Selected topics in graph theory

January 21, 2019 — topics for the oral exam

Below you can find partition topics for the oral exam. Recall that the oral exam results in a modifier from the set $\{-1.0, -0.5, 0.0, +0.5, +1.0, +1.5\}$ to you final grade.

At the exam, you sample a number of questions from the basic topics. Furthermore, you pick one of the advanced reasonings to show. For the basic topics, we prefer having a general overview of all topics rather than excellent knowledge on a proper subset of the topics. In particular,

- fair answers to all basic questions will result in a nonnegative modifier;
- near-zero knowledge on one of the sampled question will result in a nonpositive modifier.

Embeddings

1. Combinatorial embeddings: definitions. Euler genus of an embedding and of a graph.
2. Contractible and noncontractible cycles. One- and two-sided cycles.
3. Cutting along a cycle and its impact on the genus of the embedding (with proofs).
4. Edge-width of an embedding. Large-edge-width (LEW) embeddings: definition, fact that every LEW embedding is a minimum genus embedding, uniqueness of LEW embeddings for 3-connected graphs.
5. Facewidth of an embedding: definition, connection to edge-width.
6. Algorithm to find a shortest noncontractible cycle.

Graph Minors

1. WQO: definitions, basic properties. The lemma that finite subsets of a WQO form a WQO, with a proof.
2. Relation of being a minor and a topological minor, with examples.
3. Kruskal's theorem: statement, examples.
4. Threewidth. Definition, examples, cops and robber games, basic properties.
5. Brambles. Statement of the duality theorem between brambles and treewidth. Proof of the inequality in one direction: that a bramble of large order implies large treewidth.
6. Excluded grid theorem, both in the general and planar setting.
7. Algorithmic applications of the excluded grid theorem, in particular in planar graphs.
8. Robertson-Seymour theorem. Statement and algorithmic applications.
9. Structural theorem for graphs excluding a fixed minor. Definitions and statement.

Expanders

1. Edge and vertex expansion: definitions.
2. Adjacency matrix and its basic spectral properties (symmetry, eigenvalues, structure of eigenvectors) for d-regular multigraphs.
3. How being disconnected and bipartite is visible in the spectrum.
4. Tensor and cartesian product of graphs, with the corresponding changes in the spectrum.
5. Inequalities binging expansions and spectrum.
6. Zig-zag product: definitions and spectral properties.
7. Construction of an arbitrarily large expander using spectral properties of graph products as black-boxes.
8. One "bare hands" construction of an expander (e.g., Problem 65).
9. Random walks in expanders. Randomness reductions in one- and two-sided error algorithms.
10. Alon-Boppana theorem.
11. $L = SL$: statement, hint how to prove it.

Colorings

Warning: This part has been modified after the lecture on 21.01, to be inline with the presented material. This part is not mandatory for those that take the oral exam on 18.01.

1. Colorings, list colorings. $\chi(G)$, $\chi'(G)$, $ch(G)$, $ch'(G)$. Chromatic number and degeneracy.
2. Brooks' theorem and Vizing's theorem: statements.
3. 5-color theorem with proof.
4. Thomassen's theorem about list coloring of planar graphs: statement and sketch of the proof.
5. A nontrivial example of discharging.

Advanced reasonings: pick one to present

1. Proofs of two facts about LEW embeddings: every LEW embedding is a minimum genus embedding and uniqueness of LEW embeddings for 3-connected graphs.
2. Proof of the Kruskal's theorem.
3. Proof of the duality theorem between brambles and treewidth.
4. Sketch of the proof of the excluded grid theorem in general graphs.
5. Proof of the excluded grid theorem in planar graphs.
6. Rough sketch of the proof of the Robertson-Seymour theorem.
7. Proofs of all four inequalities between edge/vertex expansion and the corresponding spectral gap.

8. Proofs of the spectral properties of the zig-zag product.
9. Proof that $L = SL$ assuming spectral properties of various graph products as black-boxes.
10. Proof of the Alon-Boppana theorem.
11. Proof that planar graphs are 5-list-colorable.
12. Proof of the Vizing's theorem.