Commutative algebra

preparation for the 2nd midterm exam, for 15.01.2020

Problem 1.
Give an example of a finitely generated $\mathbb{C}$-algebra A and maximal ideals $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset A$ such that $\dim(A_{\mathfrak{p}}) \neq \dim(A_{\mathfrak{q}})$.

Problem 2.
Find a Noether normalization of $\mathbb{C}[x, y, z, t]/I$ for the ideal

$$I = (x^2 + xy - z, xy^2 - t^2 + z, x^2y + xy^2 + t^2).$$

Problem 3.
Give an example of a prime ideal $\mathfrak{p} \subset \mathbb{C}[x, y]$ such that the corresponding algebraic set $V := V(\mathfrak{p}) \subset \mathbb{C}^2$ satisfies the following condition: $V \cap \mathbb{R}^2 \subset \mathbb{R}^2$ is non-empty and is not connected in the Euclidean topology.

Problem 4.
Let $\omega \in \mathbb{C} \setminus \mathbb{R}$ be a third root of 1. Let $i\colon \mathbb{Z} \to \mathbb{Z}[\omega]$ be the canonical inclusion. Prove that the fibers of $i^*\colon \operatorname{Spec}(\mathbb{Z}[\omega]) \to \operatorname{Spec}(\mathbb{Z})$ have at most two elements. Do all the fibers have two elements?

Problem 5.
Let $\Bbbk$ be a field and $I = (y^2 - x^3 - x^2) \subset \Bbbk[x, y]$.

1. Prove that the normalization of $\Bbbk[x, y]/I$ is isomorphic to $\Bbbk[y/x]$.
2. Compute the fibers of $i^*\colon \operatorname{Spec}(\Bbbk[y/x]) \to \operatorname{Spec}(\Bbbk[x, y]/I)$ associated to the normalization $i\colon \Bbbk[x, y]/I \hookrightarrow \Bbbk[y/x]$.

Problem 6.
Let A be a discrete valuation ring and K its field of fractions. Prove that there is $a \in A$ such that $S^{-1}A = K$ for $S = \{1, a, a^2, \ldots, a^n, \ldots\}$.

Problem 7.
Let A be a discrete valuation ring and K its field of fractions. Let $B \subset K$ be a subring. Prove that if $A \subset B$, then either $B = A$ or $B = K$.

Problem 8.
Find all singular points of the surface in $\mathbb{C}^3$ defined by $xy^2 = z^2$.

Problem 9.
Let $I = (xy - z, x^2z - y^2) \subseteq \mathbb{C}[x, y, z]$.

1. Find the dimension of $V(I)$.
2. Find irreducible components of $V(I)$.
3. Find parametrizations of components.
4. Find radicals of ideals of components.