T. Hibi "Binomial ideals arising from combinatorics"

lecture notes
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1 Lecture 1

Let $S = k[x_1, ..., x_n]$ be the polynomial ring in n variables over a field k and let

$$\mathrm{Mon}(S) = \{x^{\mathbf{a}} = x_1^{a_1} ... x_n^{a_n} : \mathbf{a} = (a_1, ..., a_n) \in \mathbb{Z}^n_{\geq 0}\}$$

be the set of monomials of S.

a) Dickson's Lemma

Let $x^{\mathbf{a}} = x_1^{a_1} ... x_n^{a_n}$ and $x^{\mathbf{b}} = x_1^{b_1} ... x_n^{b_n}$. We say that $x^{\mathbf{a}}$ **divides** $x^{\mathbf{b}}$ if $a_i \leq b_i$ for all $1 \leq i \leq n$. Let $\emptyset \neq M \subset \mathrm{Mon}(S)$. We say that $x^{\mathbf{a}} \in M$ is **minimal** if for $x^{\mathbf{b}} \in M$ such that $x^{\mathbf{b}} \mid x^{\mathbf{a}}$ we have $b = a$. Let M^{min} be the set of minimal monomials in M.

Theorem 1.1 (Dickson's Lemma)**.** *Let $\emptyset \neq M \subset \mathrm{Mon}(S)$. Then M^{min} is a finite set.*

Proof. We use induction on n. For $n = 1$ the proof is easy. Let $n \geq 2$ and let $y = x_n$. Write $S = k[x_1, ..., x_{n-1}, y]$ and $B = k[x_1, ..., x_{n-1}]$. Set $N = \{x^{\mathbf{a}} \in \mathrm{Mon}(B) \mid x^{\mathbf{a}} y^b \in M \text{ for some } b \geq 0\}$. By induction we have that N^{min} is a finite set. Let $N^{min} = \{u_1, ..., u_s\}$ and $u_1 y^{b_1}, ..., u_s y^{b_s} \in M$. Let $b = \max\{b_1, ..., b_s\}$. For each $0 \leq c < b$ define

$$N_c = \{x^{\mathbf{a}} \in N : x^{\mathbf{a}} y^c \in M\} \subset N$$

Again, we know N_c^{min} is a finite set, say $N_c^{min} = \{u_1^{(c)}, ..., u_{s_c}^{(c)}\}$. Consider the following monomials:

$$u_1 y^{b_1},\ u_2 y^{b_2},\ ...\ u_s y^{b_s}$$

$$u_1^{(0)},\ ...\ u_{s_0}^{(0)}$$

$$u_1^{(b-1)} y^{b-1},\ ...\ u_{s_{b-1}}^{(b-1)}$$

It then follows easily that every monomial in M is divisible by one of the monomials on the above list ∎

b) Monomial order

A **monomial order** on S is a total order $<$ on $\mathrm{Mon}(S)$ such that

i) $1 < u$ for $1 \neq u \in \mathrm{Mon}(S)$,

ii) if $u, v \in \mathrm{Mon}(S)$ and $u < v$, then $uw < vw$ for all $w \in \mathrm{Mon}(S)$.

Example 1.2. Let $x^{\mathbf{a}} = x_1^{a_1} ... x_n^{a_n}$ and $x^{\mathbf{b}} = x_1^{b_1} ... x_n^{b_n}$.

a) (Lexicographic order) We say that $x^{\mathbf{a}} <_{lex} x^{\mathbf{b}}$ if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the leftmost non-zero component of the vector $\mathbf{a} - \mathbf{b}$ is negative. We call $<_{lex}$ the **lex order on S induced by** $x_1 > ... > x_n$.

b) (Reverse lexicographic order) We say that $x^{\mathbf{a}} <_{rev} x^{\mathbf{b}}$ if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the rightmost non-zero component of the vector $\mathbf{a} - \mathbf{b}$ is positive. We call $<_{rev}$ the **reverse lex order on** S **induced by** $x_1 > ... > x_n$.

c) (Purely lexicographic order) We say that $x^{\mathbf{a}} <_{purelex} x^{\mathbf{b}}$ if the leftmost non-zero component of the vector $\mathbf{a} - \mathbf{b}$ is negative.

The reverse purely lexicographic order is not a monomial order since $1 > x_1$.
We have:

$$x_2x_3 <_{lex} x_1x_4$$

$$x_1x_4 <_{rev} x_2x_3$$

$$x_2^5 <_{purelex} x_1^3$$

Lemma 1.3. *If $u \mid v$ and $u \neq v$ then $u < v$.*

Proof. We have $v = uw$ for some $1 \neq w \in \mathrm{Mon}(S)$. From the first property of the monomial order we have that $1 < w$. Hence, from the second property of the monomial order we have $u < uw = v$. ■

Lemma 1.4. *There exists no infinite descending sequence of monomials of the form $...u_2 < u_1 < u_0$.*

Proof. Suppose that there exists such a sequence M. Let $M^{min} = \{u_{i_0}, u_{i_1}, ..., u_{i_s}\}$ with $i_0 < i_1 < ... < i_s$. Then we have $u_{i_j} \mid u_{i_s+1}$ for some $0 \leq j \leq s$. Hence from the previous lemma we get $u_{i_j} < u_{i_s+1}$. Thus $i_j > i_s + 1$, which is a contradiction.

■

c) Gröbner bases

Fix a monomial order $<$ on S. Given a polynomial $0 \neq f = \sum_{u \in \mathrm{Mon}(S)} c_u u$ ($c_u \in k$). We define the **support** of f to be $\mathrm{supp}(f) = \{u \in \mathrm{Mon}(S) \mid c_u \neq 0\}$. Define also the **initial monomial of** f to be $in_< f$ = the biggest monomial w.r.t $<$ belonging to $\mathrm{supp}(f)$. Given an ideal $0 \neq I \subset S$ we define the **initial ideal of** I: $in_<(I) = (\{in_< f : 0 \neq f \in I\})$.

Lemma 1.5. *There exists polynomials $g_1, ..., g_s \in I$ s.t $in_<(I) = (in_< g_1, ..., in_< g_s)$.*

Proof. From 1.1 we have $\{in_< f : 0 \neq f \in I\} = \{in_< g_1, ..., in_< g_s\}$ for some polynomials $g_1, .., g_s$. It follows that $in_<(I) = (in_< g_1, ..., in_< g_s)$. ■

Let $0 \neq I \subset S$ be an ideal. A **Gröbner basis** of I w.r.t. the monomial order $<$ is a finite set $\mathcal{G} = \{g_1, ..., g_s\}$ of polynomials where each $0 \neq g_i \in I$, such that $in_<(I) = (in_< g_1, ..., in_< g_s)$.
A Gröbner basis always exists but cannot be unique.

d) Hilbert's basis theorem

Fix a monomial order $<$ on S.

Theorem 1.6. *If $\mathcal{G} = \{g_1, ..., g_s\}$ is a Gröbner basis of an ideal $0 \neq I \subset S$, then I is generated by $g_1, ..., g_s$. In other words, every Gröbner basis of I is a system of generators of I.*

Proof by Gordan. Let $0 \neq f \in I$. Since $in_< f \in in_< I$ one has $in_< g_{i_0} \mid in_< f$ for some $1 \leq i_0 \leq s$. Let $in_< f = w_0 in_< g_{i_0}$ for $w_0 \in \mathrm{Mon}(S)$. Set $h_0 = f - c_{i_o}^{-1} c_0 w_0 g_{i_0} \in I$ where $LT(f) = c_0 in_< f$ and $c_{i_0} in_< g_{i_0} = LT(g_{i_0})$. If $h_0 = 0$, then $f \in (g_1, ..., g_s)$. If $h_0 \neq 0$, then $in_< h_0 < in_< f$. Continue this procedure and use Lemma 1.4 to finish the proof. ■

Corrolary 1.7 (Hilbert's basis theorem)**.** *Every ideal of the polynomial ring is finitely generated.*

e) Macaulay' theorem

Notation 1.8. $S = K[x_1, ..., x_n], 0 \neq I \subset S$ ideal, $<$ monomial order

Definition 1.9. A monomial $u \in Mon(S)$ is called *standard* with respect to $in_<(I)$ if $u \notin in_<(I)$

Theorem 1.10 (1.8 Macaulay)**.** *The set of standard monomials with respect to $in_<(I)$ is a K-basis of S/I.*

Proof. Let $B = \{\bar{u} = u + I \in S/I : u \in Mon(S)$ is standard with respect to $in_<(I)\}$ We show that B is a K - basis of S/I.

- B is linearly independent :

  let $c_1\bar{u}_1 + ...c_n\bar{u}_n = 0$ in S/I where $c_i \in K$ and $u_1 < u_2 < ...u_n$ are standard. Then $0 \neq f = c_1u_1 + ...c_nu_n \in I$ and $in_<(f) = u_n \in in_<(I)$. This is impossible since u_n is standard

- S/I is spanned by B:

  Let $\langle B \rangle$ denote the subspace of S/I spanned by B. Let $0 \neq f \in S$. We show $\bar{f} \in \langle B \rangle$ by using induction (lemma 1.4) on $in_<(f)$:

  Suppose $\bar{u} = \overline{in_<(f)} \in B$. By assumption of induction we know $\overline{f - cu} \in \langle B \rangle$ (coefficient of min f). Since $u \in B$, one has $f \in \langle B \rangle$

  Suppose $\bar{u} = \overline{in_<(f)} \notin B$. Then u is not standard, i.e. $u \in in_<(I)$. Hence $\exists_{0 \neq g \in I}\ u = in_<(g)$. Then (by induction) $\overline{c'f - cg} \in \langle B \rangle$. However in S/I $\overline{c'f} = \overline{c'f - cg} \in \langle B \rangle$. Thus $c'f \in \langle B \rangle$ and $f \in \langle B \rangle$.

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Corrolary 1.11 (1.9)**.** *$0 \neq I \subset S$ ideal, $<$ monoid order, $h_1, ..., h_s \in I$ with each $h_i \neq 0$. Let $\mathcal{H} = \{u \in Mon(S) : \forall_{1 \leq i \leq s} in_<(h_i) \nmid u\}$*

Suppose $\mathcal{H}$ is linearly independent over K in S/I. Then $\{h_1, ..., h_s\}$ is a GB of I w.r.t. $<$. In particular $\{h_1, ...h_s\}$ is a system of generators of I

Example 1.12 (1.10)**.** Consider the semigroup ring $A = K[t, xt, yt, xyt, yzt, xyzt] \subset k[x, y, z, t]$. Define the surjective ring homomorphism $u : S = k[x_1, x_2, ..., x_6] \to A$ by setting $u : x_1 \mapsto t, x_2 \mapsto xt, ..., x_6 \mapsto xyzt$, $I = \ker(u)$. We know $T_1 = x_2x_3 - x_1x_4, T_2 = x_2x_5 - x_1x_6, T_3 = x_4x_5 - x_3x_6 \in I = (T_1, T_2, T_3)$ (this equality is not obvious).

By using (1.9) we can show that $\{T_1, T_2, T_3\}$ a GB w.r.t. rev. lex. order induced by $x_1 > x_2 > ...$ (Problem 1)

Problem 1.13. *In (1.10) show that $\{T_1, T_2, T_3\}$ is a GB, w.r.t $<_{rev}$*

Solution:

$\mathcal{H} = \{x_1^{a_1}x_2^{a_2}x_4^{a_4}x_6^{a_6}, x_1^{b_1}x_3^{b3}x_4^{b_4}x_6^{b_6}, x_1^{c_1}x_3^{c_3}x_5^{c_5}x_6^{c_6}\}$ is linearly independent if:

$u, v \in \mathcal{H}, u \neq v \Rightarrow \pi(u) \neq \pi(v)$

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Problem 1.14. *(chsugi) $S = K[x_1, ..., x_{10}], I = (T_1, ..., T_5)$, where $T_1 = 18 - 26, T_2 = 29 - 37, T_3 = 310 - 48, T_4 = 46 - 59, T_5 = 57 - 110$. Show that $\nexists$ monomial order $<$ on S for which $\{T_1, ..., T_5\}$ is a GB of I w.r.t. $<$*

Solution:

Suppose, on the contrary, that there exists a monomial order $<$ on S such that $G = f_1, ..., f_5$ is a Grobner basis of I with respect to $<$. First, note that each of the five polynomials: $x_1x_8x_9 - x_3x_6x_7, x_2x_9x_{10} - x_4x_7x_8, x_2x_6x_{10} - x_5x_7x_8, x_3x_6x_{10} - x_5x_8x_9, x_1x_9x_{10} - x_4x_6x_7$ belongs to I.

Let, say, $x1x8x9 > x3x6x7$. Since $x1x8x9 \in in_<(I)$, there is $g \in G$ such that $in_<(g)$ divides $x1x8x9$. Such $g \in G$ must be $f1$. Hence $x1x8 > x2x6$. Thus $x2x6 \notin in_<(I)$. Hence there exists no $g \in G$ such that $in_<(g)$ divides $x2x6x10$. Hence $x2x6x10 < x5x7x8$. Thus $x5x7 > x1x10$. Continuing these arguments yields $x1x8x9 > x3x6x7, x2x9x10 > x4x7x8, x2x6x10 < x5x7x8, x3x6x10 > x5x8x9, x1x9x10 < x4x6x7$ and $x1x8 > x2x6, x2x9 > x3x7, x3x10 > x4x8, x4x6 > x5x9, x5x7 > x1x10$. Hence $(x1x8)(x2x9)(x3x10)(x4x6)(x5x7) > (x2x6)(x3x7)(x4x8)(x5x9)(x1x10)$. However, both sides of the above inequality coincide with $x1x2 \cdot \cdot \cdot x10$. This is a contradiction.

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2 Toric rings and toric ideals

a) Configuration matrix

$A = (a_{ij})_{1\leq i\leq d, 1\leq j\leq n} \in \mathbb{Z}^{d\times n}$, column vector $a_j = [a_{1,j}, ..., a_{d,j}]^T, 1 \leq j \leq n$

Definition 2.1. We call A a configuration matrix if $\exists_{0\neq c\in\mathbb{R}^d}$ for which $\forall_{1\leq j\leq n} a_j \cdot c = 1$ usual inner product in $\mathbb{R}^d \iff cA = 0$

Example 2.2. Given $A \in \mathbb{Z}^{(d-1)\times n}$, define $A^{\#} \in \mathbb{Z}^{d\times n} = \begin{bmatrix} & A & \\ & & \\ 1 & 1 & 1 \end{bmatrix}$ matrix with A on top, ones below. Then $A^{\#}$ is a configuration matrix with $c = [0, ..., 0, 1]$

Example 2.3. If $a_{1j} + ...a_{dj} = h \neq 0 \forall_{1\leq j\leq n}$ then A is a configuration matrix with $c = [1/h, ...1/h]$

b) toric ideal

Definition 2.4. A binomial is a polynomial of the form $u - v$ where u and v are monomials with $\deg u = \deg v$

A binomial ideal of $S = k[z_1, ..., z_n]$ is an ideal of S generated by binomials

Given a configuration matrix $A \in \mathbb{Z}^{d\times n}$ define $Ker_{\mathbb{Z}} A = \{b \in \mathbb{Z}^n : Ab = 0\}$

Lemma 2.5 (2.2)**.** *If* $b = [b_1, ..., b_n] \in Ker_{\mathbb{Z}} A$*, then* $b_1 + ..., b_n = 0$.

Proof. Since A is a configuration matrix, one has $0 \neq c \in \mathbb{R}^d$ with $\forall_{1\leq j\leq n} a_j c = 0$. Since $Ab = 0$, one has $\sum_{j=1}^n b_j a_j = 0$. Hence $0 = (\sum_{j=0}^n b_j a_j)c = \sum_{j=0}^n b_j(a_j c) = \sum b_j$. ■

Definition 2.6. Now, for each $b = [b_1, ..., b_n] \in ker_{\mathbb{Z}} A$, define the binomial $f_b = \prod_{b_n>0} x_i^{b_i} - \prod_{b_i<0} x_i^{b_i} = f_b^+ - f_b^- \in S$. By (2.2) one has $\deg f_b^+ = \deg f_b^-$.

Let us define $I_A := (\{f_b : b \in ker_{\mathbb{Z}} A\})$

Example 2.7. $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{Z}^{4\times 5}$ configuration matrix. One has $A * [-1, 1, 1, 1, -2]^T =:$ $Ab = 0, b \in ker_Z A$, $f_b = x_2x_3x_4 - x_1x_5^2$. One can show that $I_A = (f_b)$

c) Toric ring

Definition 2.8. $A = (a_{ij}) \in \mathbb{Z}^{d\times n}$ is a configuration matrix if $\exists 0 \neq \mathbf{c} \in \mathbb{R}^d$ such that for all horizontal vector $\mathbf{a}_j, \mathbf{a}_j\mathbf{c} = 1 \Leftrightarrow \mathbf{c}A = [1, ..., 1]$

$\mathbf{t}^{a_j} := t_1^{a_{1j}} t_2^{a_{2j}} ...t_d^{a_{dj}} \in K[t_1, t_1^{-1}, ..., t_d, t_d^{-1}]$

Definition 2.9. The toric ring of A is the subring $K[A] \subset k[t_1, t_1^{-1}...t_d, t_d^{-1}]$ generated by $t^{a_1}, t^{a_2}, ..., t^{a_n}$

$K[A] = K[\mathbf{t}^{a_1}, t^{a_2}, ..., t^{a_n}](\subset k[t_1, t_1^{-1}, ..., t_d, t_d^{-1}])$

Example 2.10. $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$K[A] = K[t_1, t_3, t_2t_3, t_1t_2t_3]$

Now define the surjective ring homomorphism $\pi : S = k[x_1, ..., x_n] \to k[A] = k[t^{a_1}, ..., t^{a_n}]$ $x_i \mapsto t^{a_i}$ $\subset k[t_1, t_1^{-1}, ..., t_d, t_d^{-1}]$

Theorem 2.11. $I_A = \ker(\pi)$.

Corrolary 2.12. I_A *is a prime ideal.*

Proof of Theorem 2.11. (First Step)

We will show that, for $u, v \in \mathrm{Mon}(S)$, if $\pi(u) = \pi(v)$, then $\deg u = \deg v$. Let $u = \prod_{j=1}^n x_j^{c_j}, v = \prod_{j=1}^n x_j^{d_j}$. Then $\pi(u) = \prod_{j=1}^n (\mathbf{t}^{a_j})^{c_j}, \pi(v) = \prod_{j=1}^n (\mathbf{t}^{a_j})^{d_j}$ In other words, $\pi(u) = \mathbf{t}^{\sum_{j=1}^n c_j a_j}, \pi(v) = \mathbf{t}^{\sum_{j=1}^n d_j a_j}$. If $\pi(u) = \pi(v)$, then $\sum_{j=1}^n c_j a_j = \sum_{j=1}^n d_j a_j$. Thus $(\sum_{j=1}^n c_j a_j) \cdot c = (\sum_{j=1}^n d_j a_j) \cdot c = \sum_{j=1}^n d_j = \sum_{j=1}^n c_j$. Hence $\deg u = \deg v$.

(Second step) We will show that $\ker(\pi)$ is a binomial ideal. Write $f \in S = k[x_1, ..., x_n]$ as $f = f_1 + ... + f_t$ where each $f_i \in S$ and for monomials $u \in \mathrm{supp}(f_i)$ and $v \in \mathrm{supp}(f_j)$, one has $\pi(u) = \pi(v)$ if and only if $i = j$.

Let $f_i = \sum_{k=1}^{s_i} c_{ik} u_{ik}$ where $0 \neq c_{ik} \in k$, $u_{ik} \in \mathrm{Mon}(S)$. Since $\pi(u_{i1}) = \pi(u_{ik})$ for $k = 2, ..., s_i$ it follows that $\pi(f_i) = \sum_{k=1}^{s_i} c_{ik}\pi(u_{ik}) = (\sum_{k=1}^{s_i} c_{ik})\pi(u_{i1})$. Hence $\pi(f) = \pi(f_1) + ... + \pi(f_t) = \sum_{i=1}^{t}(\sum_{k=1}^{s_i} c_{ik})\pi(u_{i1})$.

If $i \neq j$, then $\pi(u_{i1}) \neq \pi(u_{j1})$. Thus, if $f \in \ker(\pi)$ then $\sum_{k=1}^{s_i} c_{ik} = 0$ for all $1 \leq i \leq t$. Hence $c_{i1} = -\sum_{k=2}^{s_i} c_{ik}$. We have $f_i = \sum_{k=1}^{s_i} c_{ik} u_{ik} = \sum_{k=2}^{s_i} c_{ik}(u_{ik} - u_{i1})$. Thus $f = \sum_{i=1}^{t}(\sum_{k=2}^{s_i} c_{ik}(u_{ik} - u_{i1}))$. We have $u_{ik} - u_{i1} \in \ker(\pi)$. Therefore, first step shows that $u_{ik} - u_{i1}$ is a binomial. Hence $\ker(\pi)$ is generated by those binomials $u - v$ with $\pi(u) = \pi(v)$.

(Third step) We will show that $I_A = \ker(\pi)$. Let $f = \prod_{j=1}^n x_j^{c_j} - \prod_{j=1}^n x_j^{d_j}$ be a binomial. Then $\pi(f) = \prod_{j=1}^n (\mathbf{t}^{a_j})^{c_j} - \prod_{j=1}^n (\mathbf{t}^{a_j})^{d_j} = \mathbf{t}^{\sum a_j c_j} - \mathbf{t}^{\sum a_j d_j}$. Hence $\pi(f) = 0$ if and only if $\sum_{j=1}^n c_j \mathbf{a}_j = \sum_{j=1}^n d_j \mathbf{a}_j$ if and only if $f_b = f$, $b = \mathbf{c} - \mathbf{d} \in \mathrm{Ker}_{\mathbb{Z}} A$. Thus binomials belonging to $\ker(\pi)$ must belong to I_A. The converse is clear. Hence $I_A = \ker(\pi)$. ∎

d) Toric ideals arising from finite graphs

Let G be a finite connected simple graph on the vertex set $[d] = \{1, 2, ..., d\}$ with the set of edges $E(G) = \{e_1, ..., e_n\}$. For each edge e_i connecting vertices p_i and $q_i \in [d]$ define $\mathbf{t}^{e_i} = t_{p_i} t_{q_i} \in k[t_1, ..., t_d]$. The **toric ring** (or **edge ring**)of G is $k[G] = k[\mathbf{t}^{e_1}, ..., \mathbf{t}^{e_n}]$. Define $\pi : S = k[x_1, ..., x_n] \to k[G]$, by $x_i \mapsto \mathbf{t}^{e_i}$. We call $\mathbf{ker}(\pi)$ the **toric ideal** of G and we denote it by I_G.

Graph terminology: (even, odd) cycle, chord, (closed) walk.

Problem 2.13. *Find a configuration matrix A with $I_A = I_G$*

Solution:

The matrix $M \in \mathbb{Z}^{n \times n}$ where n is a number of vertices. In columns there are 0 and 1, every column corresponds to a one edge, ones are in vertices of the edge. Proof follows from the definition of surjections in I_G and I_A.

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Problem 2.14. *If Γ is an even closed walk, then show that $f_\Gamma \in I_G$*

Solution:

going through the graph and taking even edges we will take edges with all vertices possible. the same with even edges. Closed walk was even so there is the same number of even and odd edges. When we multiplicate we obtain the same monomials. Hence $f_\gamma \in I_G$.

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Problem 2.15. *Show that I_G is generated by those binomials f_Γ, where Γ is an even closed walk*

Solution:

Let I'_G denote the binomial ideal generated by these binomials f_Γ, where Γ is an even closed walk of G. Choose a binomial $f = \prod_{k=1}^q x_{i_k} - \prod_{k=1}^q x_{j_k} \in I_G$. We prove $f \in I'_G$ by induction on $q = degf$. One can assume that $i_k \neq j_{k'}$ for all k and k'. Let say, $\pi(x_{i_l}) = t_1 t_2$. Since $\pi(\prod_{k=1}^q x_{i_k}) = \pi(\prod_{k=1}^q x_{j_k})$ one has $\pi(x_{j_m}) = t_2 t_r$ for some m with $r \neq 1$. Say $m = 1, r = 3$. Thus $\pi(x_{j_1}) = t_2 t_3$. Then $\pi(x_{i_l}) = t_3 t_s$ for some l with $s \neq 2$. Say $l = 2, s = 4$. Repeated application of these procedure yields an even closed walk $\Gamma' = (e_{i_1}, e_{j_1}, ..., e_{i_l}, e_{j_l})$ with $f_{\Gamma'} = \prod_{k=1}^p x_{i_k} - \prod_{k=1}^p x_{j_k} \in I'_g$. This one has $\pi(\prod_{k=p+1}^q x_{i_k}) = \pi(\prod_{k=p+1}^q x_{j_k})$. Hence $g = \prod_{k=p+1}^q x_{i_k} - \prod_{k=p+1}^q x_{j_k} \in I_G$. By induction one has $g \in I'_G$. Now one has: $f = (\prod_{k=f+1}^q x_{i_k}) f_{\Gamma'} + (\prod_{k=1}^p x_{j_k}) g$. $f_{\Gamma'}, g \in I'_G$. Thus $f \in I'_G$ as desired

■

Problem 2.16. *We say that an even closed walk Γ is primitive if there is no even closed walk Γ' with $\Gamma' \neq \Gamma$ such that $f^+_{\Gamma'} | f^+_{\Gamma}$ and $f^-_{\Gamma'} | f^-_{\Gamma}$. Show that I_G is generated by those binomials f_Γ, where Γ is a primitive even closed walk.*

Problem 2.17. *Find a "minimal" system of binomial generators of I_G for the following graphs $G_1 = <><>$, $G_2 =$ "hexagon with the diameter".*

Solution:
for <><>: we take binomials representing <> and <> - it's enough
for hexagon:
we don't need to take the whole hexagon, we only need both halfs as a cycles. $x_1x_3 - x_2x_7, x_5x_7 - x_4x_6$

■

Problem 2.18. *Let G be a finite connected simple bipartite graph. Show that I_G is generated by those binomials f_C, where C is an even cycle without chord (cięciwa)*

3 Regular triangulation of lattice polytopes

a) Triangulation of lattice polytopes (integral polytopes

Definition 3.1. A convex polytope is a convex hull of a finite set

A convex polytope $P \subset \mathbb{R}^d$ of dimension d is called a lattice (or integral) polytope if each vertex $\in \mathbb{Z}^d$. Let $P \in \mathbb{R}^d$ be a lattice polytope of dim $= d$ and $P \cap \mathbb{Z}^d = \{\mathbf{a}_i, \mathbf{a}_2, ..., \mathbf{a}_n\}$

Write $A(P) \subset \mathbb{Z}^{(d+1)\times n}$ for the configuration matrix $A(P) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & ... & \mathbf{a}_n \\ 1 & 1 & ... & 1 \end{bmatrix} \in \mathbb{Z}^{(d+1)\times n}$ since $dimP = d$ one have rank $A(P) = d + 1$

Example 3.2. for two tetrahedrons with common base $\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

Definition 3.3. A simplex belonging to P of a dimension $s - 1$ is a subset $F = \{a_{i_1}, ..., a_{i_s}\} \subset P \cap \mathbb{Z}^d$ for which $\begin{bmatrix} a_{i_1} \\ 1 \end{bmatrix} \begin{bmatrix} a_{i_2} \\ 1 \end{bmatrix} ... \begin{bmatrix} a_{i_s} \\ 1 \end{bmatrix}$ are linearly independent over $\mathbb{Q}$

In particular $\emptyset$ is a simplex belonging to P of dimension -1

A maximal simplex $= \{a_{i_1}, ..., a_{i_{d+1}}\}$ belonging to P is a simplex of dimension d. A maximal simplex is called fundamental if

$\mathbb{Z}A(P) = \mathbb{Z}A(F)$, where $\mathbb{Z}A(F) := \mathbb{Z}\begin{bmatrix} a_{i_1} \\ 1 \end{bmatrix} + \mathbb{Z}\begin{bmatrix} a_{i_2} \\ 1 \end{bmatrix} + ... + \mathbb{Z}\begin{bmatrix} a_{i_{d+1}} \\ 1 \end{bmatrix}$ and $\mathbb{Z}A(P) := \mathbb{Z}\begin{bmatrix} a_1 \\ 1 \end{bmatrix} + \mathbb{Z}\begin{bmatrix} a_2 \\ 1 \end{bmatrix} + ... + \mathbb{Z}\begin{bmatrix} a_n \\ 1 \end{bmatrix} \subset \mathbb{Z}^{d+1}$

Definition 3.4. A collection Δ of simplices belonging to P is called a triangulation of P if the following conditions are satisfied::

1. If $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$
2. If $F, G \in \Delta$, then $conv(F) \cap conv(G) = conv(F \cap G)$
3. $P = \bigcup_{F \in \Delta} conv(F)$ (convex hull of F in R^d)

Definition 3.5. A triangulation $\Delta$of P is called unimodular if every maximal simplex $F \in \Delta$ is fundamental.

b) Regular triangulations

$P \subset \mathbb{R}^d$ lattice polytope of dimension d, $P \cap \mathbb{Z}^d = \{a_1, ..., , a_n\}$, $A(P) = \begin{bmatrix} a_1 & a_2 & ... & a_n \\ 1 & 1 & ... & 1 \end{bmatrix} \in \mathbb{Z}^{(d+1)\times n}$

$K[A(P)] = K[t^{a_1}s, ..., t^{a_n}s] \subset K[t, t^{-1}, ..., t_d, t_d^{-1}, s]$ toric ring, $\pi : S = K[x_1, ..., x_n] \to K[A(P)]$ $\pi(x_i) = t^{a_i}s$, $I_{A(P)} = ker(\pi)$ toric ideal

Fix monomial order on S and let $in_<(I_{A(P)})$ denote initial ideal

Recall that the radical of $in_<(I_{A(P)})$ is the ideal of S generated by those polynomials $f \in S$ with $f^N \in in_<(I_{A(P)})$ for some $N = N_f > 0$

Example 3.6. if $in_<(I_{A(P)}) = (x_1^3x_2x_3^5x_4, x_2^3x_5x_6^2)$, then $\sqrt{in_<(I_{A(P)})} = (x_1x_2x_3x_4, x_2x_5x_6)$ generated by square free monomials

Lemma 3.7 (3.1). *A subset $F \subset P \cap \mathbb{Z}^d$ is a simplex belonging to P if $\prod_{a_j \in F} x_j \notin \sqrt{in_<(I_{A(P)})}$*

Sketch of proof. Let $F = \{a_{i_1}, ..., a_{i_s}\}$. Suppose that F satisfies the inclusion. We show that $\begin{bmatrix} a_{i_1} \\ 1 \end{bmatrix} \begin{bmatrix} a_{i_2} \\ 1 \end{bmatrix} ... \begin{bmatrix} a_{i_{d+1}} \\ 1 \end{bmatrix}$ are linear independent. If not, then $\exists (0, ..., 0) \neq (a_{i_1}, ..., a_{i_s}) \in \mathbb{Z}^s$ such that: $a_{i_1} \begin{bmatrix} a_{i_1} \\ 1 \end{bmatrix} + a_{i_2} \begin{bmatrix} a_{i_2} \\ 1 \end{bmatrix} ... + a_{i_s} \begin{bmatrix} a_{i_s} \\ 1 \end{bmatrix} = 0$. Then one can easily show that $0 \neq \prod_{q_k>0} x_{j_k}^{q_n} - \prod_{q_n<0} x_{j_k}^{-q_n} =: u - v \in I_{A(P)}$ Thus u or $v \in in_<(I_{A(P)})$. Hence $\prod_{q_k>0} x_{j_k} \in \sqrt{in_<(I_{A(P)})}$ or $\prod_{q_k<0} x_{j_k} \in \sqrt{in_<(I_{A(P)})}$. This contradicts $\prod_{a_j \in F} x_j \notin \sqrt{in_<(I_{A(P)})}$. ∎

Definition 3.8. Let $\Delta(in_<(I_{A(P)})) := \{F \subset P \cap \mathbb{Z}^d : \prod_{a_j \in F} x_j \notin \sqrt{in_<(I_{A(P)})}\}$

Theorem 3.9 (3.2 Strumfels). *$\Delta(in_<(I_{A(P)}))$ is a triangulation of P.*

We omit the proof

Example 3.10. in the example of tetrahedrons with common base: $I_{A(P)} = (x_2x_3x_4 - x_1x_5^2)$, $in_<(I_{A(P)}) = (x_1x_5^2)$, $\sqrt{(in_<(I_{A(P)}))} = (x_1x_5)$

Definition 3.11. A triangulation Δ of P is called regular if $\Delta = \Delta(in_<(I_{A(P)}))$ for some monomial order $<$

Theorem 3.12. *$\Delta(in_<(I_{A(P)}))$ is unimodular $\Leftrightarrow in_<(I_{A(P)}) = \sqrt{in_<(I_{A(P)})}$ ($\Leftrightarrow in_<(I_{A(P)})$ is generated by square free monoids)*

Commutative Algebra

$\exists$ unimodular triangulation $\Rightarrow$ toric ring $K[A(P)]$ is normal and Cohen - Macaulay.

4 The join-meet ideals of finite lattice

a) Review on classical lattice theory

Definition 4.1. A lattice is a poset L in which any two elements a and b of L has a meet $a \wedge b$ and a join $a \vee b$. In particular, a finite lattice has both the minimal element $\hat{0}$ and the maximal element $\hat{1}$

A finite lattice L is called **distributive** if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

A finite L is called **modular** if $a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

Every distributive lattice L is modular. In fact if L is distributive lattice and $a \leq c$, then $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c$

Problem 4.2. *Let G be a finite group and $L(G)$ the set of normal subgroups of G. We can regard $L(G)$ as a poset ordered by inclusion. Show that:*

1. *$L(G)$ is a lattice*
2. *$L(G)$ is a modular lattice*

3. *For G a finite abelian group: $L(G)$ is a distributive lattice $\Leftrightarrow$ G is a cyclic group*

Solution:

[for 1.c)] If $G \simeq \mathbb{Z}/n\mathbb{Z}$. Subgroups are $\mathbb{Z}/k\mathbb{Z}$ for $k|n$ $\mathbb{Z}/k_1\mathbb{Z} \cap \mathbb{Z}/k_2\mathbb{Z} = \mathbb{Z}/lcm(k_1,k_2)\mathbb{Z}, \mathbb{Z}/k_1\mathbb{Z} + \mathbb{Z}/k_2\mathbb{Z} = \mathbb{Z}/gcd(k_1,k_2)\mathbb{Z}$. Enough to check gcd, lcm satisfies distributive laws.

G not cycle $\Rightarrow F = \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times ...$ such that $n_1|n2|....$ Let's take 3 subgroups isomorphic to $\mathbb{Z}/n_1\mathbb{Z}$

$H_1 := \langle(1,0,0,...)\rangle, H_1 := \langle(0, \frac{n_2}{n_1}, 0, ...)\rangle, H_1 := \langle(1, \frac{n_2}{n_1}, 0, ...)\rangle$ so we obtain a sublattice of "diamond" type, so it's not distributive.

■

Definition 4.3. N_5 pentagon lattice

M_5 the diamond lattice (quadrangle with a diagonal and a vertex on a diagonal)

Fact 4.4. *N_5 is not modular (in fact, even though $a < c$, one has $a \vee (b \wedge c) = a \vee 0 = a, (a \vee b) \wedge c = 1 \wedge c = c$)*

M_5 modular, but not distributive (In fact $a \wedge (b \vee c) = a \wedge 1 = a, (a \wedge) \vee (a \wedge c) = 0 \vee 0 = 0$)

Theorem 4.5 ((4.1.) Dedekind)**.**

1. *a finite lattice L is modular $\Leftrightarrow$ no sublattice of L is N_5*
2. *a modular lattice L is distributive $\Leftrightarrow$ no sublattice of L is M_5*
3. *a finite lattice L is distributive $\Leftrightarrow$ neither N_5 nor M_5 is a sublattice of L.*

Definition 4.6. Let $P = \{p_1, .., p_n\}$ be a finite poset with a partial order $<$. A **poset ideal** of P is a subset $\alpha \subset P$ such that if $p_i \in \alpha, p_j \leq p_i$,then $p_j \in \alpha$.

Let $J(P)$ denote the **set of poset ideals of** P.

Fact 4.7. *If α and β are poset ideals, $\alpha \cup \beta$ and $\alpha \cap \beta$ are also poset ideals of P. Hence $J(P)$ can be a finite lattice, ordered by inclusion. It is distributive.*

Theorem 4.8 (4.2 Birkhoff)**.** *Give a finite distributive lattice L, there is a unique poset P such that $L = J(P)$.*

Definition 4.9. Join irreducible element of a lattice is an element which has only one arrow going down.

b) Join-meet ideals of finite lattices

Let L be a finite lattice and $K[L] := K[\{x_a : a \in L\}]$ the polynomial ring in $|L|$ - variables over a field K.

Given $a, b \in L$, define the binomial $f_{a,b}$ by setting $f_{a,b} = x_a x_b - x_{a \wedge b} x_{a \vee b}$

In particular $f_{a,b} = 0 \Leftrightarrow a$ and b are comparable (either $a \leq b$ or $b \leq a$)

Definition 4.10. The **join-meet ideal** of L is finite binomial ideal $I_L := (\{f_{a,b} : a$ and b are incomparable$\})$

Example 4.11. $I_{N_5} = (x_a x_b - x_0 x_1, x_b x_c - x_0 x_1)$

$I_{M_5} = (x_a x_b - x_0 x_1, x_a x_c - x_0 x_1, x_b x_c - x_0 x_1,)$

Definition 4.12. A monomial order $<$ on $K[L]$ is called **compatible** if for any a and b of L for which a and b are incomparable, one has $in_<(f_{a,b}) = x_a x_b$

Example 4.13. $L = \{x_1, x_2, ..., x_n\}, x_i < x_j$ in $L \Rightarrow i > j$. Then $<_{rev}$ induced by $x_1 > ... > x_n$. Then $<_{rev}$ is a compatible monomial order on $K[L]$. We called it rank reversed lexicographic order.

Theorem 4.14 ((4.4))**.** *Let L be a finite lattice and fix a compatible monomial order $<$ on $K[L]$. Let $G_L := \{f_{a,b} : a, b \in L$ are incomparable$\}$ Then the following are equivalent:*

1. *G_L is a Grobner basis with respect to $<$*
2. *L is distributive*

Theorem 4.15 (4.5)**.** *Give a finite lattice L. The following conditions are equivalent:*

1. *I_L is a prime ideal*
2. *L is distributive*

In both upper theorems implication from top to bottom is easy - exercise

c) Toric ring $R_K[L]$ with $L = J(P)$ distributive lattice

Let $P = \{p_1, p_2, ..., p_n\}$ be a finite poset and $L = J(P)$ the distributive lattice consisting of all poset ideals of P, ordered by the inclusion. Let $S = K[x_1, x_2, ..., x_n, t]$ denote the polynomial ring in $(n+1)$ variables over a field K. Give a poset ideal $\alpha \in L = J(P)$. We introduce the monomial u_α by setting $u_\alpha = (\prod_{p_i \in \alpha} x_i)t \in S$. In particular $u_\emptyset = t, u_P = x_1 \cdots x_n t$. Let $R_K[L]$ denote the toric ring $R_K[L] := K[\{u_\alpha : \alpha \in L = J(P)\}]$.

Example 4.16. rysunki

Define the surjective ring homomorphism $\pi : K[L] = K[\{x_\alpha : \alpha \in L = J(P)\}] \to R_k[L]$ by setting $\pi(x_\alpha) = u_\alpha$ for all $\alpha \in L = J(P)$.

Lemma 4.17 (4.6)**.** *$I_L \subset \ker(\pi)$*

Proof. $\alpha, \beta \in L = J(P), \alpha \vee \beta = \alpha \cup \beta, \alpha \wedge \beta = \alpha \cap \beta, \pi(x_{\alpha\cap\beta} x_{\alpha\cup\beta}) = (\prod_{p_i \in \alpha\cap\beta} x_i)(\prod_{p_i \in \alpha\cup\beta} x_i)t^2, \pi(x_\alpha x_\beta) = u_\alpha u_\beta = (\prod_{p_i \in \alpha} x_i)(\prod_{p_i \beta} x_i)t^2$ Hence $u_\alpha u_\beta = u_{\alpha\cap\beta} u_{\alpha\cup\beta}$ in $R_K[L]$. Thus $x_\alpha x_\beta - x_{\alpha\wedge\beta} x_{\alpha\vee\beta} \in \ker(\pi)$ ■

Theorem 4.18 (4.7)**.** *Let $L = J(L)$ and fix compatible monomial order $<$ on $K[L]$. Then $G_L := \{f_{\alpha,\beta} : \alpha, \beta \in L = J(P)$ are incomparable$\}$ is a Grobner basis of $\ker(\pi)$ with respect to $<$. In particular $I_L = \ker(\pi)$, so 2 $\Rightarrow$ 1 in Theorems 4.4 and 4.5.*

Proof. the technique of corollary of Macaulay's theorem in paragraph 1. can be applied. Let $In_<(G_L) := \{in_<(f_{\alpha,\beta}) : f_{\alpha,\beta} \in G_L\}$. In other words $in_<(G_L)$ is the set of monomials $x_\alpha x_\beta \in K[L]$ for which α and β are incomparable. Lemma (4.6) says that $in_<(G_L) \subset in_<(\ker \pi)$. Let B denote the set of those monomials $w \in K[L]$ such that $\forall_{x_\alpha x_\beta \in in_<(G_L)}\ x_\alpha x_\beta \nmid w$ and B' those monomials $w \in K[L]$ with $\omega \notin in_<(\ker \pi)$. Recall that, Macaulay's theorem $\Rightarrow$ B' is a $K-$basis of $R_K[L] = K[L]/\ker \pi$. Since $B' \subset B$, in order to show that $B' = B$, our work is to show that B is linearly independent in $R_K[L] = K[L]/\ker \pi$. Now, we prove, for $w, w' \in B$ with $w \neq w'$ one has $\pi(w) \neq \pi(w')$:

Let $w = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_p}, w' = x_{\beta_1} x_{\beta_2} \cdots x_{\beta_q}$ and $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_p, \beta_1 \leq \beta_2 \leq ... \leq \beta_q$. $\pi(w) =$ (monomials in $x_i)t^p, \pi(w') =$ (monomials in $x_i)t^q$. We may assume that $p = q$. Induction on $\deg w$ ($= \deg w'$) one can assume that $\forall_{i,j} \alpha_i \neq \beta_j$. Thus $\alpha_1 \not\subset \beta_1$. Take $p_{i_0} \in \alpha_1 \setminus \beta_1$. As subsets of P one has $\alpha_1 \subseteq \alpha_2 \subseteq ... \subseteq \alpha_p, \beta_1 \subseteq \beta_2 \subseteq ... \subseteq \beta_p$. Since $\forall_{1 \leq i \leq p} p_{i_0} \in \alpha_i$, $\pi(x_{i_0})^p$ appears in $\pi(w) = \pi(x_{\alpha_1})\pi(x_{\alpha_2}) \cdots \pi(x_{\alpha_p})$. However, since $p_{i_0} \notin \beta_1$, the power r of x_{i_0} for which $\pi(x_{i_0})^r$ appears in $\pi(w')$ is at most $p - 1 \Rightarrow \alpha_1 = \beta_1$. Contradiction. ■

Problem 4.19.

1. *Find a configuration matrix A with $I_L = I_A$ where lattice L = tree squares connected to look like a sign ">".*
2. *Find a finite poset P with $L = J(P)$ where L = two cubes with common edge.*

Solution:

1)

$x_\emptyset \mapsto t, x_1 \mapsto x_1 t, x_{1,2} \mapsto x_1 x_2 t, x_{1,3} \mapsto x_1 x_2 t, x_{2,3} \mapsto x_2 x_3 t, x_{1,2,3} \mapsto x_1 x_2 x_3 t, x_{1,2,4} \mapsto x_1 x_2 x_4 t, x_{1,2,3,4} \mapsto$

$x_1 x_2 x_3 x_4 t$ so the matrix should be : $\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

lattice of ideals are always distributive, so from theorem (4.7) $I_L = \ker \pi$

■

Problem 4.20. *By using Dedekind theorem, prove $1 \Rightarrow 2$ of theorem (4.4) and $1 \Rightarrow 2$ of theorem (4.5)*

5 Order polytopes of finite posets

a) Order polytopes

$P = \{p_1, p_2, ..., p_n\}$ finite poset, $e_1 = [1, 0, ...0]^T, e_2 = [0, 1, 0, ..., 0]^T, ..., e_n = [0, ..., 0, 1]^T \in \mathbb{R}^n$.
$\alpha \in J(P), P(\alpha) := \sum_{p_i \in \alpha} e_i \in R^n$. In particular $P(\emptyset) = [0, ..., 0]^T \in R^n, P(P) = [1, ..., 1]^T \in R^n$

Definition 5.1. The order polytope of P is the convex polytope $O(P) \subset R^n$ which is the convex hull of $\{P(\alpha) : \alpha \in J(P)\} \in R^n$

Example 5.2. rysunek

b) Linear extensions

Definition 5.3. A permutation $i_1 i_2 ... i_n$ of $[n] = \{1, ..., n\}$ is called a **linear extension of poset** P if $p_{i_k} < p_{i_l}$ in poset P, then $k < l$

Definition 5.4. $e(P) :=$ the number of linear extension of P

Example 5.5. rysunek

Lemma 5.6 (5.2)**.**

1. *Suppose that* $i_1, i_2, ..., i_n$ *is a linear extension of* P*. Then* $\alpha_j = \{p_{i_1}, p_{i_2}, ..., p_{i_j}\} \subset P$ *is a poset ideal for all* $1 \leq j \leq n$*. Moreover,* $\emptyset = \alpha_0 < \alpha_1 < \alpha_2 < ... < \alpha_n = P$ *is a maximal chain of* $L = J(P)$

2. *If* $\emptyset = \alpha_0 < \alpha_1 < \alpha_2 < ... < \alpha_n = P$ *is a maximal chain of* $L = J(P)$ *then* $i_1, i_2, ..., i_n$ *is a linear extension of* P*. where* $p_{i_j} \in \alpha_j \setminus \alpha_{j-1}$

Proof.

1. If $p_{i_k} < p_{i_l} \in \alpha_j$, then $k < l \leq j$. Hence $p_{i_k} \in \alpha_j$

2. Let $p_{i_k} < p_{i_l}$. Since $\alpha_l = \{p_{i_1}, p_{i_2}, ..., p_{i_l}\}$ and α_l is a poset ideal of P, one has $p_{i_k} \in \alpha_l$. Hence $k < l$.

■

Example 5.7. rysunek

Corrolary 5.8 (5.3)**.** $\{$*linear extensions of* $P\} \leftrightarrow_{1:1} \{$*maximal chains of* $L = J(P)\}$*. In particular,* $e(P)$ *is equal to the number of maximal chains of* $L = J(P)$*.*

Book: R.Stanley, "Enumerative combinatorics, vol1", chapter 3.
Let $i_1, i_2, ..., i_n$ be a linear extension of P and $\alpha_j \subset \{p_{i_1}, p_{i_2}, ..., p_{i_j}\} \in L = J(P)$. Since $P(\alpha_j) = e_{i_1} + e_{i_2} + ...e_{i_j} \in O(P)$ and convex hull $conv(\{P(\emptyset), P(\alpha_1), ..., P(\alpha_n)\}) \subset O(P)$ is a standard lattice simplex in R^n. Standard means volume $= \frac{1}{n!}$

Example 5.9. rysunek

Proposition 5.10 (5.4)**.** $\dim O(P) = n$

Proposition 5.11 (5.5)**.** *The set of vertices of* $O(P)$ *is* $V(O(P)) = \{P(\alpha) : \alpha \in J(P)\}$*. In particular* $O(P)$ *is a lattice polytope.*

Lemma 5.12 (5.6)**.** $O(P) \cap \mathbb{Z}^n = V(O(P))$ *extension*

c) Toric rings of order polytopes

Recall that, in general, given a lattice polytope $P \subset R^n$ of dimension n, the toric ring of P is the toric ring $K[A(P)]$ of the configuration matrix $A(P) = \begin{bmatrix} a_1 & a_2 & ... & a_n \\ 1 & 1 & ... & 1 \end{bmatrix} \in \mathbb{Z}^{(n+1)\times N}$ where $P \cap \mathbb{Z}^n = \{a_1, a_2, .., a_N\}$. In other words, $K[A(P)] = K[x^{a_1}t, ..., x^{a_N}t] \subset K[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$, $x^{a_j} = x_1^{a_{1j}} x_2^{a_{2j}} \cdots x_n^{a_{nj}}$

Now we discuss the toric ring of $O(P)$. $O(P) \cap \mathbb{Z}^n = V(O(P)) = \{P(\alpha) : \alpha \in L = J(P)\}$. One has $x^{P(\alpha)} = x^{\sum_{p_i \in \alpha} e_i} = \prod_{p_i \in \alpha} x_i$. Hence toric ring of $O(P)$, $K[\{x^{P(\alpha)}t : \alpha \in J(P)\}] = K[\{(\prod_{p_i \in \alpha} x_i t : \alpha \in J(P)\}] \subset K[x_1, ..., x_n, t]$

Recalling section 4: $L = J(P), R_K[L] = K[\{u_\alpha t : \alpha \in J(P)\}]$ where $u_\alpha = \prod_{p_i \in \alpha} x_i$.

Example 5.13. rysunek

d) Regular triangulation of O(P)

$\pi : K[\{x_{P(\alpha)} : \alpha \in J(P)] \to K[A(O(P))]$ toric ring $x_{P(\alpha)} \mapsto x^{P(\alpha)}t = (\prod_{p_i \in \alpha} x_i)t$.

$I_{A(O(P))} = \ker \pi$, toric ideal of $O(P)$. Since $K[A(O(P))] = R_K[L]$ with $L = J(P)$ it follows that $I_{A(O(P))} = (\{x_{P(\alpha)}x_{P(\beta)} - x_{P(\alpha\wedge\beta)}x_{P(\alpha\vee\beta)} : \alpha, \beta \in L = J(P)$ are incomparable in L $\} =: G_L)$

Fix a compatible monomial order $<$ on $K[\{x_{P(\alpha)} : \alpha \in J(P)\}]$ Then we notice G_L is a Grobner base of $I_{A(O(P))}$ with respect to $<$.

$In_<(I_{A(O(P))}) = (\{x_{P(\alpha)}x_{P(\beta)} : \alpha, \beta \in L = J(P)$ are compatible $\}$. Now we discuss the regular (unimodular) triangulation $\Delta = \Delta(In_<(I_{A(O(P))}))$.

$F \subset V(O(P)) = O(P) \cap \mathbb{Z}^n$ belongs to Δ (see section 3) $\Leftrightarrow_{(definition)} \prod_{P(\delta)\in F} x_{P(\delta)} \notin \sqrt{in_<(I_A(O(P)))} = in_<(I_{A(O(P))})$ $\Leftrightarrow$ $x_{P(\alpha)}x_{P(\beta)} \nmid x_{P(\delta)\in F}$ for all $\alpha, \beta \in L = J(P)$ which are incomparable in L $\Leftrightarrow$ if $F = \{P(\alpha_{i_1}), P(\alpha_{i_2}), ..., P(\alpha_{i_s})\}$ then $\alpha_{i_1} < \alpha_{i_2} < ..., \alpha_{i_s}$ in $L = J(P)$.

Thus in particular

Proposition 5.14 (5.8)**.** *$F = \{P(\alpha_0), P(\alpha_1), .., P(\alpha_n)\}$ is a maximal simplex of Δ $\Leftrightarrow$ $\emptyset = \alpha_0 < \alpha_1, ..., \alpha_n = P$ is a maximal chain of $L = J(P)$. Furthermore $conv(F)$ is a standard lattice simplex in R^n.*

Theorem 5.15 (5.9)**.** *The volume of $O(P)$ is $\frac{e(P)}{n!}$, where $e(P)$ is the number of linear extensions of P.*

Proof. Since Δ is a triangulation of $O(P)$, one has volume of $O(P) = \sum_{F\in\Delta \text{maximal}}$ (volume of $conv(F)$) = (the number of maximal simplex of $\Delta)/n!$ = (the number of maximal chains of L)/n! = $\frac{e(P)}{n!}$ (because of corollary 5.3). ■

Problem 5.16. *Let $V \subset \mathbb{R}^n$ be a set of $(0,1)$ vectors of $\mathbb{R}^n$ and $P \subset \mathbb{R}^n$ the convex polytope which is the convex hull of V in $\mathbb{R}^n$. Show that:*

1. *The set of vertices of P coincides with V ($\Rightarrow$ (5.5))*
2. *$P \cap \mathbb{Z}^n = V$ ($\Rightarrow$ (5.6))*

Proof. Let w be a $(0,1)$ vector with $w \notin V$. Then $w \notin conv(V), V = \{v_1, ..., v_n\}$. If $w \in conv(V), w = \lambda_1 v_1 + ... + \lambda_s v_s, \lambda_1 + \lambda_2 + ...\lambda_s = 1, \lambda_i \geq 0....$ ■

Definition 5.17. $v \in P$ vertex $\Leftrightarrow$ (If $v = \frac{u+w}{2}, u, w \in P \Rightarrow v = u = w$)

Lemma 5.18. *If $P = conv(V)$, then each vertex belongs to V. Moreover, if $V(P)$ is the set of vertices of P, then $P = conv(V(P))$.*