Commutative algebra

problem set 12, for 8.01.2020
dimension theory

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Since we have not finished discussing Set 11, you can still declare problems 1, 4, 5 from Set 11.

Problem 1.
Compute the dimension of $\mathbb{C}[x_1, \dots, x_6]/I$ for

$$I = (x_5^2 - x_4x_6, x_4x_5 - x_3x_6, x_3x_5 - x_2x_6, x_2x_5 - x_1x_6, x_4^2 - x_2x_6, x_3x_4 - x_1x_6, \\ x_2x_4 - x_1x_5, x_3^2 - x_1x_5, x_2x_3 - x_1x_4, x_2^2 - x_1x_3).$$

Hint: Jacobian cirterion.

Problem 2. The tangent space
Let A be a finitely generated $\Bbbk$-algebra, where $\Bbbk$ is algerbaically closed, and $\mathfrak{m}$ its maximal ideal. Recall that in this setting $\Bbbk \simeq A/\mathfrak{m}$.

1. Show that $\operatorname{Der}_{\Bbbk}(A, A/\mathfrak{m}) = \operatorname{Der}_{\Bbbk}(A/\mathfrak{m}^2, A/\mathfrak{m}) \subseteq \{\varphi\colon A/\mathfrak{m}^2 \to A/\mathfrak{m} \mid \varphi \text{ is } \Bbbk\text{-linear and } \varphi|_{\Bbbk} = 0\}$.
2. Show that $A/\mathfrak{m}^2 = \Bbbk \oplus (\mathfrak{m}/\mathfrak{m}^2)$, describe the multiplication in $\Bbbk \oplus (\mathfrak{m}/\mathfrak{m}^2)$.
3. Prove that the inclusion in part 1 is an equality.
4. Conclude that $\operatorname{Der}_{\Bbbk}(A, A/\mathfrak{m}) = \operatorname{Hom}_{\Bbbk}(\mathfrak{m}/\mathfrak{m}^2, A/\mathfrak{m}) = \operatorname{Hom}_{\Bbbk}(\mathfrak{m}/\mathfrak{m}^2, \Bbbk) = (\mathfrak{m}/\mathfrak{m}^2)^\vee$.

Remark: the aim of this problem is to give a reason for the fact that $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ may be thought of as the tangent space to $\operatorname{Spec} A$ at $\mathfrak{m}$, assuming that we have heard about the relation between the tangent space (or bundle) and differentials.

Problem 3. [2 points]
Let A be a ring, not necessarily Noetherian.

1. Prove that $1 + \dim A \leqslant \dim A[x]$.
2. Let $f\colon A \to A[x]$ be the natural embedding. Consider $f^*\colon \operatorname{Spec} A[x] \to \operatorname{Spec} A$. Describe the fibers of f^*: find the corresponding algebra and compute its dimension.
3. Prove that $\dim A[x] \leqslant 1 + 2\dim A$.

Problem 4. [2 points]
Consider

$$(a) \quad A = \mathbb{C}[x, y]/(y^2 - x^3), \qquad\qquad (b) \quad A = \mathbb{C}[x, y]/(y^2 - x^3 - x).$$

For both algebras A describe $\Omega_{A/\mathbb{C}}$ and for any maximal ideal $\mathfrak{m} \subseteq A$ compute $\Omega_{A/\mathbb{C}}/\mathfrak{m}\Omega_{A/\mathbb{C}} \simeq \mathfrak{m}/\mathfrak{m}^2$.

Problem 5. [extra points problem, 2 points]
Let A be an integrally closed domain with the field of fractions K. Let $x \in L \supseteq K$ be algebraic over K. Prove that x is integral over A if and only if its minimal polynomial over K has all coefficients in A.