Commutative algebra

problem set 13, for 22.01.2020
graded rings and modules

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Since we have not finished discussing Set 12, you can still declare problems 1, 2, 4, 5 from Set 12.

An S-*graded* $\Bbbk$-*algebra* A is an S-graded ring A with a fixed homomorphism $\Bbbk \to A_0$, where $0 \in S$ is the neutral element.

Problem 1.
Consider the following ideals in $\Bbbk[x, y, z]$:

$$I_1 = (x^2yz + z^4 - x^2z^2, xy + y^2), \qquad I_2 = (x^2 + y, y^2, x^2 + x^2y),$$

Check whether they are homogeneous with repsect to

1. the standard $\mathbb{Z}$-grading on $\Bbbk[x, y, z]$ (that is, x, y, z have degree 1),
2. the $\mathbb{Z}^3$-grading such that the degree of x is $(1, 0, 0)$, the degree of y is $(0, 1, 0)$ and the degree of z is $(0, 0, 1)$.

Problem 2.
Let A be a $\mathbb{Z}^r$-graded $\Bbbk$-algebra such that for any non-zero homogeneous $f_1, f_2 \in A$ we have $f_1f_2 \neq 0$. Prove that

1. A is an integral domain,
2. if g_1g_2 is homogeneous for some non-zero $g_1, g_2 \in A$ then g_1 and g_2 are homogeneous,
3. all units in A are homogeneous.

Problem 3. Graded Nakayama's lemma
lady Let A be a $\mathbb{Z}_{\geqslant 0}$-graded ring and M be $\mathbb{Z}$-graded A-module. Assume that $M_i = 0$ for i small enough. Recall that the ideal A_+ is defined as $\bigoplus_{i>0} A_i$. Prove that if $A_+ \cdot M = M$ then $M = 0$. Is this true without the assumption that $M_i = 0$ for i small enough?

Problem 4. Diagonalizing algebraic torus action [2 points]
Let $V \simeq \mathbb{C}^n$ be a vector space with $\mathbb{C}^*$ action, which is linear, i.e. for any $t \in \mathbb{C}^*$ the map $\mathbb{C}^n \to \mathbb{C}^n$ is linear. Assume also that this action is algebraic, that is the map $\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$ is polynomial. Prove that there is a grading $V = \bigoplus_i V_i$ such that for any $v_i \in V_i$ the action is $t \cdot v_i = t^i v_i$.

Hint: start from finding a grading compatible with the induced action of a cyclic subgroup of $\mathbb{C}^*$.