Commutative algebra

problem set 5, for 30.10.2019
modules: localization, tensor product, Hom

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Since we have not finished discussing Set 4, you can still declare problem 3 from Set 4.

Problem 1. Exactness of localization
Let $0 \to M \to N \to P \to 0$ be an exact sequence of A-modules. Prove that for any multiplicatively closed system $S \subset A$ the sequence

$$0 \to S^{-1}M \to S^{-1}N \to S^{-1}P \to 0$$

is exact. *Hint:* $S^{-1}M = S^{-1}A \otimes_A M$.

Problem 2. Hom **and** $\otimes$ **examples [2 points]**
Let A be a ring, M an A-module and I, J ideals of A.

1. Compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ for $0 < n \in \mathbb{Z}$.
2. Describe (as precisely as you can) $\mathrm{Hom}_A(A/I, A/J)$ and $\mathrm{Hom}_A(A, M)$.
3. Compute $(\mathbb{Z}/n) \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ for $0 < n \in \mathbb{Z}$.
4. Prove directly that if $r, s \in \mathbb{Z}$ are such that $(r, s) = 1$ then $(\mathbb{Z}/r) \otimes_{\mathbb{Z}} (\mathbb{Z}/s)$ is zero.
5. Describe (as precisely as you can) $A/I \otimes_A A/J$.

Problem 3.

1. Prove the following *universal property of a free module and its basis*: an A-module M is free with a basis $X \subset M$ if and only if for any A-module N and a map $f : X \to N$ there is a unique A-module homomorphism $F : M \to N$ such that $F|_X = f$.
2. Let A be a ring and M be a free A-module of rank r. Let B be a ring with a homomorphism $\varphi : A \to B$ such that $\varphi(1_A) = 1_B$. Prove that a B-module $M \otimes_A B$ is free of rank r.
   Hint: you may start with the universal property of free modules above; for an A-basis $X \subset M$ and a B-module N with a map $f : X \to N$ construct a suitable homomorphism $g : M \times B \to N$ extending f.

Problem 4.
Let $(A, \mathfrak{m})$ be a local ring. Assume that A is a domain with the field of fractions K and let $S = A \setminus 0$. Let P be a finitely generated A-module and let $d := \dim_{A/\mathfrak{m}}(P/\mathfrak{m}P)$.

1. Show that P is generated by d elements, i.e. find a surjective homomorphism $F \twoheadrightarrow P$, where $F = A^{\oplus d}$.
2. Assume in addition that $\dim_K S^{-1}P = d$. Prove that the surjection above is an isomorphism, i.e. P is free.

Let $\Bbbk$ be a field. For a $\Bbbk$-algebra R by $\mathrm{Spec}_{\Bbbk}(R) \subset \mathrm{Spec}(R)$ we denote the set of maximal ideals $\mathfrak{m} \subset R$ such that the composition $\Bbbk \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ is an isomorphism. Such ideals are called *rational ideals.*

Problem 5. [2 points]
Let $\Bbbk$ be a field, A a $\Bbbk$-algebra, and let B and C be A-algebras.

1. Compute $\mathrm{Spec}(B \otimes_A C)$ for $B = C = \mathbb{C}$ and $A = \mathbb{R}$.
2. Prove that $\mathrm{Spec}_{\Bbbk}(B \otimes_A C) \simeq \mathrm{Spec}_{\Bbbk}(B) \times_{\mathrm{Spec}_{\Bbbk}(A)} \mathrm{Spec}_{\Bbbk}(C)$.