Selected topics in graph theory

19.11.2019 — homework, set 2

Problem 1. A set $X \subseteq V(G)$ is *irredundant* if for every $x \in X$ there exists $y \in N[x]$ such that $y \notin N[x']$ for every $x' \in X \setminus \{x\}$. Prove that one can in $2^{\mathcal{O}(\sqrt{k})} n^{1000}$ time check if a given n-vertex planar graph G contains an irredundant set of size at least k.
Remark. If your solution uses some dynamic programming algorithm as a subroutine, there is no need to describe the dynamic programming algorithm in full detail. Just a description of the table and the meaning of the cells will suffice.

Problem 2. We say that a multigraph H *immerses* into a multigraph G if we can find for every $h \in V(H)$ a distinct vertex $v_h \in V(G)$ and for every edge $gh \in E(H)$ a path P_{gh} in G with endpoints v_g and v_h such that each path P_{gh} does not pass through any vertex $v_{h'}$, $h' \in V(H) \setminus \{g, h\}$ and the paths $\{P_e \mid e \in E(H)\}$ are pairwise edge-disjoint.

A *fat path* is a multigraph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edges of the form $v_i v_{i+1}$, $1 \leq i < n$, and arbitrary multiplicity. Prove that the set of all fat paths is well quasi-ordered by the immersion relation defined above.

Problem 3. Consider the following variant of the cops and robber game. First, the cops place k cops on vertices of the graph. Second, the robber chooses his starting vertex. Then the cops and robbers take turns. In cops' turn, each cop can stay on his vertex or move to one of the adjacent vertices. Similarly, in robber's turn, the robber can stay in place or move to adjacent vertex. The cops win if one of the cops moves to a vertex occupied currently by the robber. There can be multiple cops at the same vertex.

Prove that on planar graphs 1000 cops can catch the robber.