Commutative algebra

problem set 8, for 27.11.2019
Nullstellensatz and algebraic sets, integral homomorphisms

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Since we have not finished discussing Set 7, you can still declare problems 4, 5,6 from Set 7.

A topological space is called *Noetherian* if its closed subsets satisfy the descending chain condition, i.e. every chain $V_0 \supset V_1 \supset V_2 \supset \ldots$ of closed sets eventually stabilises.

A nonempty subset Y of a topological space X is *irreducible* if it cannot be decomposed as a sum $Y = Y_1 \cup Y_2$ of two proper subsets closed in Y.

Problem 1.
Let A be a Noetherian ring. Prove that the topological space $\operatorname{Spec}(A)$ is Noetherian. Prove that the ring $A = \Bbbk[x_1, x_2, \ldots, x_n, \ldots]/(x_1^2, x_2^2, \ldots, x_n^2, \ldots)$ is not Noetherian, but $\operatorname{Spec}(A)$ is a Noetherian topological space.

Problem 2. Decomposition into irreducible components
Let X be a Noetherian topological space.

1. Prove that every nonempty closed subset $Y \subset X$ can be expressed as a finite sum $Y = Y_1 \cup \ldots \cup Y_n$ of irreducible subsets.
2. Prove that if we require that $Y_i \not\subseteq Y_j$ for $i \neq j$ then Y_i are uniquely determined.

Note that it follows from Problems 1 and 2 that closed subsets of $\operatorname{Spec}(A)$ *for a Noetherian ring A can be decomposed into irreducible components.*

Problem 3. [2 points]
Let $\Bbbk$ be an algebraically closed field. Let $A = \Bbbk[x, y]/(y^2 - x^3)$ and $i\colon A \to \widetilde{A}$ be the integral closure.

1. For every ideal $\mathfrak{m} \subset A$ calculate the dimension of the $\kappa(\mathfrak{m})$-vector space $\widetilde{A} \otimes_A \kappa(\mathfrak{m})$.
2. Describe the fibers of i^*.

Problem 4. [2 points]
Let $\Bbbk$ be a field. By an *algebraic set* in the affine space $\Bbbk^n$ we understand a set of all points on which polynomials $f_1, \ldots, f_m \in \Bbbk[x_1, \ldots, x_n]$ vanish simultaneously (equivalently, it is the set where the whole ideal $(f_1, \ldots, f_m)$ vanishes).

1. Prove that for $\Bbbk = \mathbb{R}$ the set $\{(0, \ldots, 0)\} \subset \Bbbk^n$ is described by vanishing of a single polynomial.
2. Prove that for $\Bbbk = \mathbb{R}$ any algebraic set in $\Bbbk^n$ is described by vanishing of a single polynomial.
3. Let $\Bbbk$ be any field which is **not** algebraically closed. Prove that any algebraic set in $\Bbbk^n$ is described by vanishing of a single polynomial.
   Hint: to find a polynomial vanishing only at $(0, \ldots, 0)$ start with two variables and iterate the construction.

Problem 5. [extra points problem, 2 points]
Let K be a field and $L = K(a_1, \ldots, a_s)$ be a finitely generated extension of K. We say that a subset $b_1, \ldots, b_r \in L$ is *algebraically independent over K*, if the homomorphism $f\colon K[x_1, \ldots, x_r] \to L$, given by $f(x_i) = b_i$, is injective. An algebraically independent subset $b_1, \ldots, b_r \in L$ is a *transcendental basis of L over K* (baza przestępna) if the field extension $K(b_1, \ldots, b_r) \subset L$ is algebraic. Prove that a transcendental basis of L over K exists and that every two transcendental bases have the same number of elements.

Hint: prove an analogue of the Steinitz exchange lemma (from linear algebra).