Commutative algebra

preparation for the 1st midterm exam, for 13.11.2019

Problem 1.
Let A be a ring and I its ideal. By $I[x]$ we define a set of all polynomials with coefficients in I:

$$\{a_n x^n + \ldots + a_1 x + a_0 \colon n \in \mathbb{N},\, a_0, \ldots, a_n \in I\} \subseteq A[x].$$

1. Prove that if I is prime then $I[x]$ is a prime ideal of $A[x]$.
2. Assume that I is maximal. Is $I[x]$ a maximal ideal of $A[x]$?

Problem 2.
Describe points and open subsets of $\operatorname{Spec}\big(\mathbb{C}[x,y]/(xy(x^2-y^2))\big)$.

Problem 3.
Let $\Bbbk$ be a field, A a finitely generated $\Bbbk$-algebra and $S \subset A$ a multiplicatively closed subset. Is $S^{-1}A$ a finitely generated $\Bbbk$-algebra?

Problem 4.
Let A be a domain. Assume that for any non-trivial finitely generated A-module M we have $\operatorname{Hom}_A(M, A) \neq 0$. Prove that A is a field.

Problem 5.
Give an example of a ring A, an injective A-module homomorphism $f \colon M \to M'$ and an A-module N such that the homomorphism $id \otimes f \colon N \otimes_A M \to N \otimes_A M'$ is not injective.

Problem 6.
Let A be a ring, $S \subset A$ a multiplicatively closed system and M a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Problem 7.
Let A be a ring and $\mathfrak{p} \in \operatorname{Spec}(A)$. Prove that the ring $A_{\mathfrak{p}}/(\mathfrak{p}A_{\mathfrak{p}})$ is a field. Describe this field for $A = \mathbb{C}[x,y]$ and
a. $\mathfrak{p} = (x,y)$ b. $\mathfrak{p} = (x)$ c. $\mathfrak{p} = (0)$.

Problem 8.
Let M_i for $1 \leqslant i \leqslant n$ be Noetherian A-modules. Prove that $\bigoplus_{i=1}^{n} M_i$ is also Noetherian.

Problem 9.
Let $R = \mathbb{C}[x^3, y^3, x^2y] \subseteq \mathbb{C}[x,y]$. Is R integrally closed in its field of fractions? If not, find its integral closure.

Problem 10.
Let $f \colon M \to N$ be an A-module homomorphism. Prove that f is injective (resp. surjective) if and only if for each prime ideal $\mathfrak{p} \subset A$ the homomorphism of localisations $f_{\mathfrak{p}} \colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective (resp. surjective).

Problem 11.
Give an example of a module M over a Noetherian ring A such that $M_{\mathfrak{p}}$ is finitely generated for each $\mathfrak{p} \in \operatorname{Spec}(A)$, but M is not finitely generated.

Problem 12.
Let A be a ring and M a finitely generated A-module. Take a surjective homomorphism $f \colon M \to M$.

1. Show that M is also an $A[x]$-module, where we define multiplication by a polynomial $p(x) \in A[x]$ as: $p(x) \cdot m := p(f)(m)$ (product x^k corresponds to the k-fold composition $f \circ \cdots \circ f$).
2. Prove that M and the ideal $(x) \subseteq A[x]$ satisfy the assumptions of the Nakayama's lemma.
3. Conclude that f is injective.