Commutative algebra

problem set 1, for 2.10.2019
rings, ideals, homomorphisms

You do not have to write the solutions. Please be prepared to present smoothly at the board your solutions of the problems which you declare as solved.
If you have any questions, please ask via email (m.donten@mimuw.edu.pl) or personally.

Recall that all rings considered here are **associative, commutative and have a unity** (łączne, przemienne, z jedynką). An ideal I in a ring R is **prime** (pierwszy), if for any $a, b \in R$ the condition $ab \in I$ implies that either $a \in I$ or $b \in I$. An ideal I is **maximal** (maksymalny) if it is maximal under inclusion, i.e. if $I \neq R$ and there is no ideal J of R such that $I \subsetneq J \subsetneq R$.

Problem 1. [2 points]
Let R be the ring of continuous functions $\mathbb{R} \to \mathbb{R}$ with pointwise addition and multiplication, that is $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$.

1. Show that for any $x \in \mathbb{R}$ the set $\mathfrak{m}_x = \{f \in R \mid f(x) = 0\}$ is a maximal ideal.
2. Let Z_0 be the set of functions converging to 0 at ∞. Is Z_0 a maximal ideal?
3. Let Z_1 be the set of functions f such that $f(x) = 0$ for x big enough. Is Z_1 a maximal ideal?

Problem 2. [2 points]
Let R be a ring, $I \subset R$ its ideal and $\pi \colon R \to R/I$ the canonical epimorphism.

1. Show that $\bar{J} \mapsto \pi^{-1}(\bar{J})$ is a bijection between the set of ideals in R/I and the set of ideals in R containing I.
2. Show that R/I is a domain (dziedzina) if and only if I is prime.
3. Show that R/I is a field (ciało) if and only if I is maximal.

Problem 3.
Let $I \subset R$ be an ideal. Assume that R/I is finite. Show that I is prime if and only if it is maximal.

Problem 4.
Let B be a ring and $\mathfrak{m} \subset B$ its maximal ideal.

1. Let $A \subset B$ be a subring. Is $\mathfrak{m} \cap A$ a maximal ideal? Is $\mathfrak{m} \cap A$ a prime ideal?
2. Let C be a ring and $f \colon C \to B$ a homomorphism. Is $f^{-1}(\mathfrak{m})$ a maximal ideal? Is $f^{-1}(\mathfrak{m})$ a prime ideal?

Problem 5. [extra points problem, 3 points]
Take $S^1 = \big\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\big\}$. Let R be a ring of continuous functions $S^1 \to \mathbb{R}$ with pointwise addition and multiplication, that is $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$.

1. For $x \in S^1$ define $\mathfrak{m}_x = \{f \in R \mid f(x) = 0\}$. Prove that $\mathfrak{m}_x \subset R$ is a maximal ideal.
2. Let $\mathfrak{n} \subset R$ be a maximal ideal such that $\mathfrak{n} \neq \mathfrak{m}_x$ for a chosen $x \in S^1$. Show that there is a function $f \in \mathfrak{n}$ such that $f(x) \neq 0$ and for all $y \in S^1$ we have $f(y) \geqslant 0$.
3. Let $\mathfrak{n} \subset R$ be a maximal ideal such that $\mathfrak{n} \notin \{\mathfrak{m}_x \mid x \in S^1\}$. Prove that there is a function $f \in \mathfrak{n}$ such that $f(x) > 0$ for all $x \in S^1$. Show that f is invertible and deduce that $\mathfrak{n} = (1)$ is not a maximal ideal. Thus, all maximal ideals of R are of the form $\mathfrak{m}_x$.
   Hint: for any $x \in S^1$ consider a function $f = f_x$ as in part 2. and an open subset
   $$U_x = \{y \in S^1 \mid f_x(y) > 0\} \subset S^1.$$
4. Is the characterisation of maximal ideals from part 3. valid for any *compact* topological space? Is it valid for any topological space?