

## Minors and Iterative Compression

### 1 Theory of graph minors

We define the following operations on graphs:

$-e$  - edge removal

$-v$  - vertex removal (with removal of incident edges)

$/e$  - edge contraction (with collapse of arising self-loops and multiple edges)

Here we should note that in some situations we won't collapse self-loops and multiple edges created by edge contraction.

**Definition 1.** Relation of *being a minor* is defined as follows.

Graph  $H$  is a minor ( $\leq_m$ ) of graph  $G$  iff  $H$  is obtainable from  $G$  by a sequence of aforementioned operations ( $-e$ ,  $-v$ ,  $/e$ ).

**Example 2.** Given the lattice graph  $5 \times 5$ , graph  $K_4$  is its minor as we can show a sequence of operations that transform the first one into the second one. However  $K_5$  isn't its minor as  $-e$ ,  $-v$  and  $/e$  preserve planarity.

**Definition 3.** Relation of *being a minor* can also be defined equivalently this way.

$H$  is a minor of  $G$  iff for each vertex  $u$  in  $H$  exists a set  $V_u$  of vertices from  $G$ , such that

1.  $\forall_{u,u' \in V(H)} u \neq u' \rightarrow V_u \cap V_{u'} = \emptyset$
2.  $\forall_{u \in V(H)} G[V_u]$  is connected
3.  $\forall_{u,u' \in V(H)} (u, u') \in E(H) \rightarrow E_G(V_u, V_{u'}) \neq \emptyset$

Such family of sets is called a model of  $H$  in  $G$ .

We can see that this definition implies the earlier one. We can contract each set  $V_u$  into singular vertices and declare a mapping from these new vertices to vertices of  $H$ .

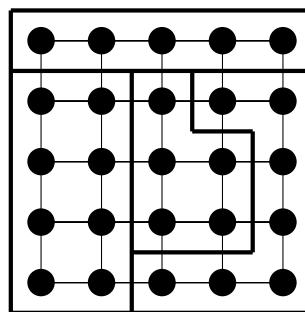


Figure 1: Model of  $K_4$  in a  $5 \times 5$  lattice.

**Definition 4.** Relation of *being a topological minor*

Graph  $H$  is a topological minor ( $\leq_{top}$ ) of graph  $G$  iff for each vertex  $u$  in  $H$  exists a vertex  $v_u$  in  $G$  such that for every pair  $u, u' \in E(H)$  exists a path  $p_{uu'}$  in  $G$  which connects  $v_u$  with  $v_{u'}$  and all such paths are vertex disjoint (except for the endpoints).

**Example 5.** Given a lattice graph  $5 \times 5$ ,  $K_4$  is its topological minor as we can show a vertex mapping and appropriate vertex disjoint paths.

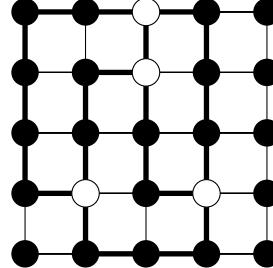


Figure 2:  $K_4$  as a topological minor in a  $5 \times 5$  lattice.

**Definition 6.** Relation of *being a topological minor* can be alternatively defined the following.

$H \leq_{top} G \iff H$  can obtained from  $G$  using  $-e$ ,  $-v$  and  $/uv$ , but we allow for edge contraction only if  $\deg(u) = 2$  or  $\deg(v) = 2$ .

Edge contraction with this constraint corresponds to contraction of paths selected in the previous definition.

Let's take a look at the complexity of checking if a graph  $H$  is a minor of  $G$ .

From now on we will assume that  $|V(H)| = k$  and  $|V(G)| = n$ . When checking if  $H \leq_m G$  we can check all possible reductions, which there are  $(k+1)^n$ . However we will find a better FPT solution.

We can check  $H \leq_{top} G$  in  $\mathcal{O}(n^k \cdot T(DPP))$ , where  $T(DPP)$  is the time complexity of the Disjoint paths problem. This is because, at first we can guess  $k$  vertices. Then we just have to solve the problem of joining these vertices with disjoint paths. This problem is known as the DISJOINT PATHS PROBLEM.

DISJOINT PATHS PROBLEM (DPP)

**Input:** A graph  $G$  and sequence of  $k$  pairs  $(s_i, t_i)$

**Question:** Sequence of vertex disjoint paths  $p_i$  where  $i$ -th path joins  $s_i$  with  $t_i$

We won't prove the next few facts.

**Theorem 7.** DIRECTED DPP (i.e. input graph is directed) is NP-hard even for  $k = 2$ [6].

There is an XP ( $n^{\mathcal{O}(k)}$ ) algorithm for DIRECTED DPP in planar graphs[11], but we don't know if this problem is in FPT.

**Theorem 8.**  $H \leq_m G$  can be checked in  $\mathcal{O}(f(k) \cdot n^3)$ [9], so this problem is FPT.

$f(k)$  in this theorem is very big - some kind of power tower of height  $k$ . This result is achieved by basing on another theorem:

**Theorem 9.** DPP can be solved in  $\mathcal{O}(f(k) \cdot n^{\mathcal{O}(1)})$ [9].

Again  $f(k)$  is very big, but for planar graphs we know that  $f(k) = 2^{2^k}$  [1].

**Theorem 10.**  $H \leq_{top} G$  is also FPT( $k$ ) with  $f(k)n^3$  [7].

$f(k)$  is similar to ones above.

### 1.1 Ordering properties of minors

Both  $\leq_m$  and  $\leq_{top}$  form an ordering. Also  $H \leq_{top} G \rightarrow H \leq_m G$ .

**Definition 11.**  $(\Omega, \leq)$  is a *quasi ordering* iff

- (i)  $a \leq b$  and  $b \leq c \rightarrow a \leq c$
- (ii)  $a \leq a$

However we **don't** require the third property of well ordering, that is if  $a \leq b$  and  $b \leq a$  then  $a = b$ .

**Definition 12.**  $(\Omega, \leq)$  is a *well quasi ordering* iff

$$\forall_{x_1, x_2, \dots} \exists_{i_1, i_2, \dots} x_{i_1} \leq x_{i_2} \leq \dots \wedge i_1 < i_2 < \dots$$

equivalently we can say:

There are no infinite antichains or infinite strictly descending sequences.

The question is whether  $\leq_m$  is WQO. We can't have infinite a strictly descending sequence with relation to  $\leq_m$ . This is because if  $G \leq_m H$  and  $G \neq H$  then  $|E(G)| + |V(G)| \leq |E(H)| + |V(H)|$  and with every step this number has to become lower until we get an empty graph after finite number of steps. There are also no infinite antichains with relation to  $\leq_m$ , but we won't prove it here.

**Theorem 13.**  $\leq_m$  is WQO [10]

However (again without proof)

**Theorem 14.**  $\leq_{top}$  isn't WQO

Now we will show an algorithm for checking planarity of a graph that uses the WQO property of  $\leq_m$ . Let  $\mathcal{P}$  be the family of all planar graphs. We know that if  $H \leq_m G$  and  $G$  is planar then  $H$  is planar as well. Therefore  $\mathcal{P}$  is a class closed under taking minors, i.e. if a graph is in  $\mathcal{P}$  then all its minors are planar as well.

**Definition 15.**  $\mathcal{K}$  - minimal non-planar graphs

$$K \in \mathcal{K} \iff K \notin \mathcal{P} \wedge \hat{K} \leq_m K \rightarrow \hat{K} \in \mathcal{P}$$

We can see that  $\mathcal{K}$  is an antichain and as  $\leq_m$  is WQO we know that this set is finite. Also, we know that graph  $G$  is planar iff  $\neg \exists_{K \in \mathcal{K}} K \leq_m G$ . Therefore this algorithm is correct

```
function ISPLANAR(G)
    for all  $K \in \mathcal{K}$  do
        if minor( $K, G$ ) then
            return false
        end if
    end for
```

```

end for
return true
end function

```

As  $\mathcal{K}$  is fixed this algorithm runs in  $\mathcal{O}(n^3)$ , as checking  $\text{minor}(K, G)$  is FPT( $|K|$ ), but possible graphs are fixed (also,  $|K|$  is constant), and we have only fixed amount of these graphs.

Of course we know that  $\mathcal{K} = \{K_{3,3}, K_5\}$ . However, this algorithm works for any class of graphs closed under taking a minor (e.g. planar graphs on a sphere with three handles).

Theory of minors will return in later lectures.

## 2 Iterative compression

### FEEDBACK VERTEX SET (FVS)

**Input:** A graph  $G$  and number  $K$ .

**Question:** If there is set of vertices  $X \subseteq V(G)$  such that  $|X| \leq k$  and  $G \setminus X$  is a forest.

We will create an algorithm for this problem of time complexity  $\mathcal{O}(5^k \cdot n^{\mathcal{O}(1)})$  using the iterative compression technique.

Let  $V(G) = \{v_1, \dots, v_n\}$  and  $G_i = G[\{v_1, \dots, v_i\}]$ . For  $G_k$  we can easily find solution  $X_k = V(G_k) = \{v_1, \dots, v_k\}$ . Given a solution  $X_{i-1}$  for the graph  $G_{i-1}$  with the property  $|X_{i-1}| \leq k$ , we can create a solution  $X_i = X_{i-1} \cup \{v_i\}$  for  $G_i$ . Of course  $|X_{i-1} \cup \{v_i\}| \leq k + 1$ .

Now if we could compress this solution, we could solve FVS by iteratively expanding and compressing solution.

### COMPRESSION FVS

**Input:** Graph  $G$ , number  $k$  and set  $Z$  of vertices from  $G$  where  $|Z| = k + 1$  and  $G \setminus Z$  is a forest.

**Question:** What is the set of vertices  $X \subseteq V(G)$ ,  $|X| \leq k$  such that  $G \setminus X$  is a forest.

To solve this problem, at first we guess  $X \cap Z$ . There are  $2^{k+1}$  possibilities and in each we get a set  $X_Z \subseteq Z$ . Then we feed  $G \setminus X_Z$ ,  $k - |X_Z|$  and  $Z \setminus X_Z$  to another subproblem — DISJOINT COMPRESSION FVS.

### DISJOINT COMPRESSION FVS

**Input:** Graph  $G'$ , number  $k'$  and set  $Z'$  of vertices from  $G'$  where  $|Z'| = k' + 1$  and  $G' \setminus Z'$  is a forest.

**Question:** Is there  $X \subseteq V(G') \setminus Z'$  such that  $|X| \leq k'$  and  $G' \setminus X$  is a forest.

First, we have to notice that if  $G'[Z']$  isn't a forest then this problem has no solution. Otherwise we will simplify this problem by considering the following possibilities, given  $v \in V(G') \setminus Z'$

- $\deg(v) = 1$  or  $\deg(v) = 0$ . In this situation  $v$  is of no concern to us as it won't be in any cycle. We can ignore  $v$  removing it from  $G'$ .
- $\deg(v) = 2$ . If none of its neighbours are in  $Z'$ , then if it were in a solution, we could just as well take any of its neighbours, therefore we contract it with its neighbour. If  $v$  has both of its neighbours in  $Z'$ , we just let it be. If it has one neighbour in  $Z'$ , then, as we don't want to forget about some cycles, we will contract  $v$  with its neighbour in  $Z'$ . If it creates multiple edges that means that we had a triangle between the contracted vertex  $v$ , some vertex in  $Z'$  and some other vertex  $u$ . We can't remove a vertex from  $Z'$ , so we have to remove  $v$  or  $u$ .

As  $v$  had no other neighbours (as it had degree 2), we can greedily add  $u$  to the solution and remove it from the graph.

Now if  $v$  is a leaf of the forest  $G' \setminus Z'$  (a vertex of degree at most 1 in  $G' \setminus Z'$ ) then there are at least two edges  $e_1, e_2$  leading to  $Z'$ .

```

if  $e_1, e_2$  lead to the same connected component of  $G'[Z']$  then
    remove  $v$ 
     $k' \leftarrow k' - 1$ 
else
    branch
        remove  $v$ 
         $k' \leftarrow k' - 1$ 
    else branch
         $Z' \leftarrow Z' \cup \{v\}$ 
    end branch
end if
```

As we can see with every branch the number of vertices decreases by one or number of connected components in  $G'[Z']$  decreases at least by one. We know that at the beginning  $G'[Z']$  has  $\leq k' + 1$  connected components. If we define a potential  $\phi = k' + (cc(G'[Z']) - 1)$  (where  $cc(G')$  is number of connected components of a graph), then we know that  $\phi \leq 2k$  and it decreases by one with every branch. Therefore, complexity of this algorithm is  $\mathcal{O}^*(2^{2k'})$ . This procedure will be called for every guessed subset of  $Z'$  from COMPRESSION FVS, which there are  $\mathcal{O}^*(2^{k'+1})$ .

However, when we call DISJOINT COMPRESSION FVS from COMPRESSION FVS with  $k' = k - k_1$ , where  $k_1 = |X_Z|$ . However, we know that the number of connected components in  $Z'$  is limited by  $k'$ . Now we can think in terms of  $k$  about complexity of COMPRESSION FVS, which is equal to

$$\sum_{k_1=0}^k \binom{k+1}{k_1} 4^{k-k_1} = \sum_{k_1=0}^k \binom{k+1}{k_1} 4^{k-k_1} \cdot 1^{k_1-1} = 5^{k+1}$$

This algorithm was first presented in 2008[4].

There is also an algorithm  $\mathcal{O}^*((1 + 2\sqrt{2})^k)$ [3], as well as a  $\mathcal{O}^*(3^k)$ [5] randomized algorithm that will be shown in later lecture.

Also, there is an old  $\mathcal{O}^*(4^k)$ [2] randomized algorithm:

1. Remove vertices with degree 0 or 1.
2. For all vertices with degree 2 contract them. If that creates self-loops remove that vertex and add it to the solution. If that creates multiple edge, branch and remove one of the ends adding it to the solution.
3. Randomly choose an end of a random edge, add it to the solution and remove it from the graph.
4. Repeat whole procedure until the graph becomes a forest or the solution becomes too big.

**Lemma 16.** *If  $\min \deg(G) \geq 3$ , then for all  $X \subseteq V$ , where  $G[V \setminus X]$  is a forest, # of edges with at least one end in  $X$  is greater or equal  $\frac{|E|}{2}$ .*

If this lemma is true we will have  $\frac{1}{2}$  probability of choosing a good edge and then probability  $\frac{1}{2}$  of choosing the right vertex.

### Proof

Let's assume that  $V \setminus X$  has  $a$  vertices. If  $a \leq \frac{|E|}{2}$  then of course the lemma is true, so we assume  $a > \frac{|E|}{2}$ . If the minimal degree is 3 then there are  $3a$  ends of edges and at most  $2a$  of them are used. Therefore  $a$  edges have to come out of  $X$ , so at least  $\frac{|E|}{2}$  edges come out of  $X$ .

Iterative compression was originally created for solving ODD CYCLE TRANSVERSAL problem[8], however we won't describe how it can be used to solve this problem here.

ODD CYCLE TRANSVERSAL

**Input:** Graph  $G$ , number  $k$ .

**Question:** Set  $X \subseteq G$  where  $|X| \leq k$  and  $G \setminus X$  is bipartite graph.

## References

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