Problem 13.4

For any $t \in \mathbb{C}^*$ we have an invertible matrix $M_t \in M_{n\times n}(\mathbb{C})$ such that for any $v \in \mathbb{C}^n$ we have $t \cdot v = M_t v$.

Observation 1. If t is of finite order as a group element, i.e. $t^k = 1$ for some $k \in \mathbb{N}$, then M_t is diagonalizable. This is because $M_t^k = id$, so the Jordan form of M_t cannot have 0 on the diagonal and cannot have 1 over the diagonal.

Observation 2. Let $\mu_k \subset \mathbb{C}^*$ be the group of the k-th roots of unity. Then for $t \in \mu_k$ the matrix M_t is diagonalizable, because t has a finite order. For all $t \in \mu_k$ matrices M_t have a common diagonal basis, because they all are powers of the group generator and the basis chosen for the generator works for all of them.

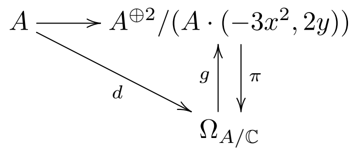
Observation 3. For any t_1, t_2 of finite order there is a common diagonal basis of M_{t_1} and M_{t_2}. This is because M_{t_1} and M_{t_2} commute. One may construct the common diagonal basis by taking t_3 of finite order such that both t_1 and t_2 are powers of t_3, and diagonalizing M_{t_3}. Thus there is a common diagonal basis for any finite set of groups $\mu_{k_1}, \ldots, \mu_{k_i}$ of roots of unity.

Observation 4. There is a common diagonal basis for elements of finite order in $\mathbb{C}^*$. To see this, for each k we define B_k as the subdivision into a direct sum of eigenspaces of $\mathbb{C}^n$ coming from the common diagonalization for $\mu_2, \ldots, \mu_k$. If $k_1 > k_2$ then B_{k_1} is a refinement of B_{k_2}. But in the chain $B_2, B_3, B_4, \ldots$ there can be only finitely many jumps, because we subdivide a finite-dimensional vector space.

Conclusion. The subset of all elements of finite order is dense in Zariski topology in $\mathbb{C}^*$, because it is infinite. Thus its product witn $\mathbb{C}^n$ is dense in Zariski topology in $\mathbb{C}^* \times \mathbb{C}^n$. Recall that the map defining the action $f\colon \mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$ is polynomial. We take the map $g\colon \mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$ corresponding to the diagonal action of the finite order elements: for an eigenvector with eigenvalue m we put t^m on the matrix diagonal, then change coordinates to the standard basis of $\mathbb{C}^n$ and write it as a map from $\mathbb{C}^* \times \mathbb{C}^n$ to $\mathbb{C}^n$. Then $f - g$ vanishes on a dense subset, so it vanishes everywhere. That is, the decomposition of $\mathbb{C}^n$ into eigenspaces which we used to describe g is the desired grading.

Problem 12.4

We didn't have a complete argument that for $A = \mathbb{C}[x,y]/(y^2 - x^3)$ the cotangent module $\Omega_{A/\mathbb{C}}$ has only one relation coming from the equation, i.e. $\Omega_{A/\mathbb{C}} \simeq A^{\oplus 2}/(2ydy - 3x^2dx)$. One can prove it imitating the argument that for $S = \Bbbk[x_1, \ldots, x_n]$ we have $\Omega_{S/\Bbbk} \simeq Sdx_1 \oplus \ldots \oplus Sdx_n$. Namely, we look at the diagram



The horizontal arrow is the gradient (in particular, a $\mathbb{C}$-linear derivation), one checks easily that it is well defined.

The map $d\colon A \to \Omega_{A/\mathbb{C}}$ sends the classes of x and y to the corresponding elements denoted dx and dy respectively. By definition $\Omega_{A/\mathbb{C}}$ this elements are generators.

The map g is unique such that the diagram commutes, from the universal property of $\Omega_{A/\mathbb{C}}$. Since the diagram commutes, $g(dx) = e_1$ and $g(dy) = e_2$.

The map π sends e_1, e_2 (classes of standard basis vectors of $A^{\oplus 2}$) to dx, dy respectively (thus, π is surjective). To see that it is well defined one has to check that $(-3x^2, 2y)$ is mapped to 0, but this is because in $\Omega_{A/\mathbb{C}}$ we have $-3x^2dx + 2ydy = d0 = 0$.

We need to show that π is one-to-one. But $g \circ \pi = id$, which can be seen on the chosen generating sets. Thus π is one-to-one and onto, hence an isomorphism.

This reasonong works without change for any quotient of a polynomial ring over a field by a principal ideal.