Y, s) = 0\).

Remark

Being a rational homology ball is the same as being a \(\mathbb{Q}\)-acyclic surface. In particular, a complement of a rational cuspidal curve \(C\) in \(\mathbb{C}P^2\) is a rational homology ball.
Corollary

If \((Y, \mathfrak{s})\) bounds a rational homology ball \(W\) (that is \(H_k(W; \mathbb{Q}) = 0\) for \(k \geq 1\)) and the spin-c structure \(\mathfrak{s}\) extends over \(W\), then \(d(Y, \mathfrak{s}) = 0\).

Remark

Being a rational homology ball is the same as being a \(\mathbb{Q}\)-acyclic surface. In particular, a complement of a rational cuspidal curve \(C\) in \(\mathbb{C}P^2\) is a rational homology ball.

Question

How to calculate \(d\)-invariants?
A *doubly pointed* Heegaard diagram is \((\Sigma, \alpha, \beta, z, w)\) with \(z, w\) disjoint from \(\alpha\) and \(\beta\).
A doubly pointed Heegaard diagram is $(\Sigma, \alpha, \beta, z, w)$ with $z, w$ disjoint from $\alpha$ and $\beta$. Connect $z$ and $w$ by an arc in both handlebodies. We get a knot in $K \subset Y$. We say that the diagram represents the knot.

Problem

Show that for any null-homologous knot $K$ in $Y$ there exists a doubly pointed Heegaard diagram representing that knot.
A **doubly pointed** Heegaard diagram is \((\Sigma, \alpha, \beta, z, w)\) with \(z, w\) disjoint from \(\alpha\) and \(\beta\). Connect \(z\) and \(w\) by an arc in both handlebodies. We get a knot in \(K \subset Y\). We say that the diagram represents the knot.

**Problem**

*Show that for any null-homologous knot \(K\) in \(Y\) there exists a doubly pointed Heegaard diagram representing that knot.*

We think of a knot as a of a doubly pointed Heegaard diagram.
The second point $w$ diagram induces a (relative) filtration on $CF^-$. 
The second point $w$ diagram induces a (relative) filtration on $CF^-$. Write $A(x) - A(y) = n_w(\phi) - n_z(\phi)$. 
The second point $w$ diagram induces a (relative) filtration on $CF^-$. Write $A(x) - A(y) = n_w(\phi) - n_z(\phi)$.

**Lemma**

We have $\sum_{x \in \mathcal{T}_\alpha \cap \mathcal{T}_\beta} (-1)^{M(x)} q^{A(x)} = \Delta(q)$. 
There are several ways to define homologies.
Floer homologies

There are several ways to define homologies. Remember! Chain complexes are good, homologies are bad!
Floer homologies

There are several ways to define homologies. Remember! Chain complexes are good, homologies are bad!

- Take generators for $\widehat{CF}$ and count only disks that do not intersect $z$ and $w$. Get $\widehat{HFK}$. 
There are several ways to define homologies. Remember! Chain complexes are good, homologies are bad!

- Take generators for $\hat{CF}$ and count only disks that do not intersect $z$ and $w$. Get $\hat{HFK}$.
- Take generators for $CF^-$ and act as above. Get $HFK^-$.
There are several ways to define homologies. Remember! Chain complexes are good, homologies are bad!

- Take generators for $\widehat{CF}$ and count only disks that do not intersect $z$ and $w$. Get $\widehat{HFK}$.
- Take generators for $CF^{-}$ and act as above. Get $HFK^{-}$.
- Take generators for $CF^{-}$ and do not change anything in the definition of $\partial$. Get $HF^{-}$ of the underlying space.
There are several ways to define homologies. Remember! Chain complexes are good, homologies are bad!

- Take generators for $\widehat{CF}$ and count only disks that do not intersect $z$ and $w$. Get $\widehat{HFK}$.
- Take generators for $CF^-$ and act as above. Get $HFK^-$. 
- Take generators for $CF^-$ and do not change anything in the definition of $\partial$. Get $HF^-$ of the underlying space.
- Do the same with $\widehat{CF}$. 

Properties of $HFK$

- Detects the genus. That is, $g(K) = \max\{i : \hat{HFK}_\star(K, i) \neq 0\}$. 
Properties of $HKF$

- Detects the genus. That is, $g(K) = \max\{i: \hat{HFK}_*(K, i) \neq 0\}$.
- In particular, it detects the unknot. The proof is much easier than for Khovanov.
Properties of $HKF$

- Detects the genus. That is, $g(K) = \max\{i : \widehat{HFK}_*(K, i) \neq 0\}$.
- In particular, it detects the unknot. The proof is much easier than for Khovanov.
- Detects fibredness, a knot $K$ is fibred if and only if $\widehat{HFK}_*(K, g) = \mathbb{Z}$. 
Properties of $HFK$

- Detects the genus. That is, $g(K) = \max\{i: \hat{HFK}_*(K, i) \neq 0\}$.
- In particular, it detects the unknot. The proof is much easier than for Khovanov.
- Detects fibredness, a knot $K$ is fibred if and only if $\hat{HFK}_*(K, g) = \mathbb{Z}$.
- The $\tau$-invariant, $\tau(K) = -\max\{s: \exists x \in HFK_-(K, s): U^i x \neq 0\}$ is a concordance invariant, equal to $2g(K)$ for all positive knots, detecting the unknotting number of positive knots.
Let $K \subset S^3$ be a knot. Take ball $B^4$ and glue to it a two–handle along $K$ with framing $q$. We obtain a 4–manifold $N$ with boundary $S^3_q(K)$. The core of the handle and a Seifert surface for $K$ form a closed surface $F$ that generates $H_2(N; \mathbb{Z})$. 
Surgeries and spin-c structures

Let $K \subset S^3$ be a knot. Take ball $B^4$ and glue to it a two–handle along $K$ with framing $q$. We obtain a 4–manifold $N$ with boundary $S^3_q(K)$. The core of the handle and a Seifert surface for $K$ form a closed surface $F$ that generates $H_2(N; \mathbb{Z})$.

Theorem

For every $m \in [-q/2, q/2) \cap \mathbb{Z}$ there exists a unique spin-c structure $s_m$ on $Y$ that extends to a spin-c structure $t_m$ on $N$ characterized by the property that $\langle c_1(t_m), F \rangle + 2m = q$
Let $K \subset S^3$ be a knot. Take ball $B^4$ and glue to it a two–handle along $K$ with framing $q$. We obtain a 4–manifold $N$ with boundary $S^3_q(K)$. The core of the handle and a Seifert surface for $K$ form a closed surface $F$ that generates $H_2(N; \mathbb{Z})$.

**Theorem**

For every $m \in [−q/2, q/2) \cap \mathbb{Z}$ there exists a unique spin-c structure $s_m$ on $Y$ that extends to a spin-c structure $t_m$ on $N$ characterized by the property that $\langle c_1(t_m), F \rangle + 2m = q$.

The bottom line: think of spin-c structures as of integers in some interval!
A $CFK^\infty$ allows us to calculate the Heegaard Floer homologies of surgeries on knots.
A $\text{CFK}^\infty$ allows us to calculate the Heegaard Floer homologies of surgeries on knots. The formula is in general very complex and involves a mapping cone on many copies of subcomplexes $\text{CFK}^\infty(i > 0)$. 
A $\text{CFK}^\infty$ allows us to calculate the Heegaard Floer homologies of surgeries on knots. The formula is in general very complex and involves a mapping cone on many copies of subcomplexes $\text{CFK}^\infty(i > 0)$. If the surgery coefficient is large, by some clever application of the adjunction inequality we can show that the formula greatly simplifies.
Theorem

Suppose $K \subset S^3$ and $q > 2g(K)$. Let $Y = S^3_q(K)$. Then

$$CF^{-}(Y, s_m) \cong CFK^\infty(K)(i < 0, j < m) \text{ and }$$

$$CF^{+}(Y, s_m) \cong CFK^\infty/(i < 0, j < m).$$
Large surgeries

Theorem

Suppose $K \subset S^3$ and $q > 2g(K)$. Let $Y = S^3_q(K)$. Then

$CF^{-}(Y, \mathfrak{s}_m) \cong CFK_{\infty}(K)(i < 0, j < m)$ and

$CF^{+}(Y, \mathfrak{s}_m) \cong CFK_{\infty} / (i < 0, j < m)$. 
Theorem

Suppose $K \subset S^3$ and $q > 2g(K)$. Let $Y = S_q^3(K)$. Then

$CF^-(Y, s_m) \cong CFK^{\infty}(K)(i < 0, j < m)$ and

$CF^+(Y, s_m) \cong CFK^{\infty}/(i < 0, j < m)$. 

Large surgeries

Theorem

Suppose $K \subset S^3$ and $q > 2g(K)$. Let $Y = S^3_q(K)$. Then

$CF^-(Y, s_m) \cong CFK^\infty(K)(i < 0, j < m)$ and

$CF^+(Y, s_m) \cong CFK^\infty/(i < 0, j < m)$. 

Maciej Borodzik (Institute of Mathematics, Polish Academy of Science)

Heegaard Floer homologies and rational cusp

Warsaw, February 2016
Large surgeries

Theorem

Suppose $K \subset S^3$ and $q > 2g(K)$. Let $Y = S^3_q(K)$. Then

$CF^-(Y, s_m) \cong CFK^\infty(K)(i < 0, j < m)$ and

$CF^+(Y, s_m) \cong CFK^\infty/(i < 0, j < m)$. 

Maciej Borodzik (Institute of Mathematics, Polish Academy of Science)

Heegaard Floer homologies and rational cusp

Warsaw, February 2016 26 / 45
Theorem

Suppose $K \subset S^3$ and $q > 2g(K)$. Let $Y = S_q^3(K)$. Then

$CF^{-}(Y, s_m) \cong CFK^\infty(K)(i < 0, j < m)$ and
$CF^{+}(Y, s_m) \cong CFK^\infty/(i < 0, j < m)$.

The grading shift of this homomorphism is

$$\frac{(q-2m)^2-q}{4q}.$$
Large surgeries

Theorem

Suppose $K \subset S^3$ and $q > 2g(K)$. Let $Y = S_q^3(K)$. Then

$\text{CF}^-(Y, s_m) \cong \text{CFK}^\infty(K)(i < 0, j < m)$ and

$\text{CF}^+(Y, s_m) \cong \text{CFK}^\infty/(i < 0, j < m)$.

The grading shift of this homomorphism is

$$\frac{(q-2m)^2 - q}{4q}.$$

All needed data is derived from the $\text{CFK}^\infty$. 
L–space knots

Definition
A knot is called an $L$–space knot if there exists a positive surgery on $K$ which is an L–space.


Definition

A knot is called an \textit{L–space knot} if there exists a positive surgery on $K$ which is an L–space.

Theorem (Hedden 2006)

\textit{Algebraic knots are L–space knots.}
Definition
A knot is called an \textit{L–space knot} if there exists a positive surgery on $K$ which is an L–space.

Theorem (Hedden 2006)
\textit{Algebraic knots are L–space knots.}

Theorem (Krcatovich 2013)
\textit{An L–space knot is prime, in particular a connected sum of two algebraic knots is not an L–space knot.}
L–space knots

Definition
A knot is called an *L–space knot* if there exists a positive surgery on $K$ which is an L–space.

Theorem (Hedden 2006)

*Algebraic knots are L–space knots.*

Theorem (Krcatovich 2013)

*An L–space knot is prime, in particular a connected sum of two algebraic knots is not an L–space knot.*

L–space knots have the $\text{CFK}^\infty$ determined from the Alexander polynomial.
The staircase

\[ \Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1. \]
The staircase

\[ \Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^{9} + t^{7} - t^{5} + t^{4} - t + 1. \]
The staircase

\[ \Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1. \]

9 = g(T_{4,7})
The staircase

\[ \Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1. \]

9 = g(T_{4,7})

18 - 17 = 1
The staircase

\[ \Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1. \]
The staircase

\[ \Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1. \]
The staircase

$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1.$$
The staircase

\[ \Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1. \]

9 = \( g(T_{4,7}) \)

18 – 17 = 1

17 – 14 = 3

14 – 13 = 1

13 – 11 = 2

... and so on
\[ \Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^9 + t^7 - t^5 + t^4 - t + 1. \]
The staircase complex
The staircase complex

Place $\mathbb{Z}_2$ for each vertex.
The staircase complex

Place $\mathbb{Z}_2$ for each vertex.

Differential is given by lines as depicted.
The staircase complex

- Place $\mathbb{Z}_2$ for each vertex.
- Differential is given by lines as depicted.
- Type A vertices.
The staircase complex

- Place $\mathbb{Z}_2$ for each vertex.
- Differential is given by lines as depicted.
- Type A vertices.
- Type B vertices.
The staircase complex

- Place $\mathbb{Z}_2$ for each vertex.
- Differential is given by lines as depicted.
- Type A vertices.
- Type B vertices.
- Bifiltration is given by coordinates.
The staircase complex

- Place $\mathbb{Z}_2$ for each vertex.
- Differential is given by lines as depicted.
- Type A vertices.
- Type B vertices.
- Bifiltration is given by coordinates.
- Absolute grading of a type A vertex is 0, of type B is 1.
Tensoring

Tensor $\text{St}(K)$ by $\mathbb{Z}_2[U, U^{-1}]$. 
Tensoring

- Tensor $\text{St}(K)$ by $\mathbb{Z}_2[U, U^{-1}]$.
- $U$ changes the filtration level by $(-1, -1)$ and the absolute grading by $-2$. 
Tensoring

- Tensor $\text{St}(K)$ by $\mathbb{Z}_2[U, U^{-1}]$.
- $U$ changes the filtration level by $(-1, -1)$ and the absolute grading by $-2$.
Tensoring

- Tensor $\text{St}(K)$ by $\mathbb{Z}_2[U, U^{-1}]$.
- $U$ changes the filtration level by $(-1, -1)$ and the absolute grading by $-2$.
- The resulting complex is $\text{CFK}^\infty(K)$ if $K$ is an algebraic knot.
The function $J(m)$

$m \in \mathbb{Z}$. Here $m = 3$. 
The function $J(m)$

- $m \in \mathbb{Z}$. Here $m = 3$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient $C_+$. 

![Diagram of a complex network with arrows and nodes, indicating a subcomplex]
The function $J(m)$

- $m \in \mathbb{Z}$. Here $m = 3$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient $C_+$. 
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.
The function $J(m)$

- $m \in \mathbb{Z}$. Here $m = 3$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient $C_+$.
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.
The function $J(m)$

- $m \in \mathbb{Z}$. Here $m = 3$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient $C_+$.
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.
The function $J(m)$

- $m \in \mathbb{Z}$. Here $m = 3$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient $C_+$.
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.
The function $J(m)$

- $m \in \mathbb{Z}$. Here $m = 3$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient $C_+$. Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.
The function $J(m)$

- $m \in \mathbb{Z}$. Here $m = 3$.
- The subcomplex $C(i < 0, j < m)$. Look at the quotient $C_+$. 
- Define $J(m)$ as the minimal absolute grading of an element non-trivial in homology of the quotient.
- We will show yet another description of $J$. 

Maciej Borodzik (Institute of Mathematics, Polish Academy of Science) 
Heegaard Floer homologies and rational cuspidal curves 
Warsaw, February 2016 31 / 45
$\text{CFK}^\infty$ for $T(2, 3) \# T(2, 3)$
$CFK^\infty$ for $T(2, 3) \# T(2, 3)$

The whole picture must be tensored by $\mathbb{Z}_2[U, U^{-1}]$. 

---

Maciej Borodzik (Institute of Mathematics, Polish Academy of Science)

Heegaard Floer homologies and rational cusps

Warsaw, February 2016 32 / 45
The whole picture must be tensored by $\mathbb{Z}_2[U, U^{-1}]$.

We have a staircase plus an acyclic complex.
The whole picture must be tensored by $\mathbb{Z}_2[U, U^{-1}]$.

We have a staircase plus an acyclic complex.

This is not always true, for example for $T(4, 5) \# T(4, 5)$. 
$CFK^\infty$ for $-T(3, 4)$
The situation is completely different than for positive $T(3, 4)$. 

$CFK^\infty$ for $-T(3, 4)$
The situation is completely different than for positive $T(3, 4)$.

A generator of homology of the complex is a sum of filtered elements.
$\text{CFK}^\infty$ for $-T(3,4)$

- The situation is completely different than for positive $T(3,4)$.
- A generator of homology of the complex is a sum of filtered elements.
- It is not a filtered element, that is an element at bifiltration element $(x, y)$ that is non-zero in the quotient by
  \[
  \text{CFK}^\infty(i \leq x - 1, j \leq y) + \text{CFK}^\infty(i \leq x, j \leq y - 1).
  \]
Semigroups of singular points

The semigroup of a singular point on a plane curve is the set of possible local intersection with the curve.

Problem

Show that for a singularity $x^p - y^q = 0$ with $p, q$ coprime, the semigroup is generated by $p$ and $q$. 
Semigroups of singular points

The semigroup of a singular point on a plane curve is the set of possible local intersection with the curve.

Problem

Show that for a singularity \( x^p - y^q = 0 \) with \( p, q \) coprime, the semigroup is generated by \( p \) and \( q \).
Semigroups of singular points

The semigroup of a singular point on a plane curve is the set of possible local intersection with the curve.

**Problem**

*Show that for a singularity $x^p - y^q = 0$ with $p, q$ coprime, the semigroup is generated by $p$ and $q$.*

- If $p = 4$, $q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \ldots)$.
Semigroups of singular points

The semigroup of a singular point on a plane curve is the set of possible local intersection with the curve.

Problem

Show that for a singularity $x^p - y^q = 0$ with $p, q$ coprime, the semigroup is generated by $p$ and $q$.

- If $p = 4, q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \ldots)$.
- The gap sequence is $G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$. 
Semigroups of singular points

The semigroup of a singular point on a plane curve is the set of possible local intersection with the curve.

Problem

Show that for a singularity $x^p - y^q = 0$ with $p, q$ coprime, the semigroup is generated by $p$ and $q$.

- If $p = 4$, $q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \ldots)$.
- The gap sequence is $G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$.
- We have $\# G_{4,7} = \mu/2$ and $\max\{x \in G_{4,7}\} = 17 = \mu - 1$. This is a special property of semigroups of singular points!
Semigroups of singular points

The semigroup of a singular point on a plane curve is the set of possible local intersection with the curve.

Problem

Show that for a singularity $x^p - y^q = 0$ with $p, q$ coprime, the semigroup is generated by $p$ and $q$.

- If $p = 4, q = 7$, the semigroup is $S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \ldots)$.
- The gap sequence is $G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\}$.
- We have $\# G_{4,7} = \mu/2$ and $\max\{x \in G_{4,7}\} = 17 = \mu - 1$. 
Semigroups of singular points

The semigroup of a singular point on a plane curve is the set of possible local intersection with the curve.

Problem

Show that for a singularity \( x^p - y^q = 0 \) with \( p, q \) coprime, the semigroup is generated by \( p \) and \( q \).

- If \( p = 4 \), \( q = 7 \), the semigroup is \( S_{4,7} := (0, 4, 7, 8, 11, 12, 14, 15, 16, 18, 19, 20, 21, \ldots) \).
- The gap sequence is \( G_{4,7} = \{1, 2, 3, 5, 6, 9, 10, 13, 17\} \).
- We have \( \#G_{4,7} = \mu/2 \) and \( \max\{x \in G_{4,7}\} = 17 = \mu - 1 \).
The Alexander polynomial

For a semigroup $S$ with a gap sequence $G$ we define
For a semigroup $S$ with a gap sequence $G$ we define

$$\Delta_S(t) = 1 + (t - 1) \sum_{j \in G} t^j.$$
The Alexander polynomial

For a semigroup $S$ with a gap sequence $G$ we define

$$\Delta_S(t) = 1 + (t - 1) \sum_{j \in G} t^j.$$

For the semigroup $S_{4,7}$, the gap sequence is $\{1, 2, 3, 5, 6, 9, 10, 13, 17\}$, so we have
The Alexander polynomial

For a semigroup $S$ with a gap sequence $G$ we define

$$\Delta_S(t) = 1 + (t - 1) \sum_{j \in G} t^j.$$ 

For the semigroup $S_{4,7}$, the gap sequence is $\{1, 2, 3, 5, 6, 9, 10, 13, 17\}$, so we have

$$\Delta_{4,7}(t) = 1 + (t - 1) \left( t + t^2 + t^3 + t^5 + t^6 + t^9 + t^{10} + t^{13} + t^{17} \right)$$
The Alexander polynomial

For a semigroup $S$ with a gap sequence $G$ we define

$$\Delta_S(t) = 1 + (t - 1) \sum_{j \in G} t^j.$$  

For the semigroup $S_{4,7}$, the gap sequence is \{1, 2, 3, 5, 6, 9, 10, 13, 17\}, so we have

$$\Delta_{4,7}(t) = 1 + (t - 1) \left(t + t^2 + t^3 + t^5 + t^6 + t^9 + t^{10} + t^{13} + t^{17}\right)$$

or:

$$\Delta_{4,7} = 1 - t + t^4 - t^5 + t^7 - t^9 + t^{11} - t^{13} + t^{14} - t^{17} + t^{18}.$$
The Alexander polynomial

For a semigroup $S$ with a gap sequence $G$ we define

$$\Delta_S(t) = 1 + (t - 1) \sum_{j \in G} t^j.$$ 

For the semigroup $S_{4,7}$, the gap sequence is \{1, 2, 3, 5, 6, 9, 10, 13, 17\}, so we have

$$\Delta_{4,7}(t) = 1 + (t - 1) \left( t + t^2 + t^3 + t^5 + t^6 + t^9 + t^{10} + t^{13} + t^{17} \right)$$

or:

$$\Delta_{4,7} = 1 - t + t^4 - t^5 + t^7 - t^9 + t^{11} - t^{13} + t^{14} - t^{17} + t^{18}.$$ 

This is the Alexander polynomial of the knot of the singularity.
The gap function

Definition

The gap function is defined as

\[ l(m) := \# \{ x \in \mathbb{Z}, \ x \geq m, \ x \notin S \} \]
The gap function

Definition
The gap function is defined as

\[ l(m) := \# \{ x \in \mathbb{Z}, \ x \geq m, \ x \notin S \}. \]

We have

\[ l_{4,7}(5) = \# \{ 5, 6, 9, 10, 13, 17 \} = 6. \]
The gap function

Definition

The gap function is defined as

\[ I(m) := \# \{ x \in \mathbb{Z}, \ x \geq m, \ x \not\in S \}. \]

We have

\[ l_{4,7}(5) = \# \{ 5, 6, 9, 10, 13, 17 \} = 6. \]

Always \( l(0) = \mu/2, \ l(x) = 0 \) for \( x \geq \mu \) and \( l(-n) = n + \mu/2 \) for \( n > 0 \).
The gap function

Definition

The gap function is defined as

\[ I(m) := \#\{x \in \mathbb{Z}, \ x \geq m, \ x \notin S\} \].

We have

\[ I_{4,7}(5) = \#\{5, 6, 9, 10, 13, 17\} = 6. \]

Always \( I(0) = \mu/2 \), \( I(x) = 0 \) for \( x \geq \mu \) and \( I(-n) = n + \mu/2 \) for \( n > 0 \).

Theorem

For an algebraic knot \( J(m) = -2I(m + g) \), where \( g = \mu/2 \) is the genus.
A connected sum of algebraic knots is not an L–space knot. But some part of the theory survives.
Gap function for connected sums

A connected sum of algebraic knots is not an L–space knot. But some part of the theory survives.

Definition

For two functions $l_1, l_2 : \mathbb{Z} \to \mathbb{Z}$ bounded from below define their \textit{infimal convolution} by $l_1 \diamond l_2(k) = \min_{n \in \mathbb{Z}} l_1(n) + l_2(k - n)$. 
Gap function for connected sums

A connected sum of algebraic knots is not an L–space knot. But some part of the theory survives.

**Definition**

For two functions $I_1, I_2 : \mathbb{Z} \to \mathbb{Z}$ bounded from below define their infimal convolution by $I_1 \diamond I_2(k) = \min_{n \in \mathbb{Z}} I_1(n) + I_2(k - n)$.

**Theorem**

Let $K = K_1 \# \ldots \# K_n$ be a connected sum of algebraic knots. Gap functions are $I_1, \ldots, I_n$. Set $I = I_1 \diamond \ldots \diamond I_n$. Then $J(m) = -2I(m + g)$, where $J$ is the minimal grading . . .
Proposition

Let $K$ be a connected sum of algebraic knots. Then

\[ d(S^3_q(K), s_m) = \frac{(q - 2m)^2 - q}{4q} - 2l(m + g). \]
**Proposition**

Let $K$ be a connected sum of algebraic knots. Then

\[ d(S^3_q(K), s_m) = \frac{(q - 2m)^2 - q}{4q} + J(m). \]
Boundary of a rational cuspidal curve

$C$ is a rational cuspidal curve with singularities $z_1, \ldots, z_n$. 
Boundary of a rational cuspidal curve

$C$ is a rational cuspidal curve with singularities $z_1, \ldots, z_n$. $d = \deg C$. 
C is a rational cuspidal curve with singularities $z_1, \ldots, z_n$. $d = \deg C$. $K_1, \ldots K_n$ are links of singularities.
C is a rational cuspidal curve with singularities $z_1, \ldots, z_n$. $d = \deg C$. $K_1, \ldots K_n$ are *links of singularities*. Define $K = K_1 \# \ldots \# K_n$. 
C is a rational cuspidal curve with singularities $z_1, \ldots, z_n$. $d = \deg C$. $K_1, \ldots K_n$ are links of singularities. Define $K = K_1 \# \ldots \# K_n$.

**Proposition**

Let $N$ be the tubular neighborhood of $C$ and let $Y = \partial N$. Then $Y$ is a $d^2$ surgery on $K$. 
Boundary of a rational cuspidal curve

$C$ is a rational cuspidal curve with singularities $z_1, \ldots, z_n$. $d = \deg C$. $K_1, \ldots K_n$ are links of singularities. Define $K = K_1 \# \ldots \# K_n$.

**Proposition**

Let $N$ be the tubular neighborhood of $C$ and let $Y = \partial N$. Then $Y$ is a $d^2$ surgery on $K$.

Then $d(Y, s) = 0$ for every spin-c structure on $Y$ that extends over $\mathbb{C}P^2 \setminus N$. 
C is a rational cuspidal curve with singularities $z_1, \ldots, z_n$. $d = \deg C$. $K_1, \ldots K_n$ are links of singularities. Define $K = K_1 \# \ldots \# K_n$.

**Proposition**

Let $N$ be the tubular neighborhood of $C$ and let $Y = \partial N$. Then $Y$ is a $d^2$ surgery on $K$.

Then $d(Y, s) = 0$ for every spin-c structure on $Y$ that extends over $\mathbb{C}P^2 \setminus N$.

**Proposition**

The spin-c structure $s_m$ extends over $\mathbb{C}P^2 \setminus N$ if $m = jd$ for $j \in \mathbb{Z}$ if $d$ is odd and $m = (j + \frac{1}{2})d$ if $d$ is even.
The FLMN conjecture

Combining results we obtain the following result.

**Theorem (—, Livingston, 2013)**

For \( j = 0, \ldots, d - 3 \) we have

\[
I(jd + 1) = \frac{(d - j - 1)(d - j - 2)}{2}.
\]
The FLMN conjecture

Combining results we obtain the following result.

**Theorem (—, Livingston, 2013)**

For \( j = 0, \ldots, d - 3 \) we have

\[
I(jd + 1) = \frac{(d - j - 1)(d - j - 2)}{2}.
\]

For \( n = 1 \) and \( n = 2 \) this is equivalent to the original FLMN conjecture (for \( n = 2 \) the translation is non-trivial and done by Bodnár–Némethi and Nayar–Pilat).
The FLMN conjecture

Combining results we obtain the following result.

**Theorem (—, Livingston, 2013)**

For \( j = 0, \ldots, d - 3 \) we have

\[
I(jd + 1) = \frac{(d - j - 1)(d - j - 2)}{2}.
\]

For \( n = 1 \) and \( n = 2 \) this is equivalent to the original FLMN conjecture (for \( n = 2 \) the translation is non-trivial and done by Bodnár–Némethi and Nayar–Pilat). For \( n \geq 3 \) the original conjecture is false, but the above result is a natural plan B.
Theorem (–, Hedden, Livingston and Bodnár, Celoria, Golla, 2014)

A set of inequalities of the semigroup function for the genus $g$ curve with cuspidal singularities.
Generalization

Theorem (–, Hedden, Livingston and Bodnár, Celoria, Golla, 2014)

A set of inequalities of the semigroup function for the genus $g$ curve with cuspidal singularities. They are of form

$$0 \leq l(jd + 1) - \frac{(d-j-1)(d-j-2)}{2} \leq g.$$
Theorem (–, Hedden, Livingston and Bodnár, Celoria, Golla, 2014)

A set of inequalities of the semigroup function for the genus g curve with cuspidal singularities. They are of form

\[ 0 \leq I(jd + 1) - \frac{(d-j-1)(d-j-2)}{2} \leq g. \]

Theorem (–, Moe, 2014)

Generalization for rational cuspidal curves in Hirzebruch surfaces. Only one side of inequality is obtained.
Generalization

Theorem (–, Hedden, Livingston and Bodnár, Celoria, Golla, 2014)

A set of inequalities of the semigroup function for the genus $g$ curve with cuspidal singularities. They are of form
\[ 0 \leq l(jd + 1) - \frac{(d-j-1)(d-j-2)}{2} \leq g. \]

Theorem (–, Moe, 2014)

Generalization for rational cuspidal curves in Hirzebruch surfaces. Only one side of inequality is obtained.

Theorem (–, 2015)

Generalization for rcc in surfaces with $p_g = 0$. The condition implies that the complement of a rcc is a negative definite manifold.
If time permits

Theorem (FLMN)

Suppose that $C$ is a curve in $\mathbb{CP}^2$ of degree $d$. Let $z \in C$ be a singular point and $S$ its semigroup. Then for $j = 1, \ldots, d - 1$ we have

$$\# S \cap [0, jd + 1) \geq \frac{1}{2} (j + 1)(j + 2).$$
If time permits

**Theorem (FLMN)**

Suppose that $C$ is a curve in $\mathbb{C}P^2$ of degree $d$. Let $z \in C$ be a singular point and $S$ its semigroup. Then for $j = 1, \ldots, d - 1$ we have

$$\# S \cap [0, jd + 1) \geq \frac{1}{2} (j + 1)(j + 2).$$

This is one part of the FLMN conjecture.
Theorem (FLMN)

Suppose that $C$ is a curve in $\mathbb{C}P^2$ of degree $d$. Let $z \in C$ be a singular point and $S$ its semigroup. Then for $j = 1, \ldots, d - 1$ we have

$$\# S \cap [0, jd + 1) \geq \frac{1}{2} (j + 1)(j + 2).$$

- This is one part of the FLMN conjecture.
- The right hand side is the dimension of space of polynomials of degree $j$, $H^0(\mathbb{C}P^2, \mathcal{O}(jH))$. 
If time permits

Theorem (FLMN)

Suppose that $C$ is a curve in $\mathbb{C}P^2$ of degree $d$. Let $z \in C$ be a singular point and $S$ its semigroup. Then for $j = 1, \ldots, d - 1$ we have

$$\# S \cap [0, jd + 1) \geq \frac{1}{2} (j + 1)(j + 2).$$

- This is one part of the FLMN conjecture.
- The right hand side is the dimension of space of polynomials of degree $j$, $H^0(\mathbb{C}P^2, \mathcal{O}(jH))$.
- The quantity $\# S \cap [0, k)$ is the number of conditions of a curve $D$ to intersect $C$ at $z$ with multiplicity at least $k$. 
Theorem (FLMN)

Suppose that $C$ is a curve in $\mathbb{C}P^2$ of degree $d$. Let $z \in C$ be a singular point and $S$ its semigroup. Then for $j = 1, \ldots, d - 1$ we have

$$\# S \cap [0, jd + 1) \geq \frac{1}{2} (j + 1)(j + 2).$$

- This is one part of the FLMN conjecture.
- The right hand side is the dimension of space of polynomials of degree $j$, $H^0(\mathbb{C}P^2, \mathcal{O}(jH))$.
- The quantity $\# S \cap [0, k)$ is the number of conditions of a curve $D$ to intersect $C$ at $z$ with multiplicity at least $k$.
- If the inequality is violated, then there exists a curve $D$ of degree $j$ intersecting $C$ with multiplicity $jd + 1$ or higher. Contradiction.
The result of MB and Moe and then of MB show that Heegaard Floer theory gives the same set of inequalities than Bézout (or Riemann–Roch).
The result of MB and Moe and then of MB show that Heegaard Floer theory gives the same set of inequalities than Bézout (or Riemann–Roch).

Problem (You’re encouraged to work at it)

Prove the FLMN inequalities using the line of FLMN for almost complex manifolds replacing $H^0(\mathbb{C}P^2, \mathcal{O}(jH))$ by some moduli space of $J$-holomorphic curves.
The result of MB and Moe and then of MB show that Heegaard Floer theory gives the same set of inequalities than Bézout (or Riemann–Roch).

**Problem (You’re encouraged to work at it)**

*Prove the FLMN inequalities using the line of FLMN for almost complex manifolds replacing $H^0(\mathbb{C}P^2, \mathcal{O}(jH))$ by some moduli space of $J$-holomorphic curves. Explain the similarity between the two approaches as a variant of GW–SW correspondence.*
Work in progress

Joint project with Hom and Schinzel.
Work in progress

Joint project with Hom and Schinzel.
Use Involutive Floer homology for finer obstruction.
Work in progress

Joint project with Hom and Schinzel.
Use Involutive Floer homology for finer obstruction.

Definition
For an algebraic knot $K$ with a staircase, the *stretch* is the length of the middle step of a staircase. A knot is called *even* or *odd* if the staircase has an even or odd number of steps.
Joint project with Hom and Schinzel. Use Involutive Floer homology for finer obstruction.

Definition
For an algebraic knot $K$ with a staircase, the *stretch* is the length of the middle step of a staircase. A knot is called *even* or *odd* if the staircase has an even or odd number of steps.
Joint project with Hom and Schinzel.

Use Involutive Floer homology for finer obstruction.

**Definition**

For an algebraic knot $K$ with a staircase, the *stretch* is the length of the middle step of a staircase. A knot is called *even* or *odd* if the staircase has an even or odd number of steps.
Joint project with Hom and Schinzel. Use Involutive Floer homology for finer obstruction.

**Definition**

For an algebraic knot $K$ with a staircase, the *stretch* is the length of the middle step of a staircase. A knot is called *even* or *odd* if the staircase has an even or odd number of steps.

The stretch of the staircase for $T(4, 5)$ is 2. This knot is odd.
Theorem (Hom, Schinzel, –)

Let $p, q$ be coprime. Write the continuous fraction expansion $q/p = [a_0; a_1; \ldots; a_k]$. Then the stretch of $T(p, q)$ is equal to $\left\lfloor \frac{a_k - 1}{2} \right\rfloor + 1$. 

Maciej Borodzik (Institute of Mathematics, Polish Academy of Science)
Heegaard Floer homologies and rational cusp
Warsaw, February 2016 45 / 45
Theorem (Hom, Schinzel, −)

Let \( p, q \) be coprime. Write the continuous fraction expansion \( \frac{q}{p} = [a_0; a_1; \ldots; a_k] \). Then the stretch of \( T(p, q) \) is equal to \( \left\lfloor \frac{a_k - 1}{2} \right\rfloor + 1 \).

Theorem (Hom, Schinzel, −)

Let \( C \) be a rational cuspidal curve with knots \( K_1, \ldots, K_n \). Suppose \( K_1 \) is odd. Then the stretch of \( K_1 \) is less or equal than \( g(K_2) + \ldots + g(K_n) \).
Bound from the IH

Theorem (Hom, Schinzel, –)
Let $p, q$ be coprime. Write the continuous fraction expansion $q/p = [a_0; a_1; \ldots; a_k]$. Then the stretch of $T(p, q)$ is equal to $[\frac{a_k-1}{2}] + 1$.

We have only this result for curves of odd degree.

Theorem (Hom, Schinzel, –)
Let $C$ be a rational cuspidal curve with knots $K_1, \ldots, K_n$. Suppose $K_1$ is odd. Then the stretch of $K_1$ is less or equal than $g(K_2) + \ldots + g(K_n)$.

Remark
This obstructs some cases with one ‘big’ singularity and some small.