Forbidden Induced Subgraphs for ~~Bounded Shrub-Depth~~ Bounded SC-Depth with Applications in Logic

Nikolas Mählmann, University of Warsaw

LOGALG 2025

The order-property

Fix: logic $\mathcal{L} \in \{\mathrm{FO}, \mathrm{MSO}\}$, $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, graph class $\mathcal{C}$

φ has the *order-property* on $\mathcal{C}$, if for every $\ell \in \mathbb{N}$ there is a graph $G \in \mathcal{C}$ and a sequence $\bar{a}_i, \ldots, \bar{a}_\ell$ of tuples of vertices of G, such that for all $i, j \in [\ell]$

$$G \models \varphi(\bar{a}_i, \bar{a}_j) \quad \Leftrightarrow \quad i \leq j.$$

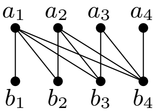
The order-property

Fix: logic $\mathcal{L} \in \{\mathrm{FO}, \mathrm{MSO}\}$, $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, graph class $\mathcal{C}$

φ has the *order-property* on $\mathcal{C}$, if for every $\ell \in \mathbb{N}$ there is a graph $G \in \mathcal{C}$ and a sequence $\bar{a}_i, \ldots, \bar{a}_\ell$ of tuples of vertices of G, such that for all $i, j \in [\ell]$

$$G \models \varphi(\bar{a}_i, \bar{a}_j) \quad \Leftrightarrow \quad i \leq j.$$

Example FO: $\varphi(x, y) :=$ "$N(x) \supseteq N(y)$"



$a_1 \prec_\varphi a_2 \prec_\varphi a_3 \prec_\varphi a_4$

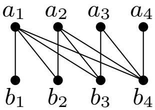
The order-property

Fix: logic $\mathcal{L} \in \{\mathrm{FO}, \mathrm{MSO}\}$, $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, graph class $\mathcal{C}$

φ has the *order-property* on $\mathcal{C}$, if for every $\ell \in \mathbb{N}$ there is a graph $G \in \mathcal{C}$ and a sequence $\bar{a}_i, \ldots, \bar{a}_\ell$ of tuples of vertices of G, such that for all $i, j \in [\ell]$

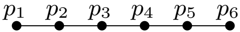
$$G \models \varphi(\bar{a}_i, \bar{a}_j) \quad \Leftrightarrow \quad i \leq j.$$

Example FO: $\varphi(x, y) :=$ "$N(x) \supseteq N(y)$"



$$a_1 \prec_\varphi a_2 \prec_\varphi a_3 \prec_\varphi a_4$$

Example MSO: $\psi(x_1 x_2, y_1 y_2) :=$ "the interval $[x_1, x_2]$ contains $[y_1, y_2]$ "



$$p_1 p_6 \prec_\psi p_2 p_6 \prec_\psi p_3 p_6 \prec_\psi \cdots \prec_\psi p_6 p_6$$

Stability

For a logic $\mathcal{L}$, a graph class $\mathcal{C}$ is *$\mathcal{L}$-stable*, if no $\mathcal{L}$-formula has the order-property on $\mathcal{C}$.

Stability

For a logic $\mathcal{L}$, a graph class $\mathcal{C}$ is *$\mathcal{L}$-stable*, if no $\mathcal{L}$-formula has the order-property on $\mathcal{C}$.

FO-stable:

- planar graphs
- map graphs
- bounded tree-width
- bounded degree

FO-unstable:

- half-graphs
- the class of all 1-subdivided graphs

Stability

For a logic $\mathcal{L}$, a graph class $\mathcal{C}$ is *$\mathcal{L}$-stable*, if no $\mathcal{L}$-formula has the order-property on $\mathcal{C}$.

FO-stable:

- planar graphs
- map graphs
- bounded tree-width
- bounded degree

FO-unstable:

- half-graphs
- the class of all 1-subdivided graphs

MSO-stable:

- bounded tree-depth
- bounded shrub-depth

MSO-unstable:

- paths

Stability

For a logic $\mathcal{L}$, a graph class $\mathcal{C}$ is *$\mathcal{L}$-stable*, if no $\mathcal{L}$-formula has the order-property on $\mathcal{C}$.

FO-stable:

- planar graphs
- map graphs
- bounded tree-width
- bounded degree

FO-unstable:

- half-graphs
- the class of all 1-subdivided graphs

MSO-stable:

- bounded tree-depth
- bounded shrub-depth

MSO-unstable:

- paths

Hereditary FO-stable classes are very well-behaved:
various combinatorial characterizations, fpt FO model checking...

Stability

For a logic $\mathcal{L}$, a graph class $\mathcal{C}$ is *$\mathcal{L}$-stable*, if no $\mathcal{L}$-formula has the order-property on $\mathcal{C}$.

FO-stable:

- planar graphs
- map graphs
- bounded tree-width
- bounded degree

FO-unstable:

- half-graphs
- the class of all 1-subdivided graphs

MSO-stable:

- bounded tree-depth
- bounded shrub-depth

MSO-unstable:

- paths

Hereditary FO-stable classes are very well-behaved:
various combinatorial characterizations, fpt FO model checking...

What about hereditary MSO-stable classes?

Main result

For every hereditary graph class $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ has bounded SC-depth (shrub-depth)

5. $\mathcal{C}$ is MSO-stable

Main result

For every hereditary graph class $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ has bounded SC-depth (shrub-depth)

3. $\exists t$ such that $\mathcal{C}$ excludes all flipped H_t and tP_t

5. $\mathcal{C}$ is MSO-stable

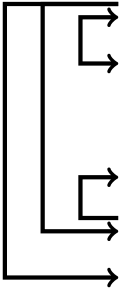
Main result

For every hereditary graph class $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ has bounded SC-depth (shrub-depth)
2. $\mathcal{C}$ is ∞-flip-flat
3. $\exists t$ such that $\mathcal{C}$ excludes all flipped H_t and tP_t
4. $\mathcal{C}$ does not FO-interpret all paths
5. $\mathcal{C}$ is MSO-stable
6. FO and MSO coincide on $\mathcal{C}$

Main result

For every hereditary graph class $\mathcal{C}$, the following are equivalent:



1. $\mathcal{C}$ has bounded SC-depth (shrub-depth)
2. $\mathcal{C}$ is ∞-flip-flat
3. $\exists t$ such that $\mathcal{C}$ excludes all flipped H_t and tP_t
4. $\mathcal{C}$ does not FO-interpret all paths
5. $\mathcal{C}$ is MSO-stable
6. FO and MSO coincide on $\mathcal{C}$

$1 \Leftrightarrow 2$: [Dreier, NM, Toruńczyk; 2024] and [Ossona de Mendez, Pilipczuk, Siebertz; 2022]

$1 \Rightarrow 5$: [Ganian, Hliněný, Nešetřil, Obdržálek, Ossona de Mendez, Ramadurai; 2012]

$1 \Rightarrow 6$: [Gajarský and Hliněný; 2015], [Chen and Flum; 2020] $5 \Rightarrow 4$: [trivial]

Main result

For every hereditary graph class $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ has bounded SC-depth (shrub-depth)
2. $\mathcal{C}$ is ∞-flip-flat
3. $\exists t$ such that $\mathcal{C}$ excludes all flipped H_t and tP_t
4. $\mathcal{C}$ does not FO-interpret all paths
5. $\mathcal{C}$ is MSO-stable
6. FO and MSO coincide on $\mathcal{C}$

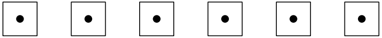
1 $\Leftrightarrow$ 2: [Dreier, NM, Toruńczyk; 2024] and [Ossona de Mendez, Pilipczuk, Siebertz; 2022]

1 $\Rightarrow$ 5: [Ganian, Hliněný, Nešetřil, Obdržálek, Ossona de Mendez, Ramadurai; 2012]

1 $\Rightarrow$ 6: [Gajarský and Hliněný; 2015], [Chen and Flum; 2020]   5 $\Rightarrow$ 4: [trivial]

SC-depth

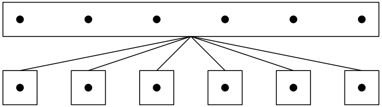
$\mathsf{SC}_0 := \{K_1\}$, $\quad \mathsf{SC}_{k+1} :=$ *set complements* of a disjoint unions of graphs from SC_k.

SC-depth 0 

SC-depth

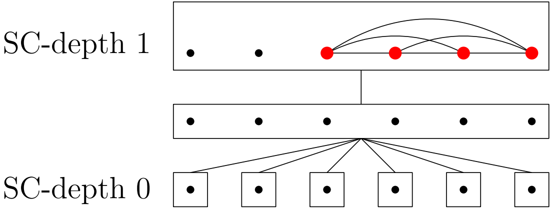
$\mathrm{SC}_0 := \{K_1\}$, $\mathrm{SC}_{k+1} :=$ *set complements* of a disjoint unions of graphs from SC_k.

SC-depth 0



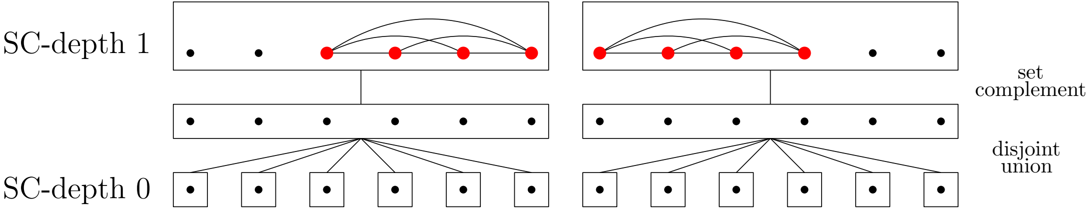
SC-depth

$\mathsf{SC}_0 := \{K_1\}$, $\mathsf{SC}_{k+1} :=$ *set complements* of a disjoint unions of graphs from SC_k.



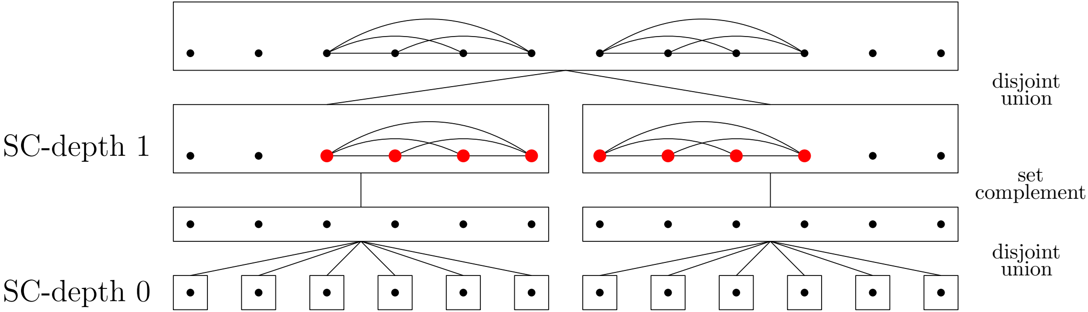
SC-depth

$\mathrm{SC}_0 := \{K_1\}, \quad \mathrm{SC}_{k+1} :=$ *set complements* of a disjoint unions of graphs from SC_k.



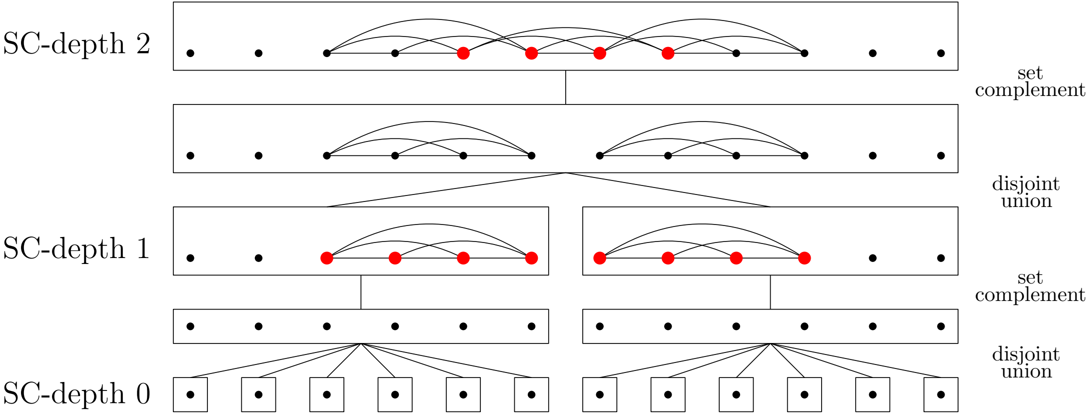
SC-depth

$SC_0 := \{K_1\}, \quad SC_{k+1} :=$ *set complements* of a disjoint unions of graphs from SC_k.



SC-depth

$\mathrm{SC}_0 := \{K_1\}$, $\mathrm{SC}_{k+1} :=$ *set complements* of a disjoint unions of graphs from SC_k.

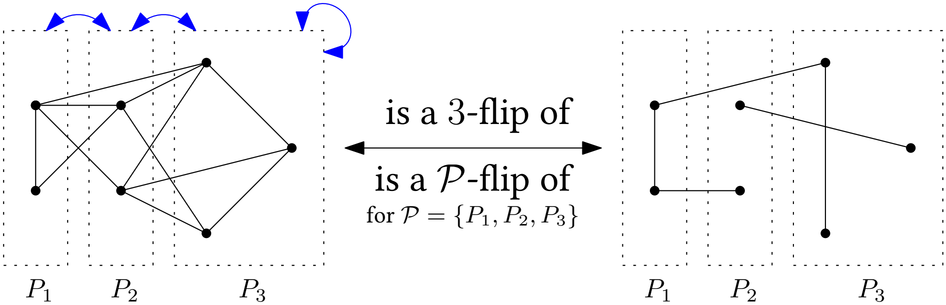


Main result

For every hereditary graph class $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ has bounded SC-depth (shrub-depth)
2. $\mathcal{C}$ is ∞-flip-flat
3. $\exists t$ such that $\mathcal{C}$ excludes all flipped H_t and tP_t
4. $\mathcal{C}$ does not FO-interpret all paths
5. $\mathcal{C}$ is MSO-stable
6. FO and MSO coincide on $\mathcal{C}$

Flips



P_1
P_2
P_3
is a 3-flip of
is a $\mathcal{P}$-flip of
for $\mathcal{P} = \{P_1, P_2, P_3\}$
P_1
P_2
P_3

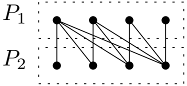
Theorem

A class $\mathcal{C}$ has bounded SC-depth iff there is $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped H_t and all flipped tP_t as induced subgraphs.

Theorem

A class $\mathcal{C}$ has bounded SC-depth iff there is $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped H_t and all flipped tP_t as induced subgraphs.

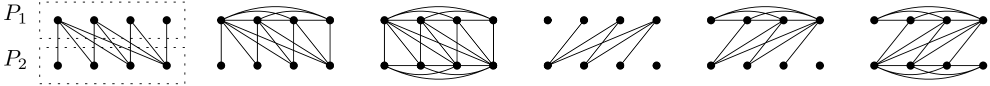
H_t = half-graph of order t;   flipped $H_t = \{P_1, P_2\}$-flip of H_t.



Theorem

A class $\mathcal{C}$ has bounded SC-depth iff there is $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped H_t and all flipped tP_t as induced subgraphs.

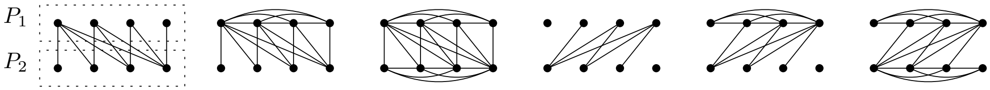
H_t = half-graph of order t; flipped H_t = $\{P_1, P_2\}$-flip of H_t.



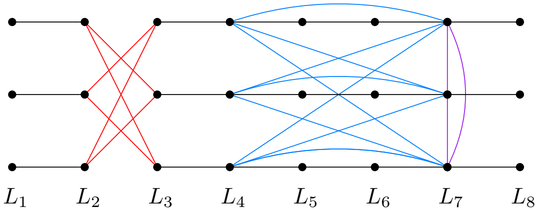
Theorem

A class $\mathcal{C}$ has bounded SC-depth iff there is $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped H_t and all flipped tP_t as induced subgraphs.

H_t = half-graph of order t; flipped H_t = $\{P_1, P_2\}$-flip of H_t.



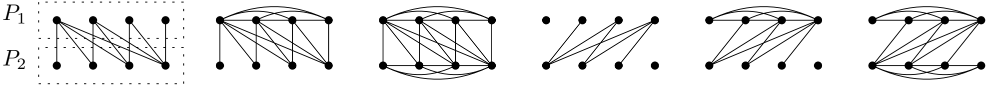
tP_t = disjoint union of t many P_t; flipped tP_t = flip of tP_t respecting *layers*.



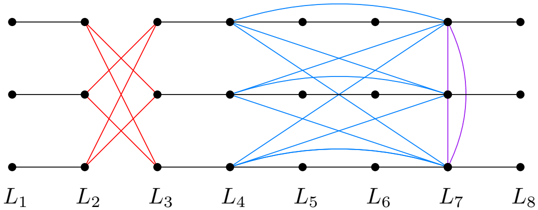
Theorem

A class $\mathcal{C}$ has bounded SC-depth iff there is $t \in \mathbb{N}$ such that $\mathcal{C}$ excludes all flipped H_t and all flipped tP_t as induced subgraphs.

H_t = half-graph of order t;   flipped H_t = $\{P_1, P_2\}$-flip of H_t.



tP_t = disjoint union of t many P_t;   flipped tP_t = flip of tP_t respecting *layers*.

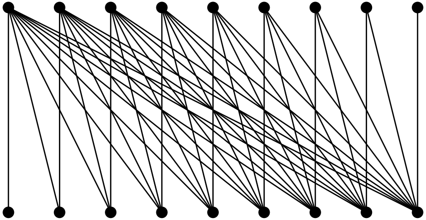


Next up: large flipped H_t and $tP_t \Rightarrow$ large SC-depth

Lemma

Huge flipped H_t or $tP_t \Rightarrow$ large flipped H_s or sP_s in every set complement.

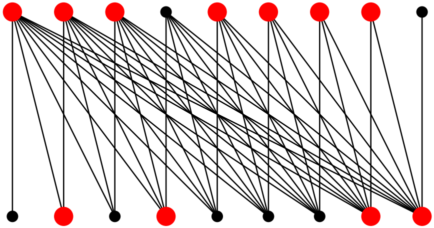
By the pigeonhole principle:



Lemma

Huge flipped H_t or $tP_t \Rightarrow$ large flipped H_s or sP_s in every set complement.

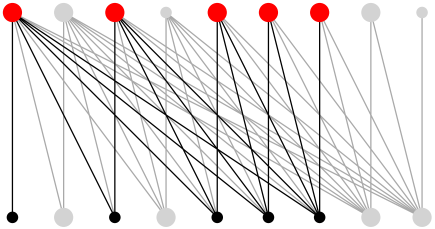
By the pigeonhole principle:



Lemma

Huge flipped H_t or tP_t $\Rightarrow$ large flipped H_s or sP_s in every set complement.

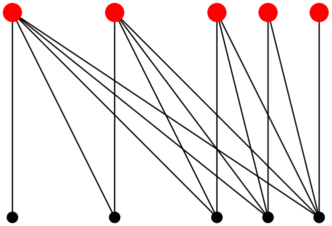
By the pigeonhole principle:



Lemma

Huge flipped H_t or $tP_t \Rightarrow$ large flipped H_s or sP_s in every set complement.

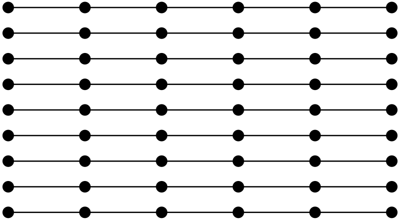
By the pigeonhole principle:



Lemma

Huge flipped H_t or $tP_t \Rightarrow$ large flipped H_s or sP_s in every set complement.

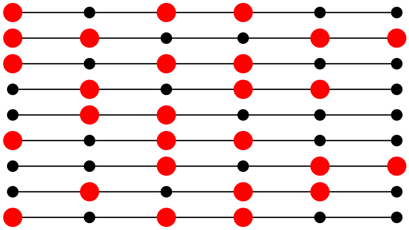
By the pigeonhole principle:



Lemma

Huge flipped H_t or tP_t $\Rightarrow$ large flipped H_s or sP_s in every set complement.

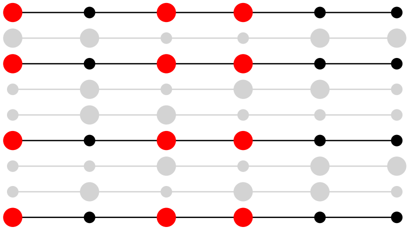
By the pigeonhole principle:



Lemma

Huge flipped H_t or $tP_t \Rightarrow$ large flipped H_s or sP_s in every set complement.

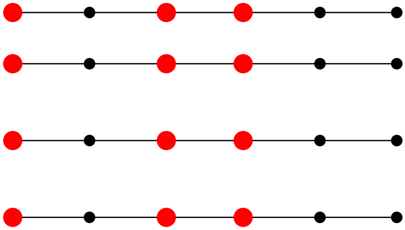
By the pigeonhole principle:



Lemma

Huge flipped H_t or $tP_t \Rightarrow$ large flipped H_s or sP_s in every set complement.

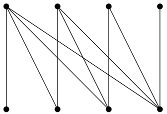
By the pigeonhole principle:

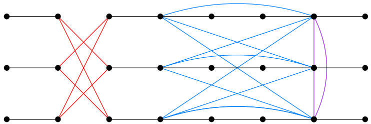


Main result

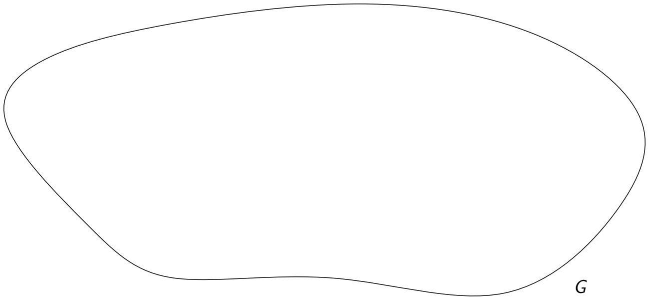
For every hereditary graph class $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ has bounded SC-depth (shrub-depth)
2. $\mathcal{C}$ is ∞-flip-flat
3. $\exists t$ such that $\mathcal{C}$ excludes all flipped H_t and tP_t
4. $\mathcal{C}$ does not FO-interpret all paths
5. $\mathcal{C}$ is MSO-stable
6. FO and MSO coincide on $\mathcal{C}$





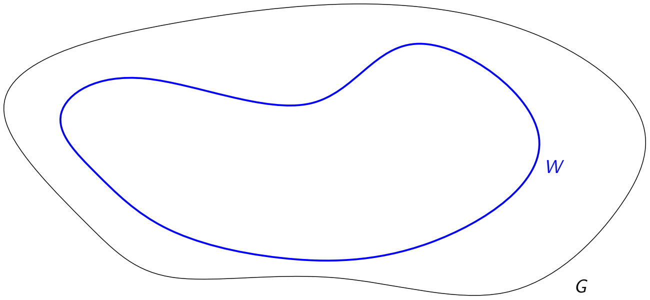
Flip-flatness



Flip-flatness (slightly informal)

A class $\mathcal{C}$ is *r-flip-flat* if in every large set W we find a still large set A that is r-scattered after performing a k-flip of G.

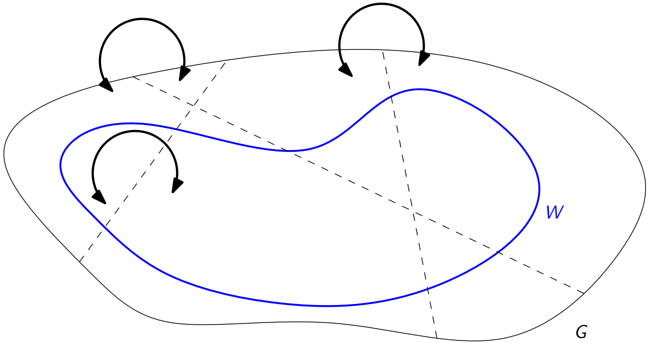
Flip-flatness



Flip-flatness (slightly informal)

A class $\mathcal{C}$ is *r-flip-flat* if in every large set W we find a still large set A that is r-scattered after performing a k-flip of G.

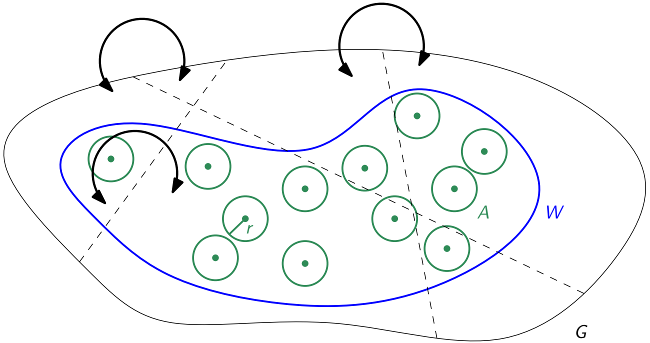
Flip-flatness



Flip-flatness (slightly informal)

A class $\mathcal{C}$ is *r-flip-flat* if in every large set W we find a still large set A that is r-scattered after performing a k-flip of G.

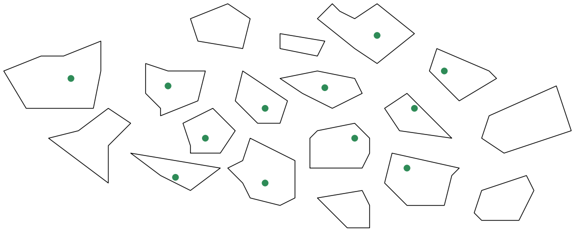
Flip-flatness



Flip-flatness (slightly informal)

A class $\mathcal{C}$ is *r-flip-flat* if in every large set W we find a still large set A that is r-scattered after performing a k-flip of G.

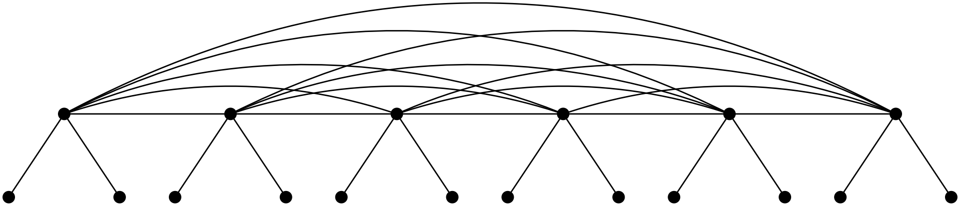
Flip-flatness



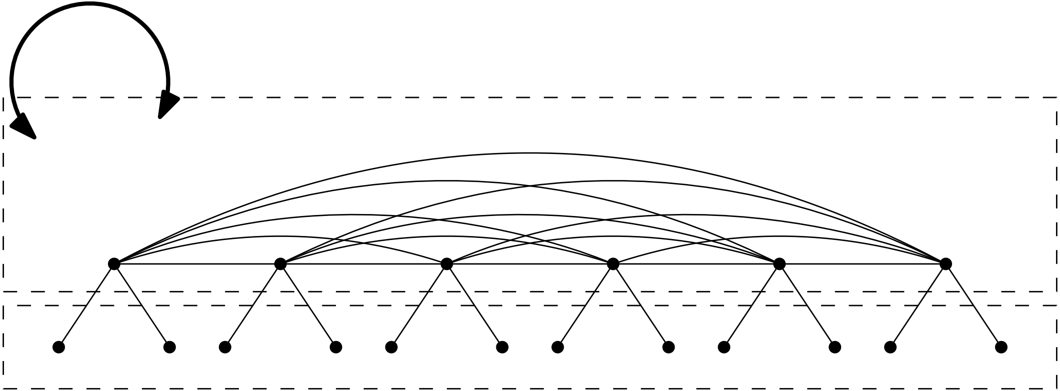
Flip-flatness (slightly informal)

A class $\mathcal{C}$ is *r-flip-flat* if in every large set W we find a still large set A that is r-scattered after performing a k-flip of G.

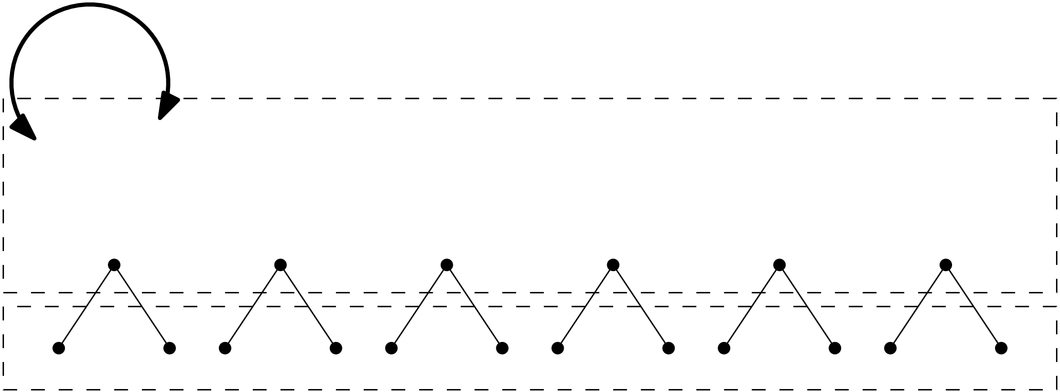
Flip-flatness example



Flip-flatness example



Flip-flatness example



Flip-flatness (slightly informal)

A class $\mathcal{C}$ is *r-flip-flat* if in every large set W we find a still large set A that is r-scattered after performing a k-flip of G.

Theorem

For every hereditary graph class $\mathcal{C}$:

- *$\mathcal{C}$ is FO-stable iff $\mathcal{C}$ is r-flip-flat for every $r \in \mathbb{N}$.* *[Dreier, NM, Siebertz, Toruńczyk; 2023]*
- *$\mathcal{C}$ has bd. SC-depth iff $\mathcal{C}$ is ∞-flip-flat.* *[Dreier, NM, Toruńczyk; 2024]* *[Ossona de Mendez, Pilipczuk, Siebertz; 2022]*

Flip-flatness (slightly informal)

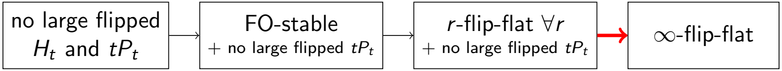
A class $\mathcal{C}$ is *r-flip-flat* if in every large set W we find a still large set A that is r-scattered after performing a k-flip of G.

Theorem

For every hereditary graph class $\mathcal{C}$:

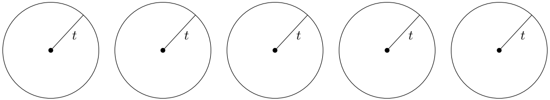
- *$\mathcal{C}$ is FO-stable iff $\mathcal{C}$ is r-flip-flat for every $r \in \mathbb{N}$.* *[Dreier, NM, Siebertz, Toruńczyk; 2023]*
- *$\mathcal{C}$ has bd. SC-depth iff $\mathcal{C}$ is ∞-flip-flat.* *[Dreier, NM, Toruńczyk; 2024]* *[Ossona de Mendez, Pilipczuk, Siebertz; 2022]*

The plan:



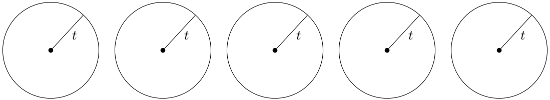
t-flip-flat + no large flipped $tP_t \Rightarrow \infty$-flip-flat

Apply t-flip-flatness. Result: many disjoint radius-t balls in a k-flip.

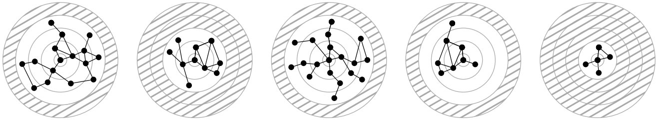


t-flip-flat + no large flipped $tP_t \Rightarrow \infty$-flip-flat

Apply t-flip-flatness. Result: many disjoint radius-t balls in a k-flip.

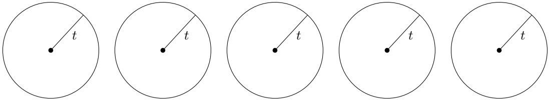


Case 1: Many balls whose outermost BFS layer is empty: this is ∞-flip-flatness ✓

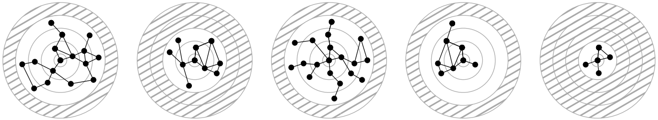


t-flip-flat + no large flipped $tP_t \Rightarrow \infty$-flip-flat

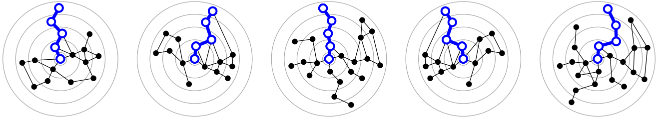
Apply t-flip-flatness. Result: many disjoint radius-t balls in a k-flip.



Case 1: Many balls whose outermost BFS layer is empty: this is ∞-flip-flatness ✓



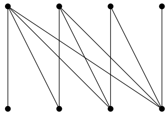
Case 2: Many balls whose outermost layer is non-empty: flipped tP_t; contradiction!

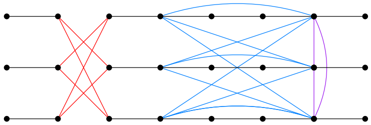


Main result

For every hereditary graph class $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ has bounded SC-depth (shrub-depth)
2. $\mathcal{C}$ is ∞-flip-flat
3. $\exists t$ such that $\mathcal{C}$ excludes all flipped H_t and tP_t
4. $\mathcal{C}$ does not FO-interpret all paths
5. $\mathcal{C}$ is MSO-stable
6. FO and MSO coincide on $\mathcal{C}$





The expressive power of MSO

FO and MSO have the same expressive power on a graph class $\mathcal{C}$ if for every MSO-sentence φ there is an FO-sentence ψ such that for all $G \in \mathcal{C}$:

$$G \models \varphi \Leftrightarrow G \models \psi.$$

The expressive power of MSO

FO and MSO have the same expressive power on a graph class $\mathcal{C}$ if for every MSO-sentence φ there is an FO-sentence ψ such that for all $G \in \mathcal{C}$:

$$G \models \varphi \Leftrightarrow G \models \psi.$$

Theorem [Gajarský and Hliněný; 2015]

FO and MSO have the same expressive power on every class of bounded SC-depth.

Theorem [this paper]

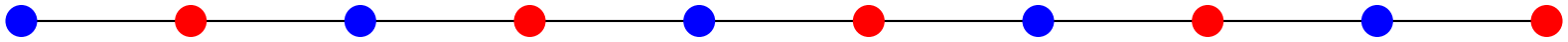
MSO is more expressive than FO on every hereditary class of unbounded SC-depth.

“Philosophical meaning”: MSO is more expressive than FO iff there are long orders.

Separating MSO and FO on flipped half-graphs

We first separate MSO and FO on the class of paths.

Even length on paths is expressible in MSO:

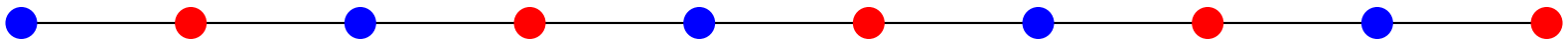


Quantify 2-coloring and check if the endpoints have different colors.

Separating MSO and FO on flipped half-graphs

We first separate MSO and FO on the class of paths.

Even length on paths is expressible in MSO:



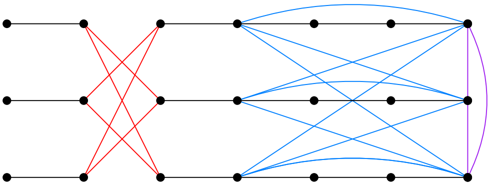
Quantify 2-coloring and check if the endpoints have different colors.

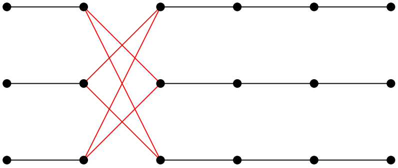
Even length on paths is not expressible in FO. (Ehrenfeucht-Fraïssé Games)

(In)expressibility lifts to flipped half-graphs. ✓

Separating MSO and FO on flipped tP_t

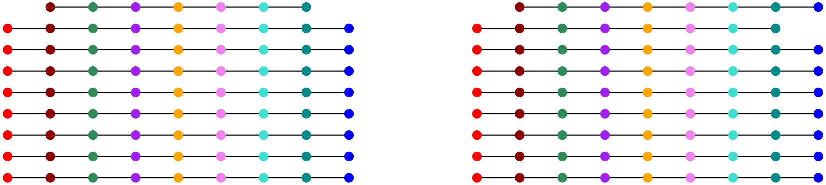
The flipped tP_t in $\mathcal{C}$ could be totally different from the flipped $(t+1)P_{t+1}$ in $\mathcal{C}$.





Separating MSO and FO on flipped tP_t

We separate two induced subgraphs of the same flipped tP_t:



FO cannot distinguish between the above two graphs (Hanf Locality), but MSO can.

Main result

For every hereditary graph class $\mathcal{C}$, the following are equivalent:

1. $\mathcal{C}$ has bounded SC-depth (shrub-depth)
2. $\mathcal{C}$ is ∞-flip-flat
3. $\exists t$ such that $\mathcal{C}$ excludes all flipped H_t and tP_t
4. $\mathcal{C}$ does not FO-interpret all paths
5. $\mathcal{C}$ is MSO-stable
6. FO and MSO coincide on $\mathcal{C}$