Commutative algebra

problem set 11, for 18.12.2019

Dedekind domains, smoothness and the Jacobian criterion

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Since we have not finished discussing Set 10, you can still declare problems 2, 4 from Set 10.

Let A be an integral domain with field of fractions K. We call an A-submodule M of K a *fractional ideal* of A if for some $x \in A$, $x \neq 0$ we have $xM \subset A$. Examples are ideals of A (for $x = 1$) and *principal fractional ideals* uA, where $u \in K^*$. An A-submodule M of K is called an *invertible ideal* if there is an A-submodule N of K such that $MN = A$.

Problem 1. The ideal class group [2 points]

Let A be an integral domain with field of fractions K.

1. Show that a finitely generated A-submodule M of K is a fractional ideal. Show that if A is Noetherian then every fractional ideal is finitely generated.
2. Let M be an invertible ideal of A. Prove that the submodule N such that $MN = A$ is uniquely determined and that M is a fractional ideal.
3. Let M be a fractional ideal of A. Show that invertibility is a local property, i.e. the following are equivalent:
   a) M is invertible,
   b) M is fintely generated and its localizations in all prime ideals are invertible,
   c) M is finitely generated and its localizations in all maximal ideals are invertible.
4. Prove that if A is a discrete valuation ring then every non-zero fractional ideal of A is invertible.
5. Prove that if A is a Dedekind domain then every non-zero fractional ideal of A is invertible. *Hint: localize.*

Thus in a Dedekind domain A the set of all non-zero fractional ideals is a group with the operation of multiplication. Its quotient by a subgroup of all principal fractional ideals is the *ideal class group* of A.

Problem 2.

Let A be a Dedekind domain with the ideal class group of h elements (in particular, the ideal class group is finite). Let $I \subseteq A$ be an ideal such that I^m is principal and m and h are coprime. Prove that I is principal.

Problem 3. Solving a Diophantine equation using the ideal class group [2 points]

Let $A = \mathbb{Z}[\sqrt{-6}]$. We skip the proofs of the following facts: A is a Dedekind domain (the proof as in problem 2 from set 10) and the ideal class group of A is $\mathbb{Z}_2$ (the proof would make another long problem). Assume that $x, y \in \mathbb{Z}$ satisfy $x^2 + 6 = y^3$.

1. Show that x is divisible neither by 2 nor by 3.
2. Show that in A we have the following equality of ideals: $(y)^3 = (x - \sqrt{-6})(x + \sqrt{-6})$.
3. Show that ideals $(x - \sqrt{-6})$ and $(x + \sqrt{-6})$ are coprime. *Hint:* $A = \mathbb{Z}[t]/(t^2 + 6)$.
4. Prove that there is an ideal J such that $J^3 = (x - \sqrt{-6})$. *Hint: unique factorization of ideals.*
5. Show that J is principal. Find all pairs $x, y \in \mathbb{Z}$ satisfying $x^2 + 6 = y^3$.

Let k be an algebraically closed field. Let $I = (f_1, \dots, f_m) \subset k[x_1, \dots, x_n]$ be a prime ideal and let $V(I)$ be an (irreducible) algebraic set of I. Let $A = k[x_1, \dots, x_n]/I$, note that A is an integral domain. Take a point $p \in V(I)$ corresponding to the maximal ideal $m_p = (x_1 - a_1, \dots, x_n - a_n) \in k[x_1, \dots, x_n]$. By $\overline{m}_p$ we denote $m_p/I \subseteq A$. We say that p is a *regular point* of $V(I)$ if $A_{\overline{m}_p}$ is a regular local ring, i.e. if $\dim_k m/m^2 = \dim A_{\overline{m}_p}$ for the maximal ideal $m = \overline{m}_p A_{\overline{m}_p}$ of $A_{\overline{m}_p}$.

Problem 4. Jacobian criterion [2 points]

In the above setting we will prove that a point $p \in V(I)$ is regular if and only if it is nonsingular, i.e. if the rank of the Jacobian matrix $J = \left[\frac{\partial f_i}{\partial x_j}(p)\right]$ is $n - r$, where r is the dimension of $V(I)$ (defined as $\dim A$).

1. Define a map $\varphi\colon k[x_1, \dots, x_n] \to k^n$ by $\varphi(f) = \left(\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p)\right)$ and prove that it gives an isomorphism between m_p/m_p^2 and k^n.
2. Find the dimension of $\varphi(I)$ and of the subspace $(I + m_p^2)/m_p^2$ of m_p/m_p^2.
3. Show that $m/m^2 \simeq m_p/(I + m_p^2)$ and $\dim(m/m^2) + rk(J) = n$.
4. Conclude that the regularity of p is equivalent to $rk(J) = n - r$.

Problem 5.

Determine all the singular points of the following curves in $\mathbb{C}^2$ (and sketch their real parts):

(a) $xy = x^6 + y^6$, (b) $x^2y^2 + x^2 + y^2 + 2xy(x + y + 1) = 0$.