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# Charakteryzacja zwartych podzbiorów krzywych o $\omega$-ciągłych pochodnych

**Praca magisterska**
**na kierunku MATEMATYKA**

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## Oświadczenie kierującego pracą

Potwierdzam, że niniejsza praca została przygotowana pod moim kierunkiem i kwalifikuje się do przedstawienia jej w postępowaniu o nadanie tytułu zawodowego.

Data Podpis kierującego pracą

## Oświadczenie autora (autorów) pracy

Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowiązującymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur związanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Oświadczam ponadto, że niniejsza wersja pracy jest identyczna z załączoną wersją elektroniczną.

Data Podpis autora (autorów) pracy

## Streszczenie

Poniższa praca poświęcona jest charakteryzacji zwartych podzbiorów krzywych posiadających regularną pochodną w przestrzeni $\mathbb{R}^n$. Dokładniej, podajemy warunki konieczne i wystarczające by zwarty zbiór w $\mathbb{R}^n$ był podzbiorem skończonej sumy rozłącznych krzywych o $\omega$-ciągłych pochodnych i bez samoprzecięć. Mówiąc nieformalnie, pokazujemy, że istnieje skończony zbiór regularnych krzywych pokrywający zwarty zbiór $K$ jeśli każda trójka punktów $K$ zachowuje się jak trójka punktów na regularnej krzywej.

Praca jest zainspirowana wynikami Jonesa, Okikiolu, Schula i innych, które charakteryzują zwarte podzbiory krzywych prostowalnych lub regularnych w sensie Ahlforsa. Ich wyniki dotyczą jednak dużo szerszej klasy krzywych i, w związku z tym, ich warunki i metody dowodowe są istotnie inne od zaprezentowanych w tej pracy.



## Słowa kluczowe

krzywa o $\omega$-ciągłych pochodnych, twierdzenie Jonesa, twierdzenie Whitneya


## Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

## Klasyfikacja tematyczna

53A04 Curves in Euclidean space

## Tytuł pracy w języku angielskim

Characterization of compact subsets of curves with $\omega$-continuous derivatives

# Spis treści

# Chapter 1

# Introduction

**Notation.** By $\mathbb{R}^n$ we denote the standard $n$-dimensional Euclidean space equipped with $l_2$ norm $|\cdot|$ and scalar product $\langle \cdot, \cdot \rangle$. Given a concave non-decreasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ we say that a function $f : \mathbb{R}^m \to \mathbb{R}^k$ is $\omega$-continuous if for any different $x, y \in \mathbb{R}^m$ we have $|f(x) - f(y)| < \omega(|x - y|)$. We say that $f : \mathbb{R}^m \to \mathbb{R}^k$ is $r$-*locally* $\omega$-continuous if $r > 0$ and the aforementioned inequality holds if $|x - y| < r$.

**Our results.** In this paper we focus on characterizing compact subsets of regular curves in $\mathbb{R}^n$. We give a characterization of compact subsets of finite sum of disjoint embedded curves with $\omega$-continuous derivative and without self-intersections. Namely, we prove the following theorems.

**Theorem 1.1.** *Let $K$ be a compact subset of $\mathbb{R}^n$ satisfying the following condition: there exists $r_0 > 0$ and a concave non-decreasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ such that for all distinct $x, y, z \in K$ if $|z - x| = \operatorname{diam}\{x, y, z\} < r_0$ then*

$$\Big| \frac{y - x}{|y - x|} - \frac{z - y}{|z - y|} \Big| < \omega(\operatorname{diam}\{x, y, z\}). \tag{1.1}$$

*Then there exists a finite family of finite-length curves without self-intersections with arc-length parameterizations $\{\gamma_i\}_{1 \leqslant i \leqslant N}$, $\gamma_i : A_i \to \mathbb{R}^n$, where $A_i$ is either a circle or a closed segment, such that their images are disjoint and for every $\gamma_i$ the derivative $\gamma_i'$ is locally $342\omega$-continuous. Moreover, one can require that the total length of all curves $\gamma_i$ is bounded by $5\mathcal{H}_1(K) + \varepsilon$, where $\mathcal{H}_1$ is the one-dimensional Hausdorff measure and $\varepsilon > 0$ is chosen arbitrarily.*

**Theorem 1.2.** *Let $\gamma : A \to \mathbb{R}^n$ be an arc-length parametrization of a finite-length curve without self-intersections, where $A$ is a closed segment or a circle, such that $\gamma'$ is locally $\omega$-continuous for some concave non-decreasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$. Then there exists $r_0 > 0$ such that for all distinct $x, y, z \in \gamma(A)$ if $|z - x| = \operatorname{diam}\{x, y, z\} < r_0$ then*

$$\Big| \frac{y - x}{|y - x|} - \frac{z - y}{|z - y|} \Big| < 6\omega(\operatorname{diam}\{x, y, z\}).$$

**Related work.** This work was inspired by theorem by Jones [6] that gives characterization of compact subsets of rectifiable curves in $\mathbb{R}^2$. Jones's results were extended to $\mathbb{R}^n$ by Okikiolu [7] and to Hilbert spaces by Schul [8]. There exists an analogue of those theorems for rectifiable curves in general metric spaces [4, 5] and Heisenberg groups [3]. Ahlfors-regular subsets are treated by Schul [8] and David and Semmes [1]. However, all aforementioned results focus on rectifiable or Ahlfors-regular sets and curves, which are much wider classes that curves with a regular arc-length parametrization.

Moreover, our characterization conditions are quite different than these given by Jones, Okikiolu and Schul [6, 7, 8]. In Section 4 we discuss Jones-like conditions applied to curves with arc-length parametrization with regular derivatives and we give two counter-examples showing that Jones-like characterization is not possible.

One may notice that results in our paper seem similar to Whitney-type [9] theorems by Fefferman [2]. One can see our results also as *there exists a* regular *curve through the whole set iff there exists a* regular *curve through every three points.*

**Organization.** In Section 2 we prove Theorem 1.2. This is quite easy and straightforward corollary from the definition of the locally $\omega$-continuous function. In Section 3 we prove Theorem 1.1, by providing a construction algorithm for a family of curves. In Section 4 we discuss Jones-like conditions for curves with arc-length parametrization with regular derivatives.

# Chapter 2

# Properties of curves with $\omega$-continuous derivative

In this section we prove Theorem 1.2. Let $\gamma : A \to \mathbb{R}^n$ be an arc-length parametrization of a finite-length curve without self-intersections, where $A$ is a circle or a closed segment. Let $\omega : [0, \infty) \to [0, \infty)$ we a concave non-decreasing function with $\omega(0) = 0$ and let $r_\omega > 0$ be such that $\gamma'$ is $r_\omega$-locally $\omega$-continuous.

Let us start with choosing $r_1 > 0$ such that $r_1 < r_\omega$, $r_1 < \frac{1}{2} \operatorname{diam} A$ and $\omega(r_1) < \frac{1}{3}$.

**Lemma 2.1.** *There exists $r_2 > 0$ such that if $|\gamma(s) - \gamma(t)| < r_2$ then $|t - s| < r_1$.*

*Proof.* Let $r_2 = \inf_{s,t \in A : |s-t| \geqslant r_1} |\gamma(s) - \gamma(t)|$. By the compactness of the set $\{(s, t) \in A \times A : |s-t| \geqslant r_1\}$ there exist $s_0, t_0 \in A$ satisfying $|s_0 - t_0| \geqslant r_1$ and $r_2 = |\gamma(s_0) - \gamma(t_0)|$. Therefore if $|\gamma(s) - \gamma(t)| < r_2$ then $|s-t| < r_1$. We only need to prove that $r_2 > 0$. But $r_2 = |\gamma(s_0) - \gamma(t_0)|$ and $\gamma$ does not have self-intersections. □

In the proof of Theorem 1.2 let us take $r_0 := \min(r_\omega, r_2)$. Take any distinct $x, y, z \in \gamma(A)$ satisfying $\operatorname{diam}\{x, y, z\} < r_0$. Let $x = \gamma(a), y = \gamma(b), z = \gamma(c)$. Since $\operatorname{diam}\{x, y, z\} < r_0 \leqslant r_2$, we have $\operatorname{diam}\{a, b, c\} < r_1 < \frac{1}{2} \operatorname{diam} A$. Therefore even if $A$ is a circle, there is a natural order of $a, b, c$ in the interior of one semicircle and w.l.o.g we can assume $a < b < c \leqslant a + r_1$. Let $v = \gamma'(a)$. Then for all $s \in [a, c]$ we have

$$|v - \gamma'(s)| < \omega(|s - a|) \leqslant \omega(|c - a|) < \frac{1}{3}. \tag{2.1}$$

Note that for all $a \leqslant s_1 \leqslant s_2 \leqslant c$ we have:

$$\left|\gamma(s_2) - \gamma(s_1) - v|s_2 - s_1|\right| = \Big| \int_{s_1}^{s_2} (\gamma'(s) - v) ds \Big| \leqslant \int_{s_1}^{s_2} |\gamma'(s) - v| ds < |s_2 - s_1| \omega(|c - a|). \tag{2.2}$$

So for all $a \leqslant s_1 < s_2 \leqslant c$ (recall that $\gamma$ is an arc-length parametrization):

$$|s_2 - s_1| \geqslant |\gamma(s_2) - \gamma(s_1)| \geqslant |v|s_2 - s_1|| - |\gamma(s_2) - \gamma(s_2) - v|s_2 - s_1|| > (1 - \omega(|c - a|))|s_2 - s_1|$$

and:

$$\Big|\frac{\gamma(s_2) - \gamma(s_1)}{|\gamma(s_2) - \gamma(s_1)|} - \frac{\gamma(s_2) - \gamma(s_1)}{|s_2 - s_1|}\Big| = \Big|1 - \frac{|\gamma(s_2) - \gamma(s_1)|}{|s_2 - s_1|}\Big| < \omega(|c - a|). \tag{2.3}$$

Therefore by Equation 2.2 and Equation 2.3

$$\begin{aligned}\Big|\frac{\gamma(s_2)-\gamma(s_1)}{|\gamma(s_2)-\gamma(s_1)|}-v\Big| &\leqslant \Big|\frac{\gamma(s_2)-\gamma(s_1)}{|\gamma(s_2)-\gamma(s_1)|}-\frac{\gamma(s_2)-\gamma(s_1)}{|s_2-s_1|}\Big|+\Big|\frac{\gamma(s_2)-\gamma(s_1)}{|s_2-s_1|}-v\Big|\\ &< 2\omega(|c-a|).\end{aligned}$$

By taking every $(s_1, s_2) \in \{(a, b), (b, c), (a, c)\}$ we obtain that $\frac{y-x}{|y-x|}$, $\frac{z-y}{|z-y|}$ and $\frac{z-x}{|z-x|}$ differ from $v$ by less than $2\omega(|c-a|)$ and we can conclude with

$$\begin{aligned}\Big|\frac{y-x}{|y-x|}-\frac{z-y}{|z-y|}\Big| &\leqslant \Big|\frac{y-x}{|y-x|}-v\Big|+\Big|v-\frac{z-y}{|z-y|}\Big|\\ &< 2\omega(|c-a|)+2\omega(|c-a|)=4\omega(|c-a|)<\frac{4}{3}<\sqrt{2}.\end{aligned}$$

Note that this in particular means the triangle with vertices $x, y, z$ has obtuse angle at vertex $y$ and therefore $|z-x| = \mathrm{diam}\{x, y, z\}$. Since $|z-x| > (1-\omega(|c-a|))|c-a| > \frac{2}{3}|c-a|$ we have (recall that $\omega$ is a concave non-decreasing function):

$$\Big|\frac{y-x}{|y-x|}-\frac{z-y}{|z-y|}\Big| < 4\omega(|c-a|) \leqslant 4\omega\Big(\frac{3}{2}|z-x|\Big) \leqslant 6\omega(|z-x|).$$

That completes the proof of Theorem 1.2.

# Chapter 3

# Characterization of subsets of curves with $\omega$-continuous derivative

In this section we prove Theorem 1.1. This is done by explicit construction of desired curve family. In Section 3.1 we investigate the condition in Equation 1.1 to prove that at small scales the set $K$ lies approximately along a straight line. We use this observation in Section 3.2 to provide an explicit construction of *one* curve with properly regular derivative that covers set $K \cap B$ for some small ball $B$. Finally, in Section 3.3 we show that all these small curves for different small balls $B$ can be merged into the desired curve family.

Let us now recall the Jones's condition. Assume $n = 2$, i.e., we are working in $\mathbb{R}^2$. For every square $Q$ let $l(Q)$ be a sidelength of $Q$, let $S_K(Q)$ be the most narrow strip covering $3Q \cap K$ and let $\beta_K(Q) = \text{width}(S_K(Q))/l(Q)$. Jones's theorem asserts that $K$ is a subset of a rectifiable curve iff

$$\sum_{\text{diadic } Q} \beta_K(Q)^2 l(Q) < \infty.$$

Notice that this condition, similarly to ours, implies that in a majority of small balls points of $K$ lie approximately along a straight line. However, our condition takes into account *every* point of $K$, where Jones's one is rather a *measure-type* condition. In particular, our condition implies that at small scale vectors $(y - x)$ for $x, y \in K$ are approximately parallel, which is far from being true in Jones's condition and rectifiable curves.

## 3.1. Preliminaries

Let us fix some global coordinate system in the whole $\mathbb{R}^n$, so we can *compare points*. We use this comparison procedure to break ties in constructing algorithm so that it is fully deterministic.

**Definition 3.1.** *We say that $x < y$ for $x, y \in \mathbb{R}^n$ if for some $1 \leqslant k \leqslant n$ we have $x_i = y_i$ for $1 \leqslant i < k$ and $x_k < y_k$ (i.e., we sort points lexicographically).*

Given an isolated point $x \in K$, when we speak about the *closest* point to $x$ from $K$ (or some compact subset of $K$), we mean that we break ties using comparison procedure from Definition 3.1, i.e., we choose the smallest point among the set of the closest points. Such point exists since $K$ is compact and a projection onto any subspace of a compact set is still a compact set.

Let $K \subset \mathbb{R}^n$ be a compact set, let $\omega : [0, \infty) \to [0, \infty)$ be a concave non-decreasing function with $\omega(0) = 0$ and let $r_0 > 0$ be such that for all distinct $x, y, z \in K$ with $|z - x| = \operatorname{diam}\{x, y, z\} < r_0$ we have

$$\Big|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\Big| < \omega(\operatorname{diam}\{x,y,z\}).$$

We can assume $r_0$ is sufficiently small to ensure that $\omega(r_0) < 0.001$. The constant 0.001 is far from optimal, we do not optimize constants in our proofs. As we shall see, we only need that all constants that appear in the proof are significantly smaller than 1.

**Lemma 3.2.** *For any pairwise distinct $x, y, z \in K$ if $|z - x| = \operatorname{diam}\{x, y, z\} < r_0$ then the angle $\measuredangle(x, y, z)$ is obtuse and*

$$\Big|\frac{y-x}{|y-x|} - \frac{z-x}{|z-x|}\Big| < \omega(\operatorname{diam}\{x,y,z\}),$$

$$\Big|\frac{z-y}{|z-y|} - \frac{z-x}{|z-x|}\Big| < \omega(\operatorname{diam}\{x,y,z\}).$$

*Proof.* Since

$$\Big|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\Big| < \omega(\operatorname{diam}\{x,y,z\}) \leqslant \omega(r_0) < 0.001$$

the triangle with vertices $x, y, z$ has obtuse angle at $y$. Therefore the angle between vectors $(y - x)$ and $(z - y)$ is acute and its measure is the sum of measures of angles between vectors $(y - x)$ and $(z - x)$ and between vectors $(z - y)$ and $(z - x)$. Therefore

$$\Big|\frac{y-x}{|y-x|} - \frac{z-x}{|z-x|}\Big| \leqslant \Big|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\Big|,$$

$$\Big|\frac{z-y}{|z-y|} - \frac{z-x}{|z-x|}\Big| \leqslant \Big|\frac{y-x}{|y-x|} - \frac{z-y}{|z-y|}\Big|$$

which completes the proof. □

Take any $x_0 \in K$ and $r < \frac{1}{2}r_0$ and let $B = \{x : |x - x_0| \leqslant r\}$ and $K_B = K \cap B$. Now we are going to prove that $K_B$ lies approximately along one line. Later, we construct desired curves for all small balls $B$ and then merge curves into the desired family.

If there is one point in $K_B$ there is nothing interesting to do, so let $x_1, x_2 \in K_B$ such that $|x_2 - x_1| = \operatorname{diam} K_B > 0$ (they exists by the compactness of $K$ and $K_B$). Let $d = \operatorname{diam} K_B < r_0$ and set coordinates with base $(e_1, e_2, \ldots, e_n)$ so that $x_1 = 0$, $x_2 = (d, 0, \ldots, 0) = de_1$. Then $\frac{x_2 - x_1}{|x_2 - x_1|} = e_1$.

**Definition 3.3.** *We say that a point $y \in K_B$ is* to the right *(*to the left*) from $x \in K_B$ iff $\langle y, e_1\rangle > \langle x, e_1\rangle$ ($\langle y, e_1\rangle < \langle x, e_1\rangle$). We say that points $y, y' \in K_B$ are* on the same side *(*on opposing sides*) of $x \in K_B$ iff $0 \neq \operatorname{sgn}(\langle y - x, e_1\rangle) = \operatorname{sgn}(\langle y' - x, e_1\rangle)$ (conversely $0 \neq \operatorname{sgn}(\langle y - x, e_1\rangle) \neq \operatorname{sgn}(\langle y' - x, e_1\rangle) \neq 0$).*

**Lemma 3.4.** *Let $x, y \in K_B$ satisfy $x \neq y$ and $\langle x, e_1\rangle \leqslant \langle y, e_1\rangle$. Then*

$$\Big|\frac{y-x}{|y-x|} - e_1\Big| < 2\omega(d) < 0.002.$$

*In particular, all points in $K_B$ have distinct first coordinate, and for any two different points $x, y \in K_B$ always one lies to the right of the other one and $\operatorname{sgn}(\langle x - y, e_1\rangle) \neq 0$.*

*Proof.* If $x = x_1$ and $y = x_2$ there is nothing to prove. By symmetry, w.l.o.g. assume $y \notin \{x_1, x_2\}$. By Lemma 3.2 for triangle $x_1, y, x_2$ (note that $|x_2 - x_1| = d = \operatorname{diam}\{x_1, y, x_2\}$):

$$\Big|\frac{y - x_1}{|y - x_1|} - e_1\Big| < \omega(d). \tag{3.1}$$

If $x = x_1$ the proof is finished. Otherwise let us focus on triangle $x_1$, $x$, $y$. By Lemma 3.2 one of the angles of triangle $x_1, x, y$ is obtuse. We now prove that this is angle $\measuredangle(x_1, x, y)$.

If $x = x_2$, $\frac{x-x_1}{|x-x_1|} = e_1$. Otherwise, by Lemma 3.2 for triangle $x_1, x, x_2$:

$$\Big|\frac{x - x_1}{|x - x_1|} - e_1\Big| < \omega(d).$$

Therefore, using Equation 3.1:

$$\begin{aligned}\Big|\frac{x - x_1}{|x - x_1|} - \frac{y - x_1}{|y - x_1|}\Big| &\leqslant \Big|\frac{x - x_1}{|x - x_1|} - e_1\Big| + \Big|e_1 - \frac{y - x_1}{|y - x_1|}\Big| \\ &< \omega(d) + \omega(d) < 0.002.\end{aligned}$$

Therefore the angle $\measuredangle(x, x_1, y)$ is acute.

By contradiction, assume that $\measuredangle(x_1, y, x)$ is obtuse. In this case by Lemma 3.2:

$$\Big|\frac{x - y}{|x - y|} - e_1\Big| \leqslant \Big|\frac{x - y}{|x - y|} - \frac{x - x_1}{|x - x_1|}\Big| + \Big|\frac{x - x_1}{|x - x_1|} - e_1\Big| < 2\omega(d) < 0.002.$$

This contradicts with the assumption that $\langle x, e_1\rangle \leqslant \langle y, e_1\rangle$.

Therefore by applying Lemma 3.2 to triangle $x_1, x, y$ we get

$$\Big|\frac{y - x}{|y - x|} - e_1\Big| \leqslant \Big|\frac{y - x}{|y - x|} - \frac{y - x_1}{|y - x_1|}\Big| + \Big|\frac{y - x_1}{|y - x_1|} - e_1\Big| < 2\omega(d).$$

□

**Corollary 3.5.** *Merging Lemma 3.2 and Lemma 3.4 we have for every $x, y, z \in K_B$ if $y \notin \{x, z\}$:*

$$\Big|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y - x}{|y - x|} - \operatorname{sgn}(\langle z - y, e_1\rangle)\frac{z - y}{|z - y|}\Big| < \omega(\operatorname{diam}\{x, y, z\}).$$

**Corollary 3.6.** *Applying Lemma 3.4 once again we have for every different $x, y \in K_B$:*

$$\Big|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y - x}{|y - x|} - e_1\Big| < 2\omega(d) < 0.002.$$

**Lemma 3.7.** *Take three pairwise different points $x, y, z \in K_B$. Then $|z - x| = \operatorname{diam}\{x, y, z\}$ iff $x$ and $z$ lie on opposing sides of $y$.*

*Proof.* First assume $x$ and $z$ lie on opposing sides of $y$. W.l.o.g. assume $x$ lies to the left and $z$ lies to the right. By Lemma 3.4

$$\Big|\frac{y - x}{|y - x|} - \frac{z - y}{|z - y|}\Big| \leqslant \Big|\frac{y - x}{|y - x|} - e_1\Big| + \Big|e_1 - \frac{z - y}{|z - y|}\Big| < 4\omega(d) < 0.004.$$

Therefore the angle $\measuredangle(x, y, z)$ is obtuse.

Now assume $|z - x| = \operatorname{diam}\{x, y, z\}$. W.l.o.g. $z$ lies to the right of $x$ and, by Lemma 3.4 and Lemma 3.2,

$$\Big|\frac{y - x}{|y - x|} - e_1\Big| \leqslant \Big|\frac{y - x}{|y - x|} - \frac{z - x}{|z - x|}\Big| + \Big|\frac{z - x}{|z - x|} - e_1\Big| < 3\omega(d) < 0.003.$$

Therefore $y$ lies to the right of $x$. Making the same calculations for $z - y$ instead of $y - x$ we see that $y$ lies to the left of $z$. □

**Lemma 3.8.** *Let $x \in K_B$, $x \notin \{x_1, x_2\}$. Then $|x - x_1| < d$, $|x - x_2| < d$, i.e., the choice of $x_1$ and $x_2$ was unique up to numbering (i.e., to a rotation of the coordinate system by $180°$).*

*Proof.* Since $\operatorname{diam}\{x_1, x, x_2\} = |x_2 - x_1|$, then by Lemma 3.2 angle $\measuredangle(x_1, x, x_2)$ is obtuse and sides $(x - x_1)$ and $(x - x_2)$ of the triangle $x_1, x, x_2$ are shorter than side $(x_1 - x_2)$. □

## 3.2. The local construction

In this section we provide a construction algorithm that allows to construct *one* curve without self-intersections, with arc-length parametrization $\gamma : [a, b] \to \mathbb{R}^n$, such that $\gamma'$ is $342\omega$-continuous and $K_B \subset \gamma([a, b])$.

**Lemma 3.9.** *Let $y \in K_B$ be an accumulation point of $K_B$ and let $(y_k)_{k=1}^{\infty}$ be a sequence of points from $K_B$ convergent to $y$, different than $y$. Then there exists*

$$\lim_{k\to\infty} \operatorname{sgn}(\langle y_k - y, e_1\rangle)\frac{y_k - y}{|y_k - y|}$$

*and this limit is a vector of length 1.*

*Proof.* Fix $\varepsilon > 0$ and let $\delta > 0$ satisfy $\omega(2\delta) < \varepsilon$. Assume $M \in \mathbb{N}$ satisfies: for all $k \geqslant M$ we have $|y_k - y| < \delta$. Let $k, l \geqslant M$. By Corollary 3.5 for triangle $y_k, y_l, y$:

$$\left|\operatorname{sgn}(\langle y_k - y, e_1\rangle)\frac{y_k - y}{|y_k - y|} - \operatorname{sgn}(\langle y_l - y, e_1\rangle)\frac{y_l - y}{|y_l - y|}\right| < \omega(\operatorname{diam}\{y, y_l, y_k\}) \leqslant \omega(2\delta) < \varepsilon.$$

Therefore this sequence converges and the limit is a vector of length 1, since all terms have length 1. □

**Definition 3.10.** *For every $y \in K_B$ we define a unit vector $v_y \in \mathbb{R}^n$ as follows: if $y$ is a accumulation point of $K_B$, then*

$$v_y := \lim_{y'\to y} \operatorname{sgn}(\langle y' - y, e_1\rangle)\frac{y' - y}{|y' - y|},$$

*i.e., $v_y$ is a vector tangent to $K_B$ at $y$. Otherwise, let $y^*$ be the closest (if tied, choose the lexicographically smallest among the closest points) point from $K_B$ to $y$. Then*

$$v_y := \operatorname{sgn}(\langle y^* - y, e_1\rangle)\frac{y^* - y}{|y^* - y|}.$$

Let us now prove some properties of the chosen vectors $v_y$.

**Lemma 3.11.** *Let $y \in K_B$. Then $|v_y - e_1| \leqslant 2\omega(d) < 0.002$.*

*Proof.* By Corollary 3.6, for any $x \in K_B$, $x \neq y$:

$$\left|\operatorname{sgn}(\langle x - y, e_1\rangle)\frac{x - y}{|x - y|} - e_1\right| \leqslant 2\omega(d).$$

But $|v_y - e_1|$ is a limit of such expressions (in case of $y$ being an accumulation point of $K_B$) or is equal to a single expression expression of that form. □

**Lemma 3.12.** *Let $x, y \in K_B$, $x \neq y$. Then*

$$\left|\operatorname{sgn}(\langle y - x, e_1\rangle)\frac{y - x}{|y - x|} - v_x\right| < 2\omega(|y - x|).$$

*Proof.* If $x$ is an accumulation point of $K_B$, then take $x^\circ$ sufficiently close to $x$ such that:

$$|x - x^\circ| < \frac{1}{2}\omega(|y-x|), \qquad \Big|\operatorname{sgn}(\langle x^\circ - x, e_1\rangle)\frac{x^\circ - x}{|x^\circ - x|} - v_x\Big| < \frac{1}{2}\omega(|y-x|).$$

Then $\operatorname{diam}\{x, y, x^\circ\} \leqslant \frac{3}{2}|y-x|$ and using Lemma 3.2 and Corollary 3.5:

$$\begin{aligned}\Big|\operatorname{sgn}(\langle y-x, e_1\rangle)\frac{y-x}{|y-x|} - v_x\Big| &\leqslant \Big|\operatorname{sgn}(\langle y-x, e_1\rangle)\frac{y-x}{|y-x|} - \operatorname{sgn}(\langle x - x^\circ, e_1\rangle)\frac{x-x^\circ}{|x-x^\circ|}\Big| + \\ &+ \Big|\operatorname{sgn}(\langle x - x^\circ, e_1\rangle)\frac{x-x^\circ}{|x-x^\circ|} - v_x\Big| \\ &< \omega\Big(\frac{3}{2}|y-x|\Big) + \frac{1}{2}\omega(|y-x|) \leqslant 2\omega(|y-x|).\end{aligned}$$

Otherwise, note that by the definition of $x^*$ we have $|x - x^*| \leqslant |x-y|$ and $\operatorname{diam}\{x, x^*, y\} \leqslant 2|x-y|$. Using once again Corollary 3.5:

$$\begin{aligned}\Big|\operatorname{sgn}(\langle y-x, e_1\rangle)\frac{y-x}{|y-x|} - v_x\Big| &= \Big|\operatorname{sgn}(\langle y-x, e_1\rangle)\frac{y-x}{|y-x|} - \operatorname{sgn}(\langle x - x^*, e_1\rangle)\frac{x-x^*}{|x-x^*|}\Big| < \\ &< \omega(2|y-x|) \leqslant 2\omega(|y-x|).\end{aligned}$$

□

**Lemma 3.13.** *Let $x, y \in K_B$, $x \neq y$. Then $|v_x - v_y| < 4\omega(|y-x|)$.*

*Proof.* We use Lemma 3.12 twice:

$$\begin{aligned}|v_x - v_y| &\leqslant \Big|\operatorname{sgn}(\langle y-x, e_1\rangle)\frac{y-x}{|y-x|} - v_x\Big| + \Big|\operatorname{sgn}(\langle y-x, e_1\rangle)\frac{y-x}{|y-x|} - v_y\Big| \\ &< 2\omega(|y-x|) + 2\omega(|y-x|) = 4\omega(|y-x|).\end{aligned}$$

□

After these preparations, we now provide a construction of a sufficiently regular curve that connects two points of $K_B$.

**Lemma 3.14.** *Let $x, y \in K_B$ and assume that $y$ is to the right from $x$. Then there exists a smooth curve with arc-length parametrization $\gamma : [a, b] \to \mathbb{R}^n$ such that*

1. *$\gamma(a) = x$, $\gamma(b) = y$, $\gamma'(a) = v_x$, $\gamma'(b) = v_y$,*
2. *$\gamma'$ is $\frac{157\omega(|y-x|)}{|y-x|}$-Lipschitz continuous,*
3. *for any $t \in [a, b]$, $|\gamma'(t) - e_1| \leqslant 138\omega(d) < 0.138$.*
4. *for any different $s, t \in [a, b]$, $|s-t| \geqslant |\gamma(s) - \gamma(t)| > 0.93|s-t|$.*

*Proof.* Let $\hat{d} = |y-x|$. We start by defining $\hat{\gamma} : [0, \hat{d}] \to \mathbb{R}^n$ satisfying: $\hat{\gamma} \in C^\infty$, $\hat{\gamma}(0) = x$, $\hat{\gamma}(\hat{d}) = y$, $\hat{\gamma}'(0) = v_x$, $\hat{\gamma}'(\hat{d}) = v_y$. However, $\hat{\gamma}$ will not be an arc-length parametrization. We will then bound $\hat{\gamma}'$ to reparametrize $\hat{\gamma}$ as desired.

Let us define $\hat{\gamma}$ as follows:

$$\hat{\gamma}(t) = \Big(\frac{v_y - v_x}{\hat{d}^2} - 2\frac{y - x - \hat{d}v_x}{\hat{d}^3}\Big)t^3 + \Big(3\frac{y - x - \hat{d}v_x}{\hat{d}^2} - \frac{v_y - v_x}{\hat{d}}\Big)t^2 + v_x t + x.$$

It is easy to check that $\hat\gamma(0) = x$ and $\hat\gamma(\hat d) = y$. Let us compute $\hat\gamma'(t)$:

$$\hat\gamma'(t) = 3\Big(\frac{v_y - v_x}{\hat d^2} - 2\frac{y - x - \hat d v_x}{\hat d^3}\Big)t^2 + 2\Big(3\frac{y - x - \hat d v_x}{\hat d^2} - \frac{v_y - v_x}{\hat d}\Big)t + v_x.$$

It is easy to check that $\hat\gamma(0)' = v_x$ and $\hat\gamma(\hat d) = v_y$. Note that by Lemma 3.12 and Lemma 3.13:

$$|v_y - v_x| < 4\omega(\hat d), \qquad\qquad \Big|\frac{y - x}{\hat d} - v_x\Big| < 2\omega(\hat d).$$

Therefore for $0 \leqslant s < t \leqslant \hat d$ we have:

$$|\hat\gamma'(t) - \hat\gamma'(s)| < \frac{t - s}{\hat d}\Big(3\frac{4\omega(\hat d)(t + s)}{\hat d} + 6\frac{2\omega(\hat d)(t + s)}{\hat d} + 6 \cdot 2\omega(\hat d) + 2 \cdot 4\omega(\hat d)\Big) \tag{3.2}$$

$$\leqslant (t - s)\frac{68\omega(\hat d)}{\hat d}. \tag{3.3}$$

In particular, for $s = 0$ we have

$$|\hat\gamma'(t) - v_x| < 68\omega(\hat d) \leqslant 68\omega(d) < 0.068. \tag{3.4}$$

And, by Lemma 3.4,

$$|\hat\gamma'(t) - e_1| \leqslant |\hat\gamma'(t) - v_x| + |v_x - e_1| < 68\omega(\hat d) + 2\omega(d) \leqslant 70\omega(d) < 0.07. \tag{3.5}$$

Therefore the projection of $\hat\gamma([0, \hat d])$ onto the first coordinate axis is injective, the curve $\hat\gamma$ goes mostly to the right (i.e., $\hat\gamma'$ points to the right) and has no self-intersections.

Moreover, by Equation 3.4 and the fact that $|v_x| = 1$ we get that $0.932 < |\hat\gamma'(t)| < 1.068$, so by standard techniques one can modify parametrization $\hat\gamma$ to obtain arc-length parametrization $\gamma$ of the curve $\hat\gamma([0, \hat d])$. Namely, if $L(t) = \int_0^t |\hat\gamma'(s)|ds$ is the length function of $\hat\gamma$, we define $\gamma : [0, L(\hat d)] \to \mathbb{R}^n$ as $\gamma(u) = \hat\gamma(L^{-1}(u))$. Then

$$\gamma'(u) = (L^{-1}(u))'\hat\gamma'(L^{-1}(u)) = \frac{\hat\gamma'(L^{-1}(u))}{|\hat\gamma'(L^{-1}(u))|}. \tag{3.6}$$

We now check all conditions for $\gamma$. Point 1 is obvious by the definition of $\hat\gamma$ and Equation 3.6. By Equation 3.4, $|\hat\gamma'(t) - \gamma'(\gamma^{-1}(\hat\gamma(t)))| < 68\omega(d)$. Therefore, by Equation 3.5, $|\gamma'(u) - e_1| < 138\omega(d) < 0.138$ and Point 3 is satisfied. To check Point 4, note that by Equation 3.5 for $0 \leqslant t_1 < t_2 \leqslant \hat d$:

$$|\hat\gamma(t_2) - \hat\gamma(t_1)| \geqslant \langle \hat\gamma(t_2) - \hat\gamma(t_1), e_1\rangle = \int_{t_1}^{t_2} \langle \hat\gamma'(s), e_1\rangle ds$$

$$> \int_{t_1}^{t_2} 0.93|\hat\gamma'(s)|ds = 0.93(L(t_2) - L(t_1)).$$

Conversely, setting $u_1 = L(t_1)$ and $u_2 = L(t_2)$:

$$|\gamma(u_2) - \gamma(u_1)| \geqslant 0.93(u_2 - u_1),$$

and Point 4 is satisfied.

What remains is Point 2, i.e., Lipschitz continuity of the new parametrization, so we are now going to prove that $\gamma'$ is $157\frac{\omega(\hat d)}{\hat d}$-Lipschitz continuous. Since all functions here are $C^\infty$, we can compute

$$\gamma''(u) = ((L^{-1})')^2(u)\hat\gamma''(L^{-1}(u)) + (L^{-1})''(u)\hat\gamma'(L^{-1}(u)).$$

Since $0.932 < |\hat{\gamma}'(t)| < 1.068$ we obtain $0.932 < L'(t) < 1.068$ and $1/1.068 < (L^{-1})'(u) < 1/0.932$. Since $\hat{\gamma}'$ is $68\frac{\omega(\hat{d})}{\hat{d}}$-Lipschitz continuous (Equation 3.2), we have that

$$\left|((L^{-1})')^2(u)\hat{\gamma}''(L^{-1}(u))\right| < \frac{68}{0.932^2}\frac{\omega(\hat{d})}{\hat{d}} < 78.285\frac{\omega(\hat{d})}{\hat{d}}.$$

Since $(L^{-1})'(u) = 1/|\hat{\gamma}'(L^{-1}(u))|$, we get that:

$$\begin{aligned}\left|(L^{-1})''(u)\hat{\gamma}(L^{-1}(u))\right| &= \left|\frac{\frac{d}{du}|\hat{\gamma}'(L^{-1}(u))|}{|\hat{\gamma}'(L^{-1}(u))|^2}\right||\hat{\gamma}'(L^{-1}(u))| \\ &< \left|\frac{1}{0.932}\cdot(L^{-1})'(u)|\hat{\gamma}''(L^{-1}(u))|\right| \\ &< \frac{1}{0.932^2}\cdot 68\frac{\omega(\hat{d})}{\hat{d}} < 78.285\frac{\omega(\hat{d})}{\hat{d}}.\end{aligned}$$

Therefore $\gamma'$ is $157\frac{\omega(\hat{d})}{\hat{d}}$-Lipschitz continuous. □

**Remark 3.15.** *Let us for a while swap the roles of $x_1$ with $x_2$, i.e., rotate the coordinate system by $180°$. If we then connect points $x$ and $y$ using Lemma 3.14, we obtain the same curve, but running backwards, since the formula for $\hat{\gamma}$ gives the unique polynomial of degree at most 3 that has fixed $\hat{\gamma}$ and $\hat{\gamma}'$ at the endpoints.*

**Lemma 3.16.** *With the assumptions of Lemma 3.14, the constructed curve $\gamma$ has $169\omega$-conti-nuous derivative.*

*Proof.* Let $\gamma : [a,b] \to \mathbb{R}^n$ connect $x, y \in K_B$ and let $s, t \in [a,b]$. By Point 4 of the conditions for $\gamma$:

$$|b-a| \geqslant |y-x| > 0.93|b-a|.$$

Since $\omega$ is non-decreasing:

$$\frac{\omega(|y-x|)}{|y-x|} < \frac{\omega(|b-a|)}{0.93|b-a|}.$$

Recall $\gamma$ is $157\frac{\omega(|y-x|)}{|y-x|}$-Lipschitz continuous and $\omega$ is concave:

$$|\gamma'(t)-\gamma'(s)| < 157\frac{\omega(|y-x|)}{|y-x|}|t-s| < \frac{157}{0.93}\omega(|b-a|)\frac{|t-s|}{|b-a|} < 169\omega(|t-s|).$$

□

Let $\pi_{e_1} : \mathbb{R}^n \to \mathbb{R}$ be the projection onto $e_1$ axis. Since $x_1, x_2 \in K_B$, $[0,d] \setminus \pi_{e_1}(K_B)$ is a finite or countable family of disjoint open segments $(p^i, q^i)$ with $p^i = \pi_{e_1}(x^i)$ and $q^i = \pi_{e_1}(y^i)$, $x^i, y^i \in K_B$. For every $i$, let us connect $x^i$ with $y^i$ by a curve $\gamma_i$ constructed in Lemma 3.14. Let $\bar{K}_B$ be the set $K_B$ together with all images of $\gamma_i$.

By the properties of $\gamma_i$, for every $p \in [0,d]$ there exists exactly one $f(p) \in \bar{K}_B$ such that $p = \pi_{e_1}(f(p))$. We now prove some properties of the function $f$. Let us extend the definition of $v_x$ to $x \in \bar{K}_B$. If $x$ is in the image of $\gamma_i$, let $v_x = \gamma_i'(\gamma_i^{-1}(x))$, i.e., tangent vector to $\gamma_i$ at $x$. This definition works properly at the endpoints $x^i$, $y^i$ of $\gamma_i$ by the definition of $\gamma_i$.

**Lemma 3.17.** *For every $x \in \bar{K}_B$ the following holds:*

$$|v_x - e_1| < 138\omega(d) < 0.138.$$

*Proof.* If $x \in K_B$, Lemma 3.11 does the job. Otherwise, use Point 3 from Lemma 3.14. $\square$

**Lemma 3.18.** *For every different $x, y \in \bar{K}_B$:*

$$|v_x - v_y| < 342\omega(|x - y|).$$

*Proof.* If $x, y \in K_B$, the statement is obvious by Lemma 3.13. Therefore let us assume $x \notin K_B$, so $x$ is in the image of curve $\gamma_i$. If $y$ is in the image of $\gamma_i$, Lemma 3.16 does the job. Otherwise, let us assume w.l.o.g. that $y$ is to the right of $x$. Then it is to the right of $y^i$ too.

If $y \in K_B$ then:

$$|v_x - v_y| \leqslant |v_x - v_{y^i}| + |v_{y^i} - v_y| \leqslant 169\omega(|x - y^i|) + 4\omega(|y^i - y|) \leqslant 173\omega(|y - x|).$$

Otherwise, let $y$ be in the image of $\gamma_j$. Then:

$$\begin{aligned}|v_x - v_y| &\leqslant |v_x - v_{y^i}| + |v_{y^i} - v_{x^j}| + |v_{x^j} - v_y| \\ &\leqslant 169\omega(|x - y^i|) + 4\omega(|y^i - x^j|) + 169\omega(|x^j - y|) \leqslant 342\omega(|y - x|).\end{aligned}$$

$\square$

**Lemma 3.19.** *For every $x \in \bar{K}_B$ the vector $v_x$ is tangent at $x$ to the set $\bar{K}_B$.*

*Proof.* For $x \in \bar{K}_B \setminus K_B$ the statement is obvious. Let then look at the case $x = f(p) \in K_B$. If $x$ is an endpoint of some curve $\gamma_i$, then $\gamma_i'$ at $x$ equals $v_x$, either when $x$ is left or right endpoint of $\gamma_i$. On the other hand, if $x_n \to x$ and $x_n \neq x$, $x_n \in K_B$, then, by the definition of $v_x$,

$$\lim_{n\to\infty} \operatorname{sgn}(\langle x_n - x, e_1\rangle)\frac{x_n - x}{|x_n - x|} = v_x.$$

This finishes the proof. $\square$

We conclude this section with the final theorem.

**Theorem 3.20.** *Assume that $K$ is a compact set and $B$ is a closed ball of radius smaller than $\frac{1}{2}r_0$. Let $K_B = K \cap B$ and let $d$ be the diameter of $K_B$. Then there exists a curve without self-intersections with arc-length parametrization $\gamma : [0, L] \to \mathbb{R}^n$, such that:*

1. *$\gamma(0) = x_1$, $\gamma(L) = x_2$, $d \leqslant L < 1.161d$;*
2. *$\gamma'$ is $342\omega$-continuous;*
3. *for all $t \in [0, L]$ we have $|\gamma'(t) - e_1| < 138\omega(d) < 0.138$;*
4. *$K_B \subset \gamma([0, L])$;*
5. *for every $t \in [0, L]$ we have $\gamma'(t) = v_{\gamma(t)}$;*
6. *if moreover the center of $B$ belongs to $K_B$, then $\gamma([0, L]) \subset B$.*

*Proof.* First note that by Lemma 3.19 function $f : [0, d] \to \mathbb{R}^n$ is continuous and differentiable and $\frac{f'(p)}{|f'(p)|} = v_{f(p)}$. Note, that by Lemma 3.17, $v_x$ points mostly to the right and $f'(p)$ is non-zero and of length at least $1 - 0.138 = 0.862$. Therefore one can parametrize $f([0, d]) = \bar{K}_B$ with an arc-length parametrization $\gamma : [0, L] \to \mathbb{R}^n$ such that $\gamma'(t) = v_{\gamma(t)}$.

Point 4 follows directly from the construction. Point 3 is a corollary from Lemma 3.17 and the fact that $d \leqslant L \leqslant 1/(1-0.138)d < 1.161d$ follows. Point 2 is a corollary from Lemma 3.18.

Now assume that the center $x_c$ of $B$ belongs to $K_B$. Integrating the inequality for $|\gamma'(t)-e_1|$ (Point 3), we obtain

$$\Big|\frac{x-x_1}{|x-x_1|}-e_1\Big| < 0.138, \qquad \Big|\frac{x_2-x}{|x_2-x|}-e_1\Big| < 0.138$$

for all $x \in \bar{K}_B$. Since this in particular holds for $x = x_c$ and $x_1, x_2 \in B$, for every $x \in \bar{K}_B$ we have $|x_c - x| \leqslant \max\{|x_c - x_1|, |x_c - x_2|\}$. □

**Remark 3.21.** *Note that we still keep the property from Remark 3.15. If we swap $x_1$ with $x_2$, i.e., rotate the local coordinate system by $180°$, we get the same curve $\bar{K}_B$, but its parametrization is running backwards. Indeed, $v_x$ changes to $-v_x$ and by Remark 3.15 the images of curves $\gamma_i$ remain unchanged. Therefore, by Lemma 3.8, the constructed curve $\bar{K}_B$ does not depend on the chosen local coordinate system $(e_1, e_2, \dots, e_n)$.*

## 3.3. The global construction

In Section 3.2 we developed a way to pass one curve through points $K \cap B$ for a closed ball $B$ of diameter smaller than $r_0$. Now we would like to extend this construction to whole $K$. Naively, we would like to say *take sum of all curves for all small balls B, the sum should look nice*. Indeed, this way we get a bit weaker result than Theorem 1.1 quite immediately:

**Theorem 3.22.** *With the assumptions of Theorem 1.1, there exists a finite family of finite-length curves such that every curve:*

1. *admits an arc-length parametrization $\{\gamma_i\}_{1\leqslant i\leqslant N}$, $\gamma_i : [a_i, b_i] \to \mathbb{R}^n$,*
2. *$\gamma_i$ is injective, i.e., its image does not contain self-intersections,*
3. *the derivative $\gamma_i'$ is $342\omega$-continuous.*

*Unlike in Theorem 1.1, the images of $\gamma_i$ and $\gamma_j$ for $i \neq j$ are not necessarily disjoint.*

*Proof.* Take $r < \frac{1}{2}r_0$ and take

$$\mathcal{B}_0 = \{\{x : |x - x_0| \leqslant r\} : x_0 \in K\}.$$

Take a finite subfamily $\mathcal{B}_0' \subset \mathcal{B}_0$ such that the interiors of balls from $\mathcal{B}_0'$ cover $K$. For every $B \in \mathcal{B}_0'$ construct curve $\gamma_B$ using Theorem 3.20. The family $\{\gamma_B : B \in \mathcal{B}_0'\}$ is the desired family. □

Now we improve this construction so that the images of different curves $\gamma_B$ are disjoint.

We start with strengthening Remark 3.21, so that the local construction in the neighborhood of some $x \in K$ is totally independent even of the choice of the ball $B$ covering the neighborhood of $x$. The problem with the construction is that the choice of $y^*$ in Definition 3.10 depends on the choice of $B$. However, this can be easily circumvented.

Take $r^* < \frac{1}{20}r_0$. Let $K_{\text{lonely}} \subset K$ be the set of isolated points from $K$ that are distant by at least $r^*$ from other points of $K$. $K_{\text{lonely}}$ is finite and we can remove it from $K$: at the end of the construction, every non-covered point from $K_{\text{lonely}}$ can be covered by a sufficiently small segment. Therefore we can assume that for every $x \in K$ there exists $y \in K$, $x \neq y$ such that $|x - y| < r^*$.

**Lemma 3.23.** *Let $B$ be a closed ball with radius $r$, $2r^* \leqslant r < \frac{1}{2}r_0$ and let $x \in K \cap \frac{1}{2}B$ be not an accumulation point of $K_B$. Then $x^*$, taken in Definition 3.10, is in fact the closest point to $x$ from the entire $K$, not only $K_B$, and among all the closest points to $x$, $x^*$ is smallest according to comparison procedure described in Definition 3.1.*

*Proof.* For this $x$ there exists $y \in K$ such that $|x - y| < r^*$. Therefore $|x - x^*| < r^*$, but, since $r \geqslant 2r^*$, ball $\{y : |x - y| \leqslant r^*\} \subset B$. Therefore all closest points to $x$ from $K$ belong to $B$ and the lemma is proven. □

**Definition 3.24.** *Let $B$ be a closed ball with radius $r < \frac{1}{2}r_0$. Assume there exist at least 2 points in $K \cap \frac{1}{2}B$. Construct curve $\gamma_B$ for ball $B$, using Theorem 3.20. Let $x$ be the first ($\frac{1}{2}B$ is closed) point of $K \cap \frac{1}{2}B$ on the image of $\gamma_B$, and let $y$ be the last point (or, equivalently, $x$ is the leftmost and $y$ is the rightmost point of $K \cap \frac{1}{2}B$). The closed arc of $\gamma_B$ from $x$ to $y$ we call the* inner curve *of $\gamma_B$ and its arc-length parametrization we denote by $\bar{\gamma}_B$.*

**Lemma 3.25.** *Let $B_1, B_2$ be closed balls with radii $r_1, r_2$, such that $2r^* \leqslant r_2 < r_1 < \frac{1}{2}r_0$ and $B_2 \subset \frac{1}{2}B_1$. Assume that $K \cap \frac{1}{2}B_2$ consists of at least 2 points and the center of $B_2$ belongs to $K$. Then the curve $\bar{\gamma}_{B_2}$ is a subset of the curve $\bar{\gamma}_{B_1}$.*

*Proof.* Construct curve $\gamma_{B_1}$ using Theorem 3.20. Let $x_l$ be the leftmost point of $K \cap B_2$ (in the coordinate system used to construct $\gamma_{B_1}$) and $x_r$ be the rightmost. Since $K \cap \frac{1}{2}B_2$ consists of at least two points, $x_l$ and $x_r$ are well defined and are different. By Corollary 3.6 for ball $B_1$, all points $x \in K$ between $x_l$ and $x_r$ (especially, the center of $B_2$) are either equal to $x_l$ or satisfy

$$\Big|\frac{x - x_l}{|x - x_l|} - e_1\Big| < 0.002. \tag{3.7}$$

Therefore, all points $x \in K \cap B_1$ between $x_l$ and $x_r$ are in $B_2$.

Moreover, angle $\measuredangle(x_l, x, x_r)$ is obtuse, and $|x_r - x_l| = \operatorname{diam} K \cap B_2$. Together with Equation 3.7, this means that, to construct $\gamma_{B_2}$ using Theorem 3.20, the $e_1$ axis connects $x_l$ with $x_r$ and the order (being to the left or right) on this axis is the same as on the $e_1$ axis in $B_1$. Note that by Lemma 3.23, for every $x \in \frac{1}{2}B_2$ the point $x^*$ belongs to $B_2 \subset \frac{1}{2}B_1$ and it is in fact the closest to $x$ point in the whole set $K$. Therefore the choice of $x^*$ is independent of whether we do construction of $\gamma_{B_1}$ or $\gamma_{B_2}$. Together with Remark 3.21, this implies that the construction process in Theorem 3.20 for the part $\bar{\gamma}_{B_2}$ is a subset of the construction process for the curve $\gamma_{B_1}$. Since $B_2 \subset \frac{1}{2}B_1$, this in fact is the subset of the curve $\bar{\gamma}_{B_1}$. □

**Lemma 3.26.** *Let $B_2$, $B_3$ be closed balls with centers $y_2, y_3 \in K$ and radii $2r^* \leqslant r_2, r_3 < \frac{1}{8}r_0$. Assume that $K \cap \frac{1}{2}B_2 \cap \frac{1}{2}B_3 \neq \emptyset$ and both $\frac{1}{2}B_2$ and $\frac{1}{2}B_3$ have at least two points from $K$ each. Then the sum of curves $\bar{\gamma}_{B_2}$ and $\bar{\gamma}_{B_3}$ is one curve with the same regularity as implied by Theorem 3.20.*

*Proof.* Just note that $B_2 \cap B_3$ can be covered with a closed ball $\frac{1}{2}B_1$ such that $B_1$ has radius smaller than $\frac{1}{2}r_0$. Then both $\bar{\gamma}_{B_2}$ and $\bar{\gamma}_{B_3}$ are subsets of the curve $\bar{\gamma}_{B_1}$. Since $K \cap \frac{1}{2}B_2 \cap \frac{1}{2}B_3 \neq \emptyset$, the sum of these curves is connected. □

We can now finish the proof of Theorem 1.1. Take any finite family $\mathcal{B}$ of closed balls with centers in $K$ and radii at least $2r^*$ and smaller than $\frac{1}{10}r_0$ such that $\{\operatorname{int}\frac{1}{2}B : B \in \mathcal{B}\}$ covers $K$. For each ball $B \in \mathcal{B}$ take $\bar{\gamma}_B$ and let $\bar{K}$ denote the union of all images of these curves. Note that the set $\bar{K}$ has finite length, since all curves had finite length.

First, note that we can remove one-by-one such balls $B$ from $\mathcal{B}$ that the image of $\bar{\gamma}_B$ is contained in images of curves $\bar{\gamma}$ for other balls in $\mathcal{B}$. Therefore we can assume that for every

$B \in \mathcal{B}$ there exists a point in the image of $\bar{\gamma}_B$ that is not in any image of curve $\bar{\gamma}_{B'}$ for $B \neq B' \in \mathcal{B}$.

Take any $x \in \mathbb{R}^n$ and take

$$\mathcal{B}_x = \{B \in \mathcal{B} : \operatorname{dist}(x, B) \leqslant r^*\}.$$

If $B_x = \{y : |y-x| \leqslant 2r^* + \frac{2}{5}r_0\}$, then $\bigcup \mathcal{B}_x \subset \frac{1}{2}B_x$ and, by Lemma 3.25 (note that the radius of $B_x$, i.e., $2r^* + \frac{2}{5}r_0$, is smaller than $\frac{1}{2}r_0$), all curves $\bar{\gamma}_B$ for $B \in \mathcal{B}$ are subsets of curve $\bar{\gamma}_{B_x}$. Intuitively, locally $\bar{K}$ looks like a single curve.

Let $x \in \bar{K}$ and assume that $x$ is contained in the images of $\bar{\gamma}$ for (at least) three different $B_1, B_2, B_3 \in \mathcal{B}$. Then all images of $\bar{\gamma}_{B_i}$ for $i = 1, 2, 3$ are contained in the image of $\bar{\gamma}_{B_x}$ and one of the images must be contained in the sum of the other two, which contradicts the previous assumption.

Now construct a graph with vertex set $\mathcal{B}$ and let $B_1$ and $B_2$ be connected in this graph if images of $\bar{\gamma}_{B_1}$ and $\bar{\gamma}_{B_2}$ coincide. By the previous observations, every vertex $B \in \mathcal{B}$ has degree at most 2: no other curve image may be contained in the image of $\bar{\gamma}_B$ and at every endpoint the image of $\bar{\gamma}_B$ can coincide with at most one other curve. Therefore our graph consists only of paths and loops. By previous observations, locally $\bar{K}$ looks like one curve, so in total it is a finite sum of curves (and some of them may be closed, i.e., they may form images of circles).

What is left to prove is that for every curve in $\bar{K}$ its arc-length parametrization has locally $342\omega$-continuous derivative. However, note that if $x, y \in \bar{K}$, $|x-y| < r^*$, then $\bar{K}$ in the neighborhood of $x, y$ is a subset of $\bar{\gamma}_{B_x}$, and this curve has $342\omega$-continuous derivative.

We conclude this section with the following lemma concerning the total length of all constructed curves.

**Lemma 3.27.** *For any $\varepsilon > 0$, by taking sufficiently small $r^*$ and by taking the family $\mathcal{B}$ more carefully in the above construction, the total length of all curves can be bounded by $5\mathcal{H}_1(K) + \varepsilon$, where $\mathcal{H}_1$ is the one-dimensional Hausdorff measure.*

*Proof.* Using definition of $\mathcal{H}_1$, let $\mathcal{B}_1$ be a set of open balls of diameters smaller than $\frac{1}{20}r_0$ such that they cover $K$ and

$$\sum_{B \in \mathcal{B}_1} \operatorname{diam}(B) < \mathcal{H}_1(K) + \frac{1}{12}\varepsilon.$$

We may assume that every ball in $\mathcal{B}_1$ contains a point from $K$. For every $B \in \mathcal{B}_1$ we take a ball $B'$ such that $B \subset B'$, the center of $B'$ belongs to $K$ and $B'$ has diameter at most twice as large as the diameter of $B$. Let $\mathcal{B}_2 = \{B' : B \in \mathcal{B}_1\}$; then every $B' \in \mathcal{B}_2$ has diameter smaller than $\frac{1}{10}r_0$, has center in $K$ and the family $\mathcal{B}_2$ covers $K$. Moreover

$$\sum_{B' \in \mathcal{B}_2} \operatorname{diam}(B') < 2\mathcal{H}_1(K) + \frac{1}{6}\varepsilon.$$

Since $K$ is a compact set, we may assume that $\mathcal{B}_2$ is finite. Let $r^* = \frac{1}{2}\min_{B' \in \mathcal{B}_2} \operatorname{diam}(B')$ and let $\mathcal{B} = \{\operatorname{cl}(2B') : B' \in \mathcal{B}_2\}$. The set $\mathcal{B}$ is a set of closed balls of radii at least $2r^*$ and smaller than $\frac{1}{10}r_0$ and $\{\operatorname{int}\frac{1}{2}B : B \in \mathcal{B}\}$ covers $K$. Moreover

$$\sum_{B \in \mathcal{B}_2} \operatorname{diam} B < 4\mathcal{H}_1(K) + \frac{1}{3}\varepsilon.$$

Since we have chosen $r^*$, we may remove now the set $K_{\text{lonely}}$ from the set $K$. Note that the set $K_{\text{lonely}}$ can be covered with segments of total length at most $\frac{1}{2}\varepsilon$.

We now construct $\bar{K}$ using our new family $\mathcal{B}$. By Point 1 of Theorem 3.20, curve $\gamma_B$ for $B \in \mathcal{B}$ has length smaller than $1.161\,\mathrm{diam}(B)$. Therefore $\bar{K}$ has total length of at most

$$\frac{1}{2}\varepsilon + 1.161 \cdot (4\mathcal{H}_1(K) + \frac{1}{3}\varepsilon) < 5\mathcal{H}_1(K) + \varepsilon.$$

□

# Chapter 4

# Counter-examples for Jones-style conditions

In this section we give two counter-examples that show that any Jones-like conditions would rather not work in the field of curves with regular derivatives. Intuitively, these curves cannot *turn* fast, and therefore the condition should involve *every* point in the set $K$, not *every but the set of measure* 0.

In this section we work only in $\mathbb{R}^2$, i.e., we are not using any high-dimensional tricks. To simplify the notation, we would treat $\mathbb{R}^2$ as the complex field $\mathbb{C}$, for example $iv$ is vector $v$ rotated by $90°$ counterclockwise.

Let us recall the main techniques used by Jones [6]. Assume we have compact set $K \subset \mathbb{R}^2$. Given a square $Q$ with sidelength $l(Q)$ by $S_K(Q)$ we denote one of the most narrow strips covering $K \cap 3Q$ and by $\beta_K(Q)$ we denote the ratio width$(S_K(Q))/l(Q)$.

The main result of Jones [6] is that a compact set $K \subset \mathbb{R}^2$ is a subset of a rectifiable curve iff the following sum is finite:

$$\beta^2(K) = \sum_{Q \text{ diadic}} \beta_K^2(Q) l(Q).$$

Our examples show that measuring $\beta_K(Q)$ does not allow to decide whether $K$ can be covered by a more regular curve.

Let $\omega : [0, +\infty) \to [0, +\infty)$ be a concave non-decreasing function with $\omega(0) = 0$. We assume that $\omega(1) < 1$; otherwise we may need some rescaling in the examples. We are about to give two compact sets in $\mathbb{R}^2$ that cannot be covered by a finite family of disjoint finite-length curves with $\omega$-continuous derivative (we call such curves *regular curves* for short). However, in both examples values $S_K(Q)$ are small and not informative. Note that both sets satisfy Jones condition and therefore they are subsets of rectifiable curves.

Moreover, the set from the second example has two more properties. First, it can be covered with four regular curves, but with an intersection, showing that the condition that curves are disjoint in our result is essential. This example shows that the question, whether a compact set $K$ can be covered with a finite sum of regular curves, but not necessarily disjoint, is significantly different than the question this paper answers.

Second, if one applies the Jones's algorithm for constructing a rectifiable curve covering the set $K$ from the second example, the obtained curve will have infinite number of turns by $90°$ and therefore cannot be easily smoothed to a curve with a regular derivative and finite length.

We are not going to give all details of the examples, but to we want to give an intuition why Jones's approach breaks down here.

## 4.1. Vertical strokes example

Let

$$K := \{(0,0)\} \cup \{[(2^{-n},0),(2^{-n},2^{-n}\omega(2^{-n}))] : n \in \mathbb{Z}_+\}.$$

This set consists of countably many vertical strokes of length $2^{-n}\omega(2^{-n})$ that converge to $(0,0)$. It is obviously compact, but for every $n$ the triangle $(0,0),(2^{-n},0),(2^{-n},2^{-n}\omega(2^{-n}))$ has diameter smaller than $2 \cdot 2^{-n}$ and right angle at vertex $(2^{-n},0)$. Therefore, by Theorem 1.1, it cannot be covered by a finite number of disjoint *regular* curves.

Let us understand, not using Theorem 1.1, why it cannot be covered. Assume, by contradiction, that it is covered by a finite family $\{\gamma_i\}_{1\leqslant i\leqslant N}$. By some easy measure arguments, for every point $(2^{-n},0)$ there must be a curve $\gamma_i$ tangent to stroke $[(2^{-n},0),(2^{-n},2^{-n}\omega(2^{-n}))]$ at this point. Therefore there must be one curve $\gamma_{i_0}$ that is tangent to infinite number of strokes at their bottom endpoints. Note that between two such places curve must turn by $180^\circ$ and, since $\gamma_{i_0}$ has $\omega$-continuous derivative and it has finite length, it cannot happen infinitely many times.

Let us now look at $K$ from the point of view of numbers $\beta_K(Q)$. If there is only one stroke in the $3Q$, then $\beta_K(Q) = 0$. Apart from diadic squares with corner at $(0,0)$, for sufficiently small diadic cubes this is the case. However, for diadic squares $Q$ such that $3Q$ intersects more than one stroke we have $\beta_K(Q) \leqslant c\omega(l(Q))$ for some universal constant $c$, which means that these values are comparable to the values $\beta_K(Q)$ for a *regular* curve. Therefore numbers $\beta_K(Q)$ yield no information in this case.

## 4.2. Snail example

Let $a_0 = 1$ and $a_{n+1} = \omega(a_n)a_n$. By the assumption $\omega(1) < 1$, the sequence $a_n$ is decreasing and since $a_n \leqslant \omega(1)^n$ it converges to 0. Let $z_0 = 0 \in \mathbb{C}$ and $z_{n+1} = z_n + a_n i^n$. Then points $z_n$ form a snail-like structure and they converge to some point $z = \lim_{n\to\infty} z_n$. Let $K = \{z\} \cup \{z_n : n \geqslant 0\}$.

Since angle $\measuredangle(z_{n-1}, z_n, z_{n+1})$ is a right angle, and $\operatorname{diam}\{z_{n-1}, z_n, z_{n+1}\} < a_{n-1} + a_n$, by Theorem 1.1, set $K$ is not a subset of a finite family of disjoint *regular* curves.

Moreover, if we construct the *rectifiable* curve through $K$, using the algorithm by Jones [6], we obtain a spiral, i.e., we connect $z_n$ to $z_{n+1}$. The Jones's algorithm takes insight into smaller and smaller cubes consecutively; in fact, it notices points $z_n$ one-by-one (since $z_{n+1}$ is much closer to $z_n$ than $z_n$ is to $z_{n-1}$) and connects them to the previously obtained curve. Therefore this construction ends up with an infinite number of $90^\circ$ turns.

On the other hand, precise calculations show that we can pass one curve through each of the sets $K_l = \{z\} \cup \{z_{4k+l} : k \in \mathbb{N}\}$ for $l = 0, 1, 2, 3$. However, these curves intersect at $z$ and, since this is a limit of the whole sequence $z_n$, the intersection there is unavoidable.

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