Commutative algebra

problem set 9, for 4.12.2019
ring extensions, algebraic sets

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Since we have not finished discussing Set 8, you can still declare problems 3, 4, 5 from Set 8.

Problem 1.
Let $\Bbbk$ be a field. In $\Bbbk[x, y, z]/(xy, xz)$ find maximal chains of prime ideals of length 1 and 2, with the same end: $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ oraz $\mathfrak{p}'_0 \subsetneq \mathfrak{p}_2$.

Problem 2.
Let $\Bbbk$ be an algebraically closed field and let S_1, S_2 be polynomial rings over $\Bbbk$. Let $I_i \subset S_i$ be ideals and $Z_i = V(I_i) \subset \Bbbk^{\dim S_i}$ corresponding algebraic sets. Take the polynomial ring $S = S_1 \otimes_{\Bbbk} S_2$. There is a natural surjection $f : S \to S_1/I_1 \otimes S_2/I_2$. Prove that the algebraic set $V(\ker f) \subset \Bbbk^{\dim S}$ is equal to $Z_1 \times Z_2$.

Problem 3.
Find a Noether normalization of $\mathbb{C}[x, y, z]/(xy + z^2, x^2y - xy^3 + z^4 - 1)$.

Problem 4. [2 points]
Let $I = (xz - y^2, x^3 - yz)$ be an ideal in $\mathbb{C}[x, y, z]$.

1. Identify points of the algebraic set $V(I)$.
2. Compute $\dim(V(I))$.
3. Decompose $V(I) = V(I_1) \cup V(I_2)$, where $\dim(V(I_1)) = \dim(V(I_2)) = \dim(V(I))$, and compute radicals of I_1, I_2.
4. Prove that $V(I_1), V(I_2)$ are irreducible. *Hint: one of them has parametrization* $t \mapsto t^3, t^4, t^5$.

Problem 5. Group actions [extra points problem, 3 points]
Let A be a ring and G a finite group of automorphisms of A. The *ring of invariants* of the action of G on A is

$$A^G = \{a \in A \mid \forall_{g \in G}\ g(a) = a\}.$$

Note that if $g \in G$ and $g(\mathfrak{p}) \in \operatorname{Spec}(A)$, then $\mathfrak{p} \subset A$ is also a prime ideal, hence G acts on $\operatorname{Spec}(A)$.

1. Prove that $i : A^G \hookrightarrow A$ is an integral ring extension.
2. Let $\mathfrak{p} \in \operatorname{Spec}(A^G)$. Prove that the action of G on $(i^*)^{-1}(\mathfrak{p})$ is transitive.
   Hint: you may prove and use the following statement: if A is a ring and $I \subset A$ its ideal, and $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \operatorname{Spec}(A)$ are prime ideals such that $I \subset \bigcup_{i=1}^n \mathfrak{p}_i$, then $I \subset \mathfrak{p}_i$ for some i.
3. Conclude that the fibers of i^* in problem 5.1 from Set 7 have at most 2 elements.