Commutative algebra

problem set 3, for 16.10.2019
localization

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Since we have not finished discussing Set 2, you can still declare problems 4,5 and 6 from Set 2.

Let A be a ring and $S \subset A$ a multiplicatively closed subset (podzbiór multiplikatywnie domknięty). Then $S^{-1}A$ denotes the localization (lokalizacja) of A in S and $i\colon A \to S^{-1}A$ is the canonical homomorphism, i.e. $i(a) = a/1$.

One may think about the localization in terms of operations on fractions up to a suitable equivalence relation, but then proving some statements becomes quite painful. One way to avoid it is using the *universal property of localization*, which you prove in Problem 1. Roughly speaking, it states that $S^{-1}A$ is the smallest ring in which elements of S are invertible. A general hint for this set is: if you get lost in formulas, try the universal property!

Problem 1. Universal property of localization
Let A be a ring and $S \subset A$ be multiplicatively closed. Recall from the lecture that if $f\colon A \to B$ is a ring homomorphism such that elements of $f(S)$ are invertible in B then there exists a unique map $f'\colon S^{-1}A \to B$ such that $f'(a/1) = f(a)$, that is, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & S^{-1}A \\ & {}_{f}\searrow & \Big\downarrow{\scriptstyle \exists!\, f'} \\ & & B \end{array}$$

1. Show that $S^{-1}A$ is the unique ring with this property. That is, take a homomorphism $j\colon A \to C$ such that elements of $j(S)$ are invertible. Assume that for any ring homomorphism $f\colon A \to B$ such that elements of $f(S)$ are invertible there exists a unique map $f'\colon C \to B$ such that $f' \circ j = f$. Then $C \simeq S^{-1}A$.
2. Is the statement above still true if we do not assume that the map f' is unique?

Problem 2. Examples of localization [2 points]
1. Let $A = \mathbb{Z}/10$. Prove that $A_2 \simeq A/5$.
2. More generally, let A be a ring and $x, y \in A$ be such that $xy = 0$ and $(x, y) = (1)$. Prove that $A_x \simeq A/y$.
3. Let $\mathfrak{p} \subset A$ be a prime ideal and $S = A \setminus \mathfrak{p}$. Show that $A_\mathfrak{p} := S^{-1}A$ is a local ring (pierścień lokalny), i.e. it has only one maximal ideal.

Problem 3.
Fix an integer $n > 1$. Let A be a ring such that $x = x^n$ for all $x \in A$. Let $S \subset A$ be a multiplicatively closed subset. Prove that $i\colon A \to S^{-1}A$ is a ring epimorphism. *(Hint: consider $B = A/(\ker i)$.)*

Problem 4.
Let $S, T \subset A$ be multiplicatively closed subsets of A. Note that $ST = \{st \mid s \in S, t \in T\}$ is also multiplicatively closed. Let $i\colon A \to S^{-1}A$ and $j\colon A \to T^{-1}A$ be canonical homomorphisms.
1. Prove that $(i(T))^{-1}(S^{-1}A) \simeq (ST)^{-1}A \simeq (j(S))^{-1}(T^{-1}A)$.
2. In particular, $(S^{-1}A)_\mathfrak{p} \simeq A_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \cap S = \emptyset$.

Problem 5. Saturating the multiplicatively closed system
Let $S \subset A$ be mutliplicatively closed subset of a ring A. The *saturation* (nasycenie) of S is

$$\overline{S} = \{t \in A \,|\, ta \in S \text{ for some } a \in A\}.$$

1. Show that $\overline{S}$ is a multiplicatively closed subset of A containing S.
2. Prove that saturation does not change localization: $\overline{S}^{-1}A \simeq S^{-1}A$.
3. What is $\widetilde{S}^{-1}A$ for a multiplicatively closed subset $\widetilde{S}$ satisfying $S \subset \widetilde{S} \subset \overline{S}$?