Commutative algebra

problem set 10, for 11.12.2019
discrete valuations, Dedekind domains

You do not have to write the solutions, but please be prepared to present your solutions smoothly at the board.

Since we have not finished discussing Set 9, you can still declare problems 2, 3, 4, 5 from Set 9.

Problem 1.
Let p be a prime number. We can write any non-zero $x \in \mathbb{Q}$ uniquely as $p^m \cdot \frac{a}{b}$, where $m, a, b \in \mathbb{Z}$ and a, b are coprime to p. Define $v_p(x) = m$.

1. Prove that v_p is a discrete valuation on $\mathbb{Q}$.
2. Find the valuation ring of v_p and its residue field (quotient by the maximal ideal).

Problem 2.
Prove that $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain, but it is not a PID.

Problem 3.
Give an example of a dimension one Noetherian domain which is not normal.

Problem 4. Reminder: the Chinese Remainder Theorem [2 points]
Let A be a Dedekind domain, take ideals $I_1, \ldots, I_n \subseteq A$ and elements $x_1, \ldots, x_n \in A$. Prove that the system of congruences $x \equiv x_k \mod I_k$ for $1 \leqslant k \leqslant n$ has a solution $x \in A$ if and only if $x_j \equiv x_k \mod (I_k + I_j)$ for $1 \leqslant j < k \leqslant n$. (Note that we do not assume that the ideals are pairwise coprime!)

Hint: you may use the sequence of modules

$$A \to \bigoplus_k A/I_k \to \bigoplus_{j<k} A/(I_j + I_k)$$

with arrows defined such that the statement above is equivalent to the exactness of the sequence. Localize in a prime ideal $\mathfrak{p}$; what can you say about $A_\mathfrak{p}$?