Commutative algebra

problem set 6, for 6.11.2019

Noetherian modules, finite and integral homomorphisms

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Since we have not finished discussing Set 5, you can still declare problems 3, 4, 5 from Set 5.

Let A be a ring and $I \subset A$ its ideal. The *radical* (radykał) of I is the ideal $\sqrt{I} := \{f \in A \colon \exists_{n\in\mathbb{N}} f^n \in I\}$ of A.

By $A[[x]]$ we denote the ring of formal power series over a ring A (pierścień szeregów formalnych nad A). Its elements are sequences $(a_0, a_1, a_2, \ldots)$ of elements of A. We identify them with formal sums $\sum_{n=0}^{\infty} a_n x^n$. Addition and multiplication are the natural extension of operations on polynomials to infinite formal sums.

Problem 1.
Let A be a Noetherian ring.

1. Let $I = (a_j : i \in J)$ be an ideal of A generated by elements a_j indexed by an infinite set J. Prove that I is generated by a finite subset of $\{a_j : j \in J\}$.
2. Prove that every ideal of A contains a power of its radical.
3. Prove that $f = \sum\limits_{n=0}^{\infty} a_n x^n \in A[[x]]$ is nilpotent if and only if all coefficients a_i are nilpotent.
   Hint/remark: only one implication requires the assumption that A is Noetherian.

Problem 2.
Let M be an A-module. Let $f \colon M \to M$ be a surjective homomorphism of A-modules.

1. Prove that if M is a Noetherian module then f is an isomorphism. *Hint: consider* $\ker(f^n)$.
2. Give an example (e.g for $A = \mathbb{C}[x]$) of M and f such that $\ker f \neq 0$. *Hint: Hilbert's hotel.*

Problem 3.
Let $\Bbbk$ be a field. For every domain A given below find its fraction field K and the integral closure.

(a) $A = \Bbbk[t]$,
(b) $A = \Bbbk[x, y]/(y^2 - x^3)$,
(c) $A = \Bbbk[x, y]/(y^2 - x^3 - x^2)$.

Hint: In (b) and (c) start from finding $t \in K \setminus A$ integral over A and prove that $\Bbbk[t]$ is the integral closure of A.

Problem 4. Operations preserving finite and integral homomorphisms [2 points]
Let A, B, C be rings.

1. Prove that if $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are finite (resp. integral) homomorphisms then also $A \xrightarrow{g\circ f} C$ is a finite (resp. integral) homomorphism.
2. Prove that if $A \xrightarrow{f} B$ is a finite (resp. integral) homomorphism and C is an A-algebra then the homomorphism $C = C \otimes_A A \xrightarrow{\mathrm{id}\otimes f} C \otimes_A B$ is a finite (resp. integral).
3. Prove that if $A \xrightarrow{f} B$ is a finite (resp. integral) homomorphism and $S \subset A$ is a multiplicatively closed set then $S^{-1}A \xrightarrow{S^{-1}f} S^{-1}B$ is a finite (resp. integral) homomorphism.
4. Prove that if $A \xrightarrow{f} B$ is a finite (resp. integral) homomorphism and $I \subseteq A$ is an ideal then $A/I \to B/IB$ is a finite (resp. integral) homomorphism.