Link concordance and immersed Morse theory

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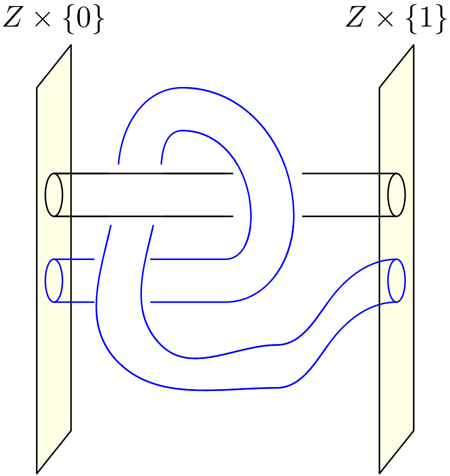
- B.–Powell–Teichner, full proof of codimension ≥ 2 case.

Link concordance via Morse theory



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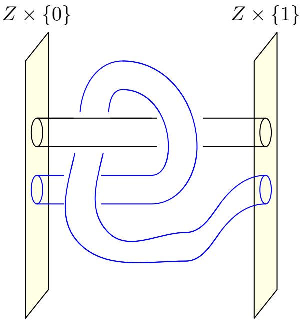

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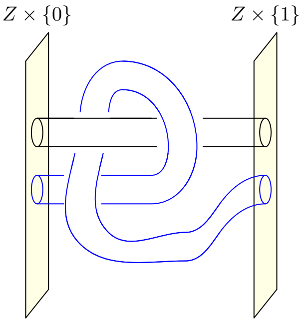

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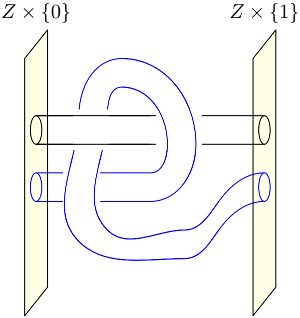

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- but the function on $\Omega = Z \times [0,1]$ can have crits.

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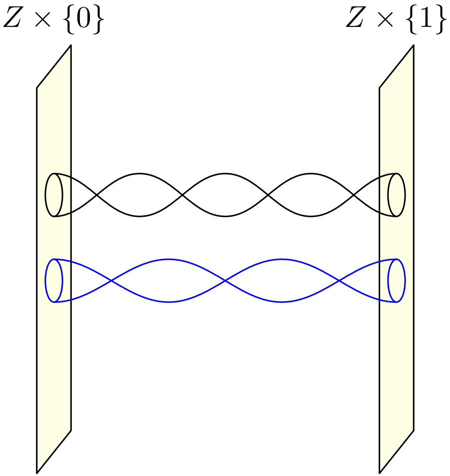

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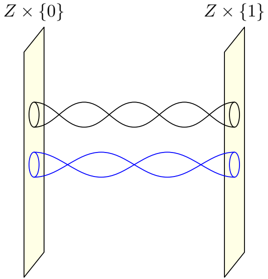

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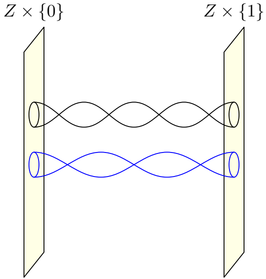

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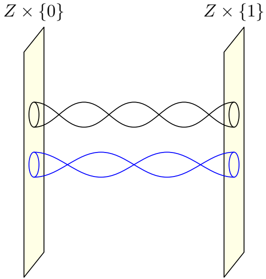

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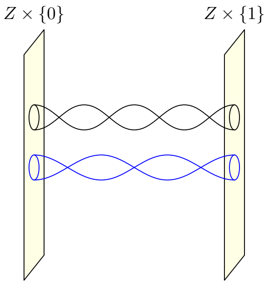

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- self-intersections of N are allowed
- can we trade crits for self-intersections?

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- This is on N, how about the ambient manifold Ω?

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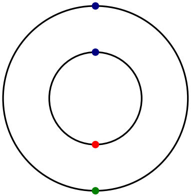
We need to assume that F_τ does not create new crits.

Path lifting theorem. Obstructions



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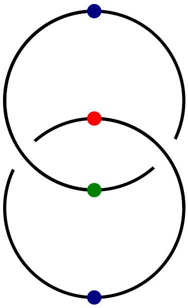

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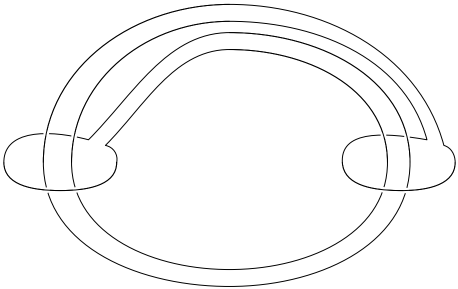
- Lifting deaths is obstructed in codim 2.

Lifting deaths. Stevedore knot



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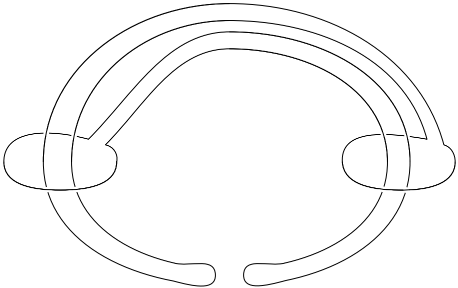

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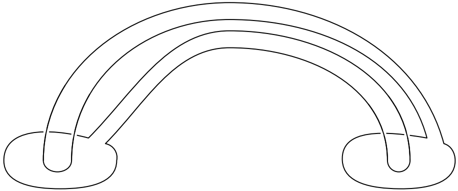

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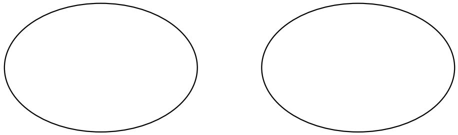

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- Need to introduce more complex vector fields, based on Sharpe (1988) and elaborated in B.,Powell 2015

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x_i coordinates on $\Omega[d]$

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$y_{i1}=\dots=y_{ik}=0$ defines i-th sheet

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Quadratic terms on $y_{11}, y_{21}, \dots, y_{d1}$

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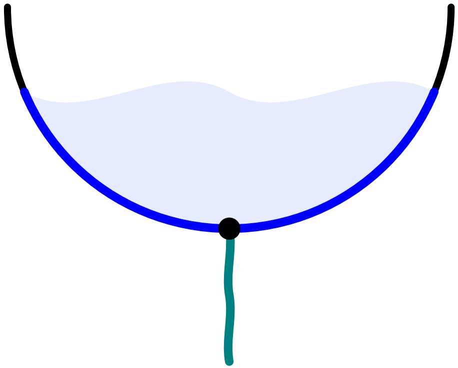

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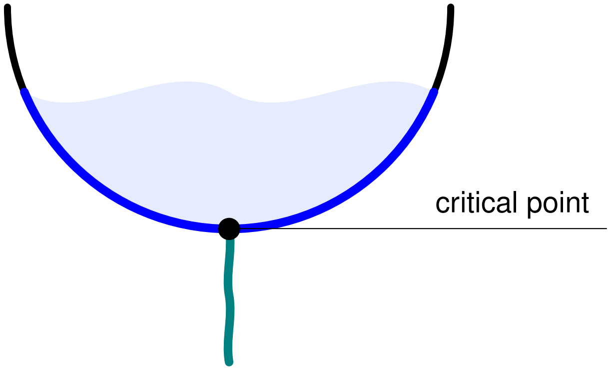

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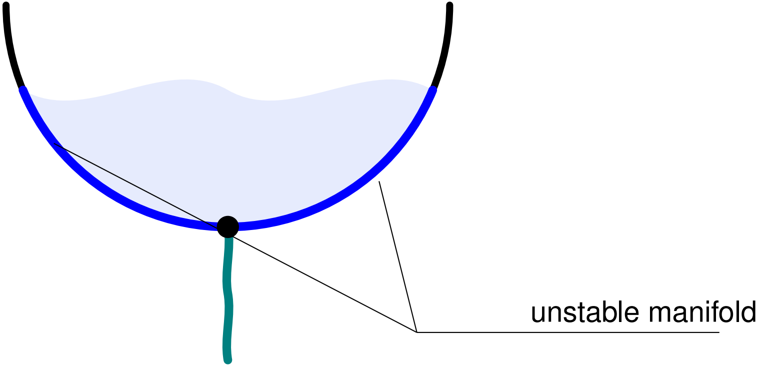

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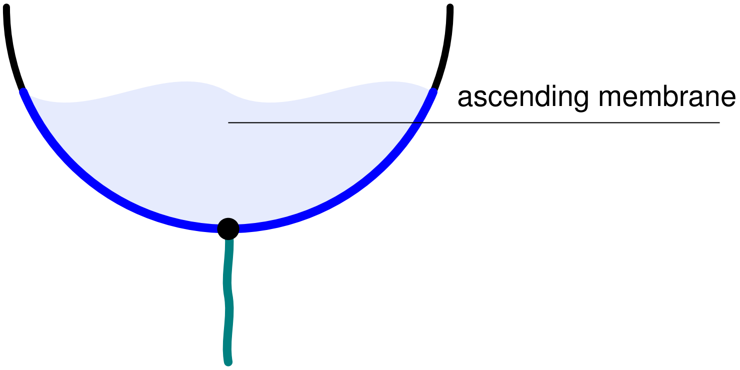

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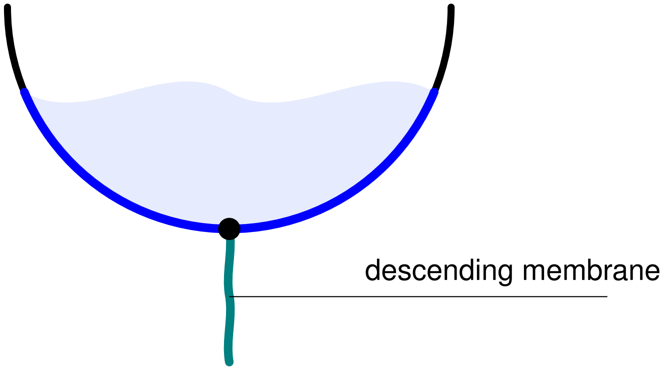

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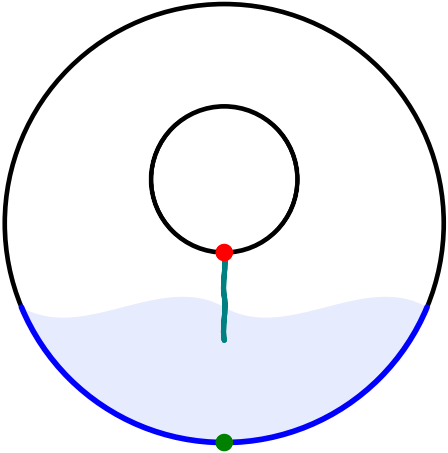
- There can be trajectories on zero-th stratum preventing cancelation.

Example. Rearrangment obstruction.



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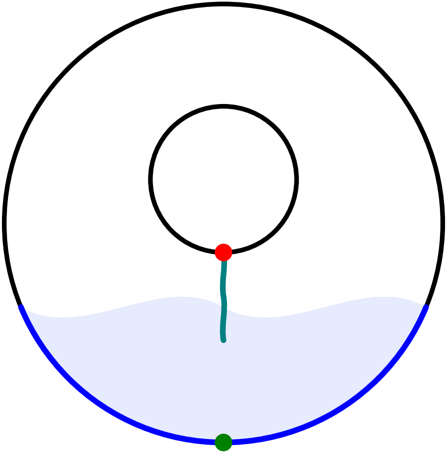
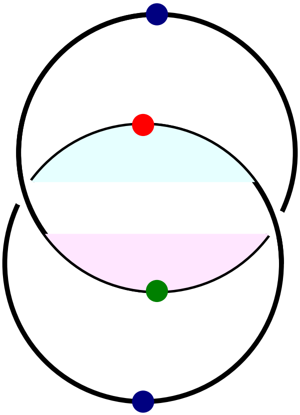


Intersection of membranes prevents rearrangement

Example. Rearrangment obstruction. 2



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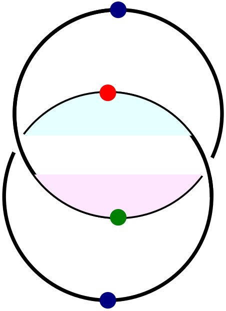

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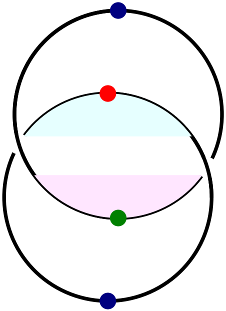

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Remark. The linking number can be defined using intersections of membranes, B. – Powell, work in progress.

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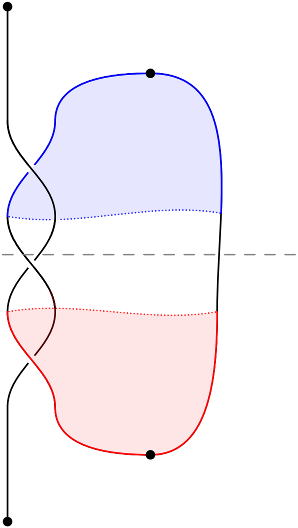

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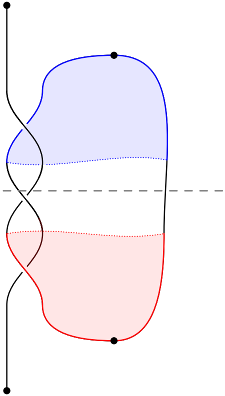
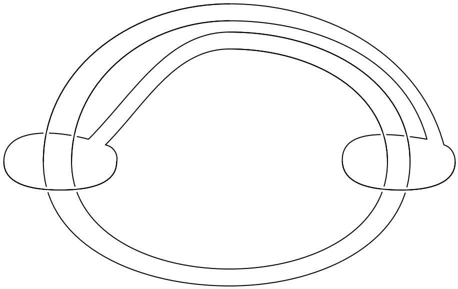


It is impossible to cancel the minimum and the maximum, because the interiors of membranes intersect.

Stevedore knot. Membranes



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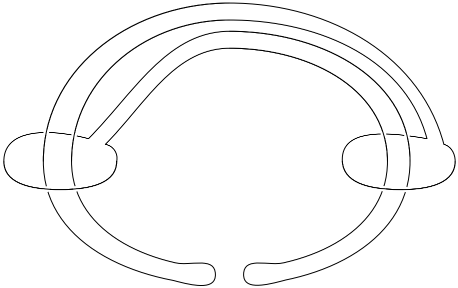

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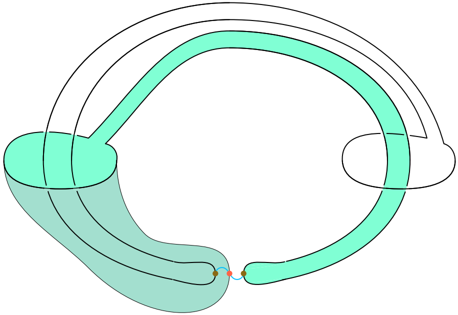

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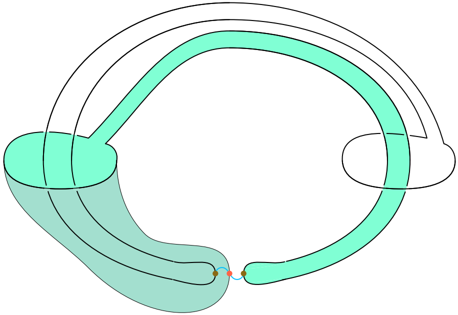

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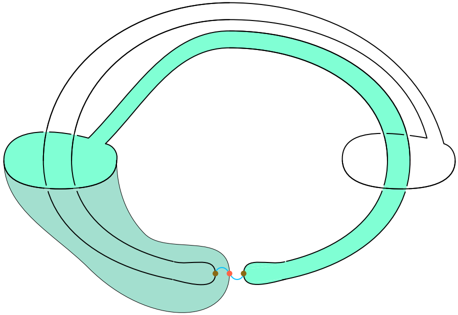

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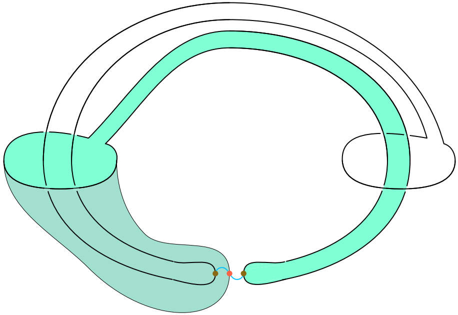

- The membranes intersect, we cannot cancel the pairs of critical points.
- It is good, because otherwise the knot would be trivial.

Finger move

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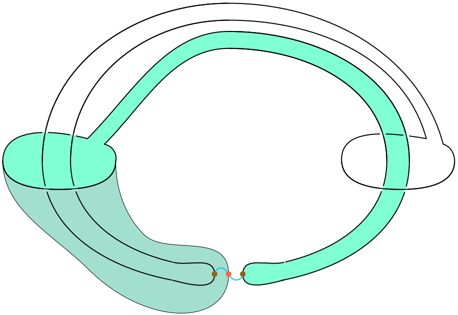
- In codimension 2, there might be finitely many intersections, finitely many trajectories.
- Finger move trades these trajectories for intersection points.
- The price we pay is that we have to leave the category of embeddings, and work with immersed Morse theory.

Example



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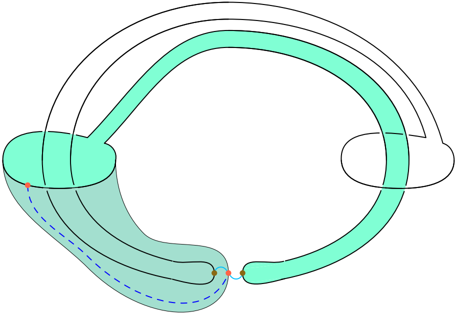

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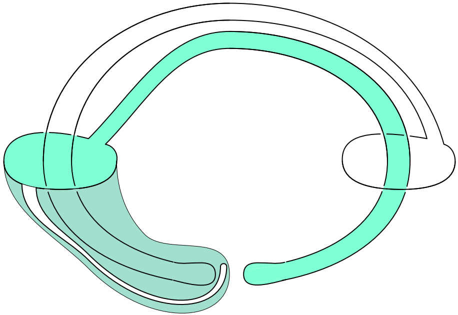

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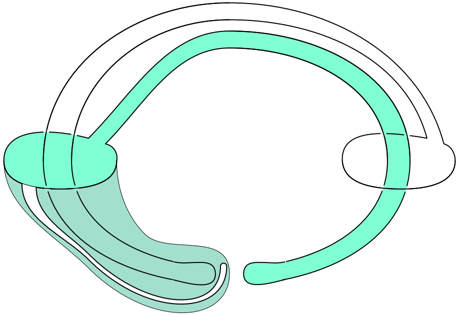

Moving the part of the knot along the guiding curve introduces double points, but cancels one trajectory.

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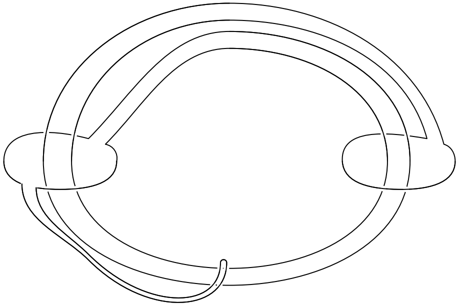

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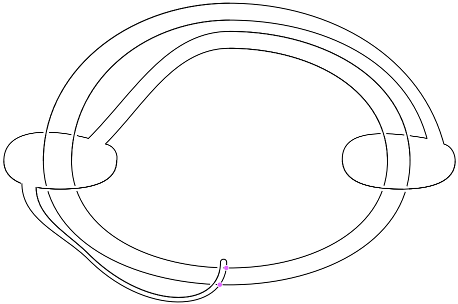

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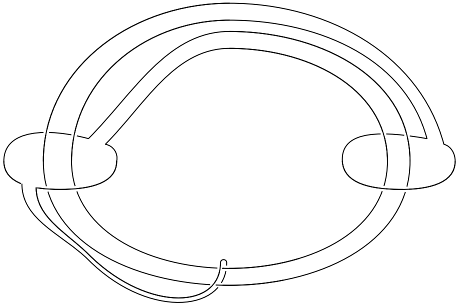
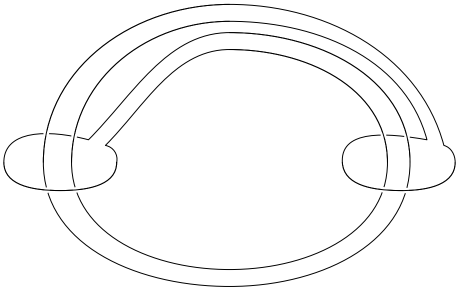


Lifting deaths. Stevedore knot. Finger move



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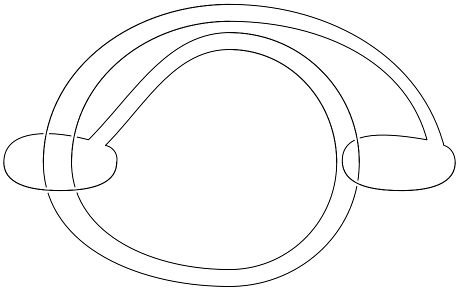

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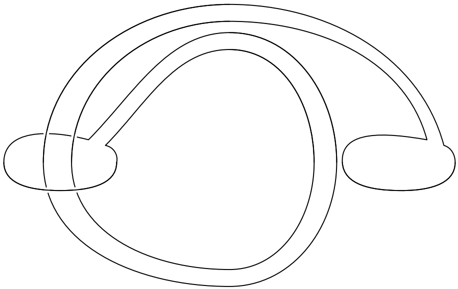

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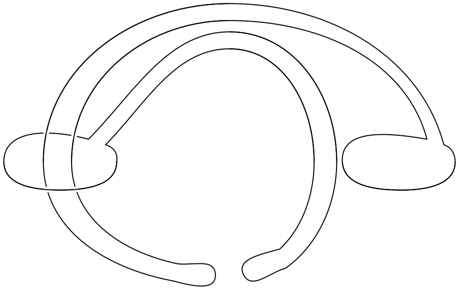

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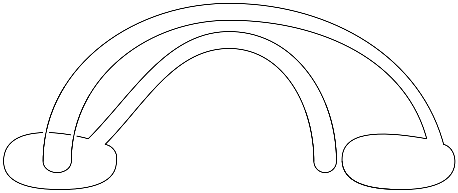

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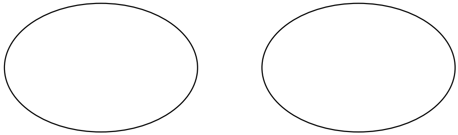

Glance into details

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- Special care: do not create new trajectories.

Path lifting theorem

Theorem (B. – Powell – Teichner, 2025)

Suppose $M = G_0(N)$ is an immersed manifold in Ω and $F\colon (\Omega, M)$ is an immersed Morse function, $\operatorname{codim} M > 2$. Suppose $f_\tau\colon N \to \mathbb{R}$ is a path of functions having births, deaths, rearrangments, but no inverse rearrangments, and $f_0 = F \circ G_0$. Then, there exists a path of functions F_τ, and a regular homotopy G_τ, such that $F_\tau \circ G_\tau = f_\tau$, and F_τ has no births or deaths on the 0-th stratum.

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This is the key tool in the proof of Link concordance implies link homotopy.

Key tool in the proof: uniqueness

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In general: no. Cerf's uniqueness has to be applied.

Cerf’s uniqueness vs. Milnor uniqueness

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Cerf's uniqueness vs. Milnor uniqueness

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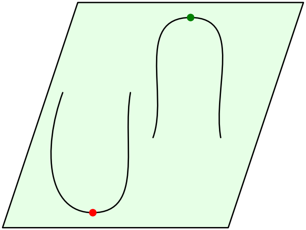
- To lift a path, we extend a gradient-like vector field on N to a grim vector field on Ω.
- The lifting has to be performed near images of stable/unstable manifolds.

Lifting rearrangements. Easy case



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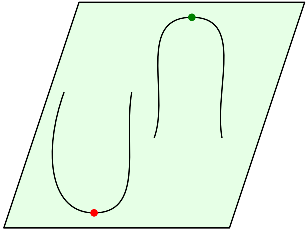

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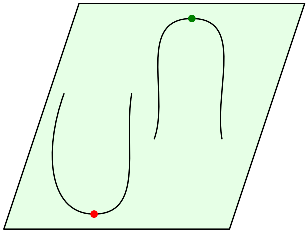
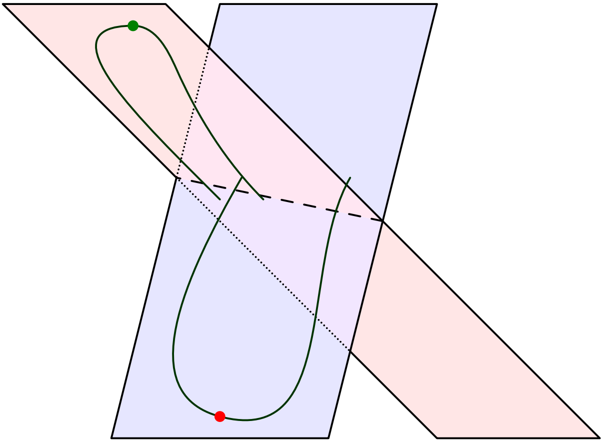


In the embedded case, stable and unstable manifolds are mapped $1-1$ on their image. We can extend the vector field and lift the path.

Lifting rearrangements. Hard case



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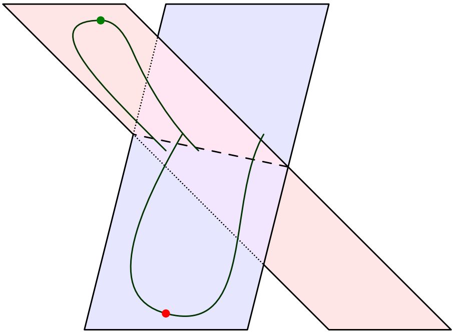

In the immersed case, stable and unstable manifolds can have self-intersections. We need to homotope the embedding to return to the previous case.

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- Plenty of freedom, so no canonical choice for the lift.
- In codim 2 even if we start with an embedding, we get self-intersections after finger move, so the situation is common.

Perspectives

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- Movie moves, even for strongly invertible knots, B. – Dai – Mallick – Stoffregen, work in progress.