Commutative algebra

problem set 2, for 9.10.2019
spectrum of the ring, Zariski topology

A lot of topology again, more algebra coming soon!
You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Let A be a ring and $X = \mathrm{Spec}(A)$. For every subset $E \subset A$ we define $V(E) = \{\mathfrak{p} \in X \mid E \subset \mathfrak{p}\}$. Note that $V(E) = V(I)$, where $I \lhd A$ is the ideal generated by E. Recall from the lecture that the family of subsets $\{V(E) \mid E \subset A\}$ satisfies the axioms of the closed sets of a topological space – this is the **Zariski topology**.

Problem 1.
Let A be a ring and $X = \mathrm{Spec}(A)$. For every $f \in A$ we define

$$X_f = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}.$$

Prove that $\{X_f \mid f \in A\}$ is a basis of open sets for the Zariski topology.

Problem 2.
1. Let A be a domain such that $\mathrm{Spec}(A)$ is a Hausdorff topological space. Show that A is a field.
2. Let A be any ring. Show that $\mathrm{Spec}(A)$ is T0 (that is, for any pair of distinct points at least one of them has an open neighbourhood not containing the other).

Problem 3. [2 points]
Let $f\colon A \to B$ be a ring homomorphism. Describe $\mathrm{Spec}(A), \mathrm{Spec}(B)$ and the map $f^*\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ for

1. $A = \mathbb{R}[x]$ and $B = \mathbb{C}[x]$, f is the inclusion.
2. $A = \mathbb{C}[x]$ and $B = \mathbb{C}[x, x^{-1}]$, f natural.
3. $A = \mathbb{C}[x]$ and $B = \mathbb{C}[x]/x(x-1)(x-2)$, f natural.

Problem 4. [2 points]
Let A be a ring and $X = \mathrm{Spec}(A)$ its spectrum with the Zariski topology.

1. Show that from any cover (pokrycie) of X by open sets one can choose a finite subcover. We say that X is *quasi-compact*. (*Hint: we can restrict to covers which consist only of sets of the form X_{f_i} – why? What can we say about the ideal generated by the f_i's?*)
2. Let $A = \mathbb{C}[x]$. Find a cover of $X = \mathrm{Spec}(A)$ by 2019 open sets such that no 2018 is sufficient to cover X.

Problem 5.
A point x of an irreducible topological space X is called a *generic point* of X if X is equal to the closure of the subset $\{x\}$.

1. Show that in a T0 topological space X every irreducible closed subset has at most one generic point.
2. Let A be a ring and $\mathfrak{p}$ its prime ideal. Prove that the closed subset $V(\mathfrak{p}) \subset \mathrm{Spec}(A)$ has a generic point.

Problem 6. [extra points problem, 3 points]
Let X be a topological space and A the ring of continuous functions $X \to \mathbb{R}$. Prove that the following conditions are equivalent:

1. X is not connected,
2. there is an element $0, 1 \neq e \in A$ such that $e^2 = e$ (called an *idempotent*),
3. there is an isomorphism $A \simeq A/I \times A/J$ for some ideals $I, J \subset A$.

Can we generalize this to the situation where A is any ring and $X = \mathit{Spec}(A)$?