Selected topics in graph theory

21.01.2019 — homework, set 4

Problem 1. For a nonnegative integer g, let $f(g)$ be the maximum possible chromatic number of a graph embeddable in a surface of Euler genus at most g. Show that $f(g) = \Theta(\sqrt{g})$.

Problem 2. Show that for every integer $g \geq 0$ there exists an integer k such that every graph G that admits an embedding in a surface of Euler genus at most g with edge-width at least k is 7-colorable.

Problem 3. Let G be a graph such that every subgraph of G has average degree strictly less than 3.

1. Show that G has either a vertex of degree at most 1 or a vertex of degree 2 with a neighbor of degree at most 5.
2. Show that there exists a coloring f of $V(G)$ with six colors such that:
    - for every color i, $f^{-1}(i)$ is an independent set in G (i.e., f is a proper coloring of G), and
    - for every two colors i, j, $f^{-1}(\{i, j\})$ is a forest.

Remark. This time, most of the problems above are easily googleable, even in stronger forms (e.g., the second problem is true even with 5 colors). They are meant as exercises in graph theory, not in googling. Thus, we make the following rule: you should not attempt googling them and you cannot use any result on colorings of graphs that was not presented at the lecture or tutorials.