Commutative algebra

problem set 7, for 20.11.2019
Artinian rings, fibers

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

A module M over a ring A is an *Artinian module* if every descending chain $M_0 \supset M_1 \supset M_2 \supset \dots$ of submodules of M stabilises. A ring A is called an *Artinian ring* if it is an Artinian A-module (i.e. descending chains of ideals stabilise).

Recall that the *nilradical* $\mathfrak{N}(A)$ of a ring A is the radical of (0), i.e. the set of all nilpotent elements.

Let $f\colon A \to B$ be a ring homomorphism and $\mathfrak{p} \in \operatorname{Spec}(A)$. Let $\kappa(\mathfrak{p}) = \operatorname{Frac}(A/\mathfrak{p}) = A_\mathfrak{p}/(\mathfrak{p}A_\mathfrak{p})$ be the *residue field* of $\mathfrak{p} \in \operatorname{Spec}(A)$. Recall that one may identify the *fiber* of $f^*\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ over a point $\mathfrak{p}$ with $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$.

Problem 1.
Let A be an Artinian ring. Prove that $\mathfrak{N}(A)$ is nilpotent.
Hint: Assume that the chain $\mathfrak{N}(A)^k$ stabilises at an ideal $I \neq 0$. Consider the set of all ideals J such that $I \cdot J \neq 0$. What can one say about a minimal element of this set?

Problem 2. [2 points for questions 1-4; 5 and 6 are extra points exercises, 1 point each]
Let A be an Artinian ring.

1. Assume that A is a domain. Prove that A is a field. *Hint: consider $x \in A$ and ideals (x^n).*
2. Prove that every prime ideal in A is maximal. (Thus $\dim A = 0$.)
3. Prove that $\operatorname{Spec}(A)$ is a finite set. Denote $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\} = \operatorname{Spec}(A)$.
   Hint: if $\mathfrak{q} \supset I_1 \cap \dots \cap I_j$ are ideals where $\mathfrak{q}$ is prime, then $\mathfrak{q} \supset I_i$ for some i.
4. Prove that $\operatorname{Spec}(A)$ is a finite discrete topological space.
5. Show that there exists n such that $\mathfrak{p}_1^n \cdot \ldots \mathfrak{p}_r^n = 0$. Conclude that there is a chain of ideals
   $$A \supsetneq I_1 \supsetneq I_2 \supsetneq \dots \supsetneq I_s = 0$$
   such that for every i the module I_i/I_{i+1} is a vector space over $A/\mathfrak{p}_{k_i}$ for some $k_i \in \{1, \dots, r\}$.
6. Use the sequence above to show that A is a Noetherian A-module. Conclude that A is a zero-dimensional Noetherian ring. *Hint: if $0 \to M \to N \to P \to 0$ is a sequence of A-modules and M, P are Noetherian then N is also Noetherian.*
   Remark: the converse is true, i.e. every zero-dimensional Noetherian ring is Artinian.

Problem 3.
Prove that if $\Bbbk$ is a field and A is a $\Bbbk$-algebra, which is a finite dimensional vector space over $\Bbbk$, then A is Artinian. Show that $|\operatorname{Spec}(A)| \leqslant \dim_\Bbbk A$.

Problem 4.
Let $f\colon A \to B$ be a ring homomorphism. Assume that B is a finite A-module, generated by $b_1, \dots, b_r$. Prove that the $\kappa(\mathfrak{p})$-module $B \otimes_A \kappa(\mathfrak{p})$ is generated by images of those elements. Show that $B \otimes \kappa(\mathfrak{p})$ is an Artinian ring and the fibers of f^* are finite sets.

Problem 5.
Let $i\colon A \hookrightarrow B$ be an integral ring extension. Consider $i^*\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$.

1. For $A = \mathbb{Z}$, $B = \mathbb{Z}[i] \subset \mathbb{C}$ find the fibers $(i^*)^{-1}(0)$, $(i^*)^{-1}(2)$, $(i^*)^{-1}(5)$ and $(i^*)^{-1}(7)$.
2. Prove that i^* is a closed map. *Hint: it was proved at the lecture that i^* is surjective.*

Problem 6.
Let $A = \mathbb{C}[t]$ and $B = \mathbb{C}[x, y, t]/(ty - x^2)$. Consider a $\mathbb{C}$-algebra homomorphism $f\colon A \to B$, $f(t) = t$. Find the fibers of $f^*\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$.