Commutative algebra

problem set 4, for 23.10.2019
modules, tensor product

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Since we have not finished discussing Set 3, you can still declare problems 4 and 5 from Set 3.

Let A be a ring and M be an A-module. We say that M is *free* (wolny) if $M \simeq A^{\oplus I}$, where I is a set of indices. That is,

$$A^{\oplus I} := \{(a_i)_{i \in I} \mid a_i \in A, a_i = 0 \text{ for all but finitely many } i\}.$$

Problem 1.

1. Let A be a ring of contiuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let $\mathfrak{m} \subset A$ be the ideal of functions vanishing at 0. Show that $A/\mathfrak{m} \simeq \mathbb{R}$ and $\mathfrak{m} = \mathfrak{m}^2$.
2. Prove that $\mathfrak{m} \subset A$ is not a finitely generated ideal. *Hint: if $\mathfrak{m}$ was finitely generated, then $\mathfrak{m}A_{\mathfrak{m}}$ would be finitely generated over $A_{\mathfrak{m}}$ and we could apply Nakayama's lemma.*
3. What changes if we consider $\mathbb{R}[x] \subset A$ instead of A?

Problem 2. Variants of Nakayama's lemma

Let A be a ring with the set of invertible elements A^*. Let $\mathcal{Q}$ be the set of all maximal ideals of A.

1. Prove that
$$J(A) := \bigcap \mathcal{Q} = \{x \in A \,|\, 1 + ax \in A^* \text{ for all } a \in A\}.$$
This ideal is called the *Jacobson radical* of A.
2. Let I be the ideal of A. Show that if $I \subset J(A)$ and if M is a finitely generated A-module with $M = IM$ then $M = (0)$.
3. Let A be a local ring, i.e. A has only one maximal ideal $\mathfrak{m}$. Let $N \subset M$ be A-modules with M finitely generated and $N + \mathfrak{m}M = M$. Prove that $N = M$.
4. Let A be a local ring with the maximal ideal $\mathfrak{m}$. Take a finitely generated A-module M and a finite set $\{x_1, \dots, x_n\} \subset M$ such that the residue classes $\{\overline{x}_1, \dots, \overline{x}_n\}$ span the vector space $M/\mathfrak{m}M$ over $A/\mathfrak{m}$. Prove that $\{x_1, \dots, x_n\}$ generate M as an A-module.

Problem 3. Free modules

1. Give an example of a quotient of a free module which is not free.
2. Give an example of a submodule of a free module which is not free.
Hint: it suffices to look at finite rings to find an example.
3. Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.
4. Prove that if A is a field then every A-module is free.

Recall the *universal property of tensor product* of A-modules M and N: for any A-module P there is a natural bijection between sets $\mathrm{Hom}_A(M \otimes_A N, P)$ and $\{\varphi \colon M \times N \to P \mid \varphi \text{ is } A\text{-bilinear}\}$. *Natural* means here that if we take a homomorphism $\psi \colon P \to P'$ of A-modules, then the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_A(M \otimes_A N, P) & \xrightarrow{\simeq} & \{\varphi \colon M \times N \to P \mid \varphi \text{ is } A\text{-bilinear}\} \\
\downarrow{\scriptstyle \psi\circ(-)} & & \downarrow{\scriptstyle \psi\circ(-)} \\
\mathrm{Hom}_A(M \otimes_A N, P') & \xrightarrow{\simeq} & \{\varphi' \colon M \times N \to P' \mid \varphi' \text{ is } A\text{-bilinear}\}
\end{array}$$

Problem 4. Tensor building blocks, part I [2 points]

Let A be a ring and M, N, P be A-modules.

1. Show that $A \otimes_A M \simeq M$.
2. Show that $M \otimes_A N \simeq N \otimes_A M$.
3. Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of A-modules M_i. Prove that
$$M \otimes_A N \simeq \bigoplus_i (M_i \otimes_A N).$$
In particular, $A^{\oplus I} \otimes_A N \simeq N^{\oplus I}$.
4. Prove that $(M \otimes_A N) \otimes_A P \simeq M \otimes_A (N \otimes_A P)$.
5. Let $A = \Bbbk$ be a field and M, N be finitely-dimensional $\Bbbk$-vector spaces. Find the dimension of $M \otimes_{\Bbbk} N$.