Combinatorial Characterizations for Monadically Stable and Monadically NIP Graph Classes

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joint work with: Jan Dreier[2], Sebastian Siebertz[3], Szymon Toruńczyk[4]

AIMoTh 2024 - 29.02.2024

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The FO Model Checking Problem

Problem: Given a graph G and an FO sentence φ, decide whether

$$G \models \varphi.$$

Example: G contains a dominating set of size k iff.

$$G \models \exists x_1 \ldots \exists x_k \forall y : \bigvee_{i \in [k]} (y = x_i \vee y \sim x_i).$$

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Question: On which classes is FO model checking fixed-parameter tractable, i.e., solvable in time $f(\varphi) \cdot n^c$?

Nowhere Dense Classes of Graphs

Definition [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is *nowhere dense*, if for every r there exists k such $\mathcal{C}$ that does not contain the r-subdivided clique of size k as a subgraph.

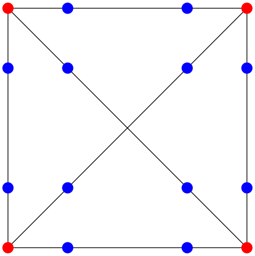

Figure: The 2-subdivided K_4.

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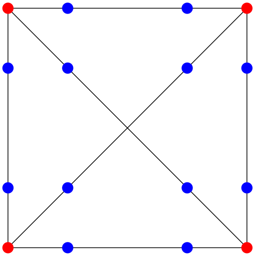

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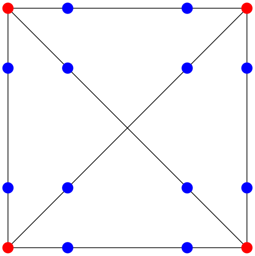

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Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a *monotone* class of graphs. If $\mathcal{C}$ is nowhere dense, then FO model checking on $\mathcal{C}$ can be done in time $f(\varphi, \varepsilon) \cdot n^{1+\varepsilon}$ for every $\varepsilon > 0$. Otherwise it is AW[$*$]-hard.

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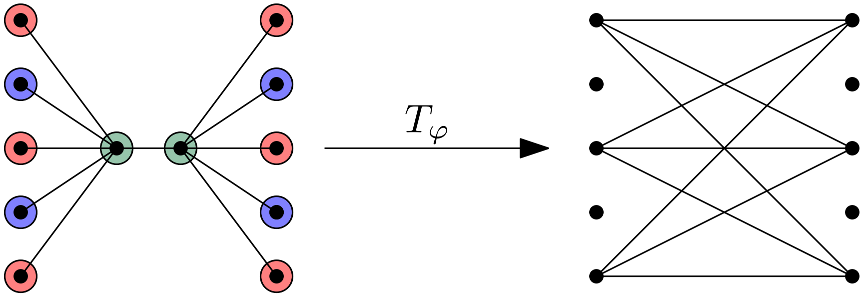
Transductions $\hat{=}$ coloring + interpreting + taking an induced subgraph



$\varphi(x, y) := \mathrm{Red}(x) \wedge \mathrm{Red}(y) \wedge \mathrm{dist}(x, y) = 3$

Monadic Stability and Monadic NIP

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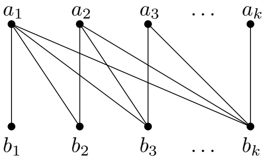

$a_i \sim b_j \Leftrightarrow i \leq j$

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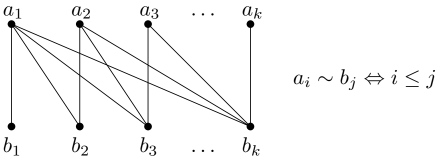

Definition

A class is *monadically NIP*, if it does not transduce the class of all graphs.

Limits of Tractability

Nowhere Dense $\subsetneq$ **Structurally Nowhere Dense** $\subseteq$ **Monadically Stable** $\subsetneq$ **Monadically NIP**

Theorem

Model checking is fixed-parameter tractable on classes that are

- nowhere dense [Grohe, Kreutzer, Siebertz, 2014]
- structurally nowhere dense [Dreier, Mählmann, Siebertz, 2023]
- monadically stable [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2023]

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Theorem [Dreier, Mählmann, Toruńczyk, 2024]

Model checking is AW[$*$]-hard on every hereditary class that is **not** monadically NIP.

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Conjecture

A hereditary class has fpt model checking iff it is monadically NIP.

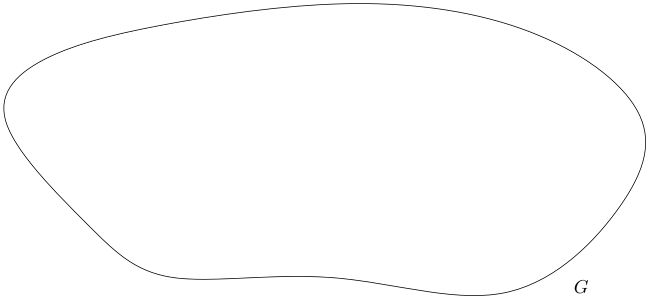
Agenda

Nowhere Dense $\subsetneq$ **Monadically Stable** $\subsetneq$ **Monadically NIP**

Goals for today:

1. Define and motivate mon. stable and mon. NIP classes. ✓
2. Give combinatorial structure characterizations of the two.
   - Build the foundation for fpt model checking.
   - Reveal connections to nowhere denseness and other graph parameters.
3. Give combinatorial non-structure characterizations of the two.
   - Various hardness results are implied.

Characterizing Nowhere Denseness: Uniform Quasi-Wideness



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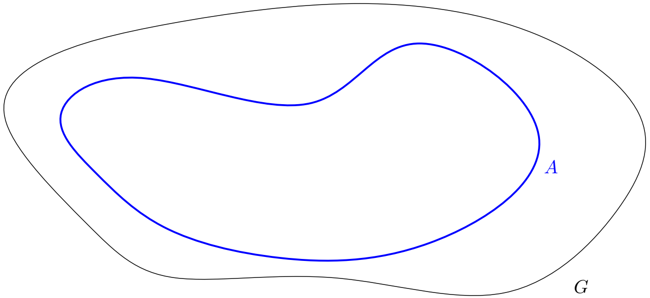

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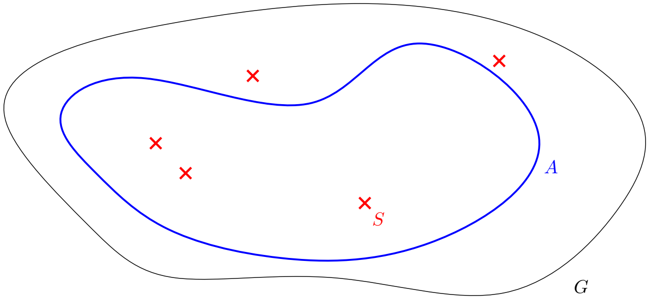

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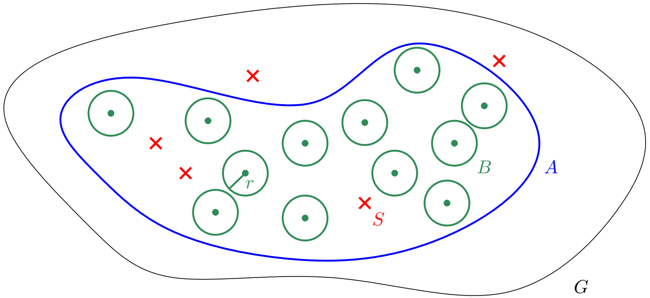

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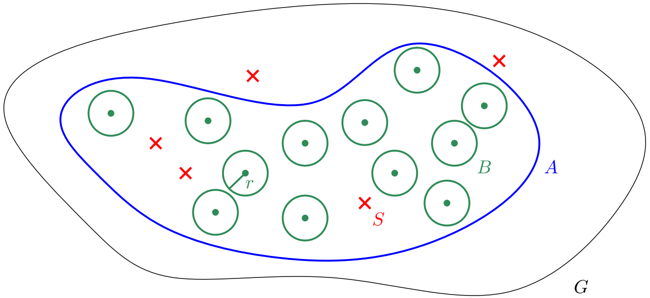
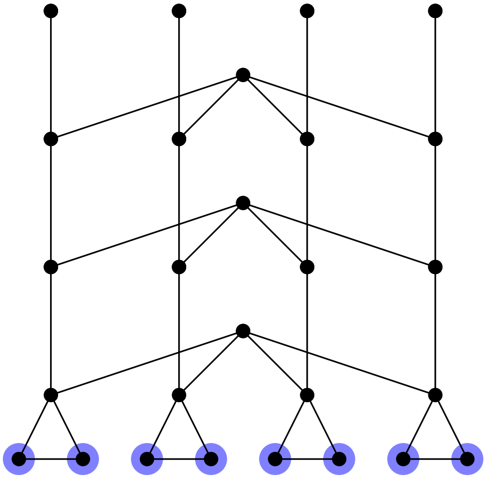


Uniform Quasi-Wideness (slightly informal)

A class $\mathcal{C}$ is *uniformly quasi-wide* if for every radius r, in every large set A we find a still large set B that is r-independent after removing a set S of constantly many vertices.

Uniform Quasi-Wideness: Example



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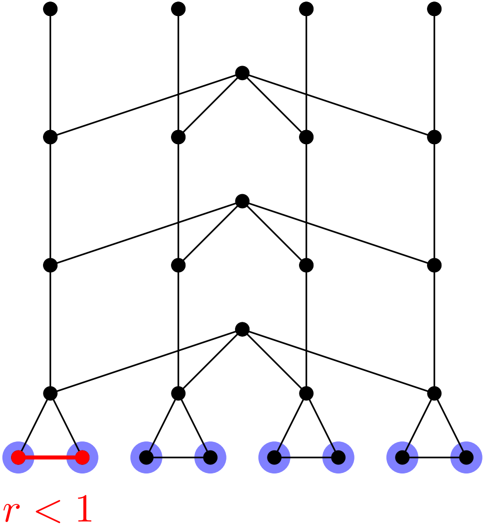

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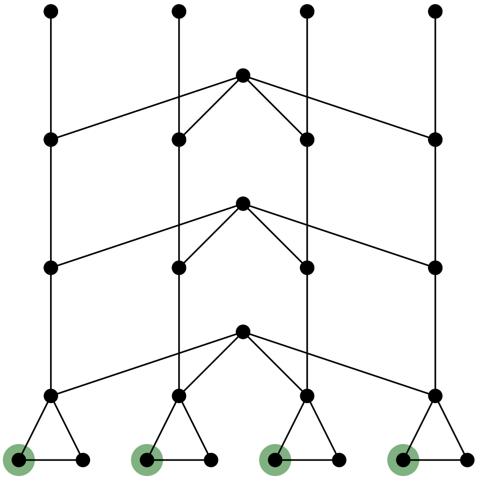

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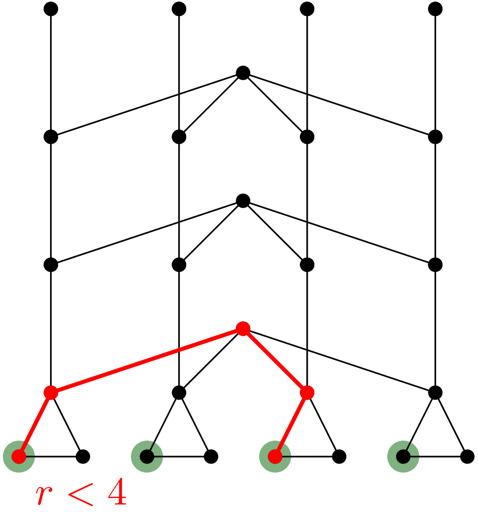

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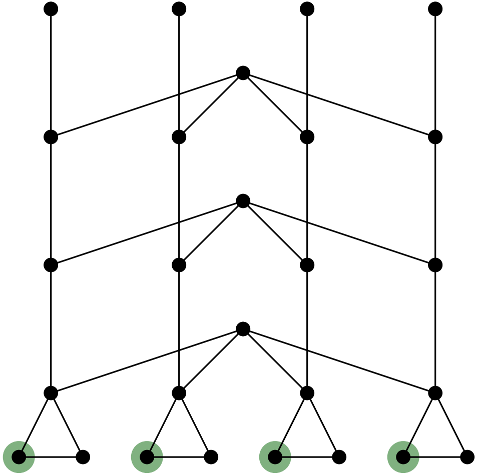

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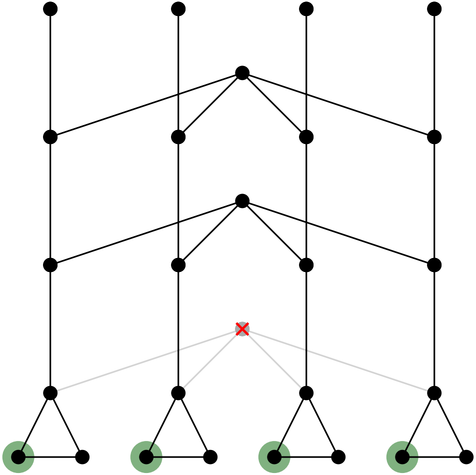

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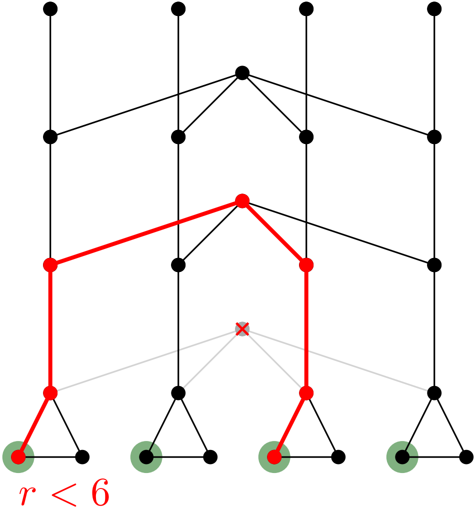

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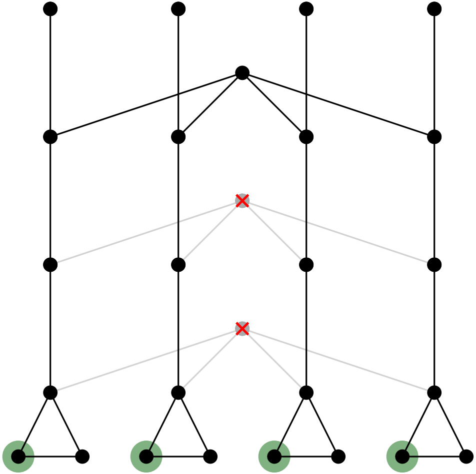

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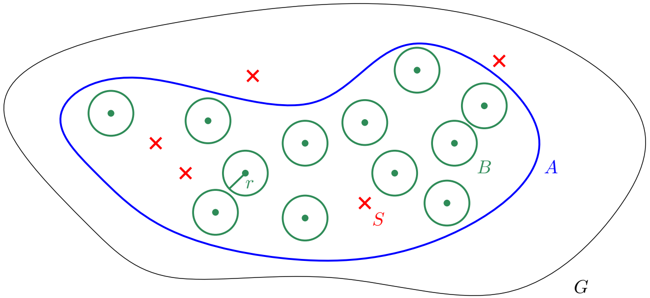

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Theorem [Něsetřil, Ossona de Mendez, 2011]

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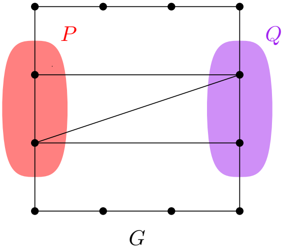

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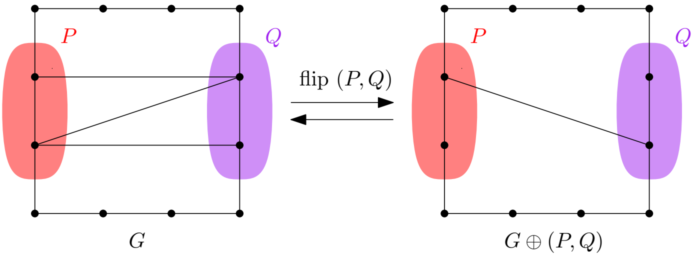

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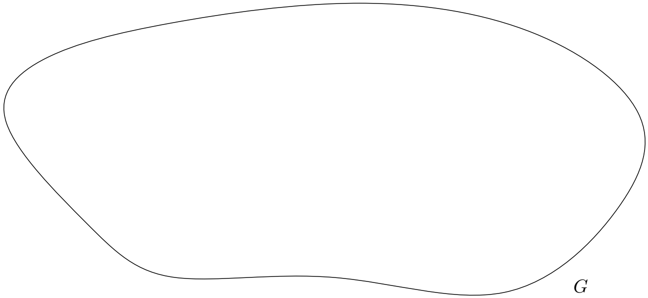

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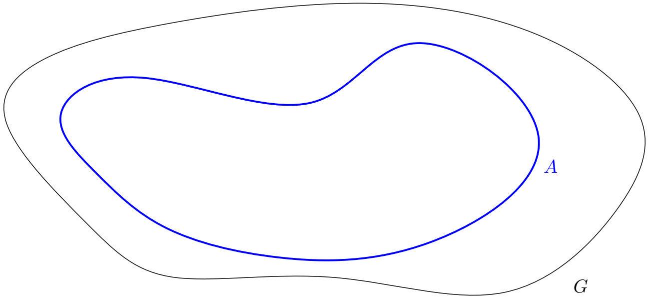

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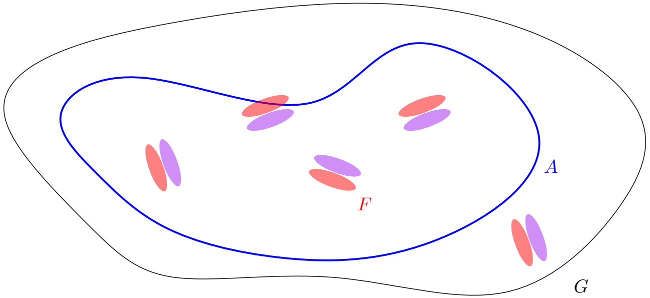

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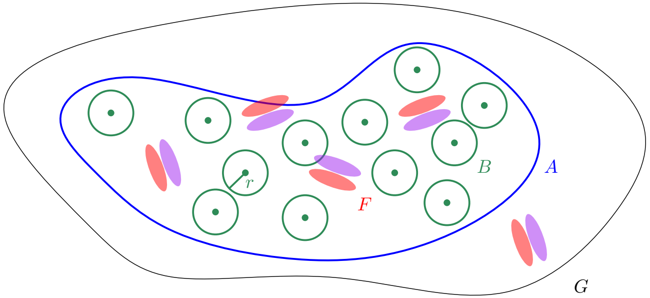

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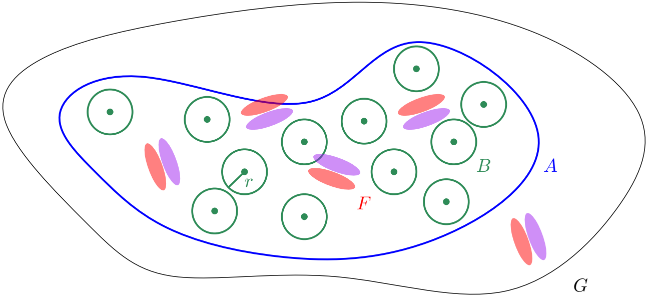
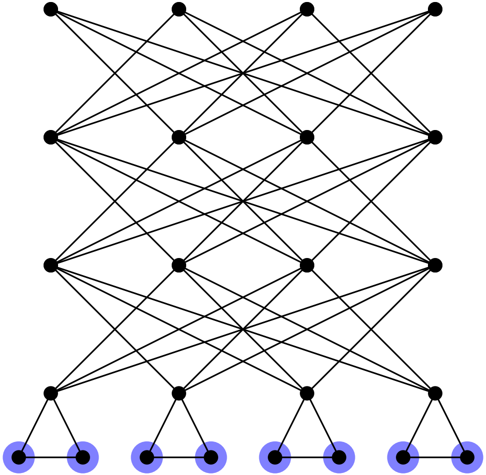


Flip-Flatness (slightly informal) [Gajarský, Kreutzer]

A class $\mathcal{C}$ is *flip-flat* if for every radius r, in every large set A we find a still large set B that is r-independent after performing a set F of constantly many flips.

Flip-Flatness: Example



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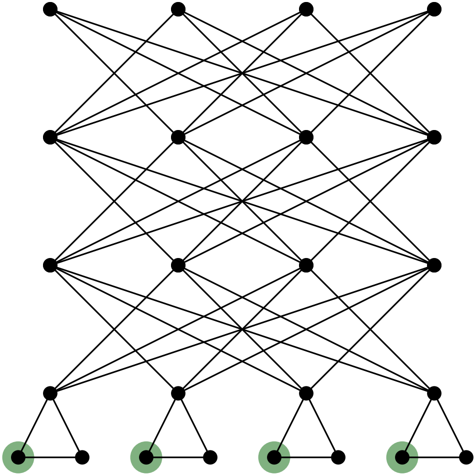

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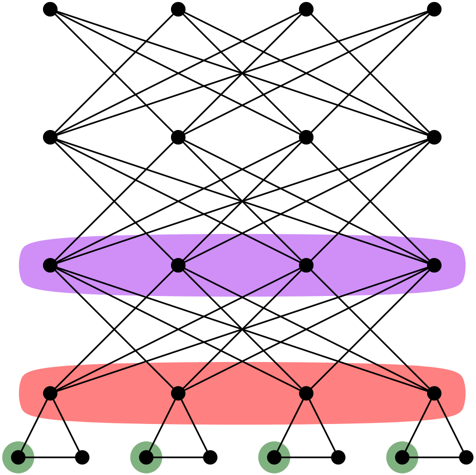

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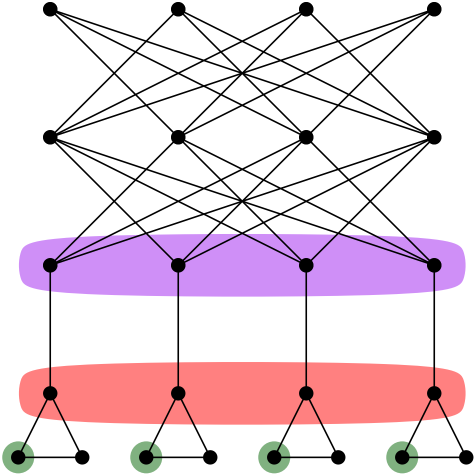

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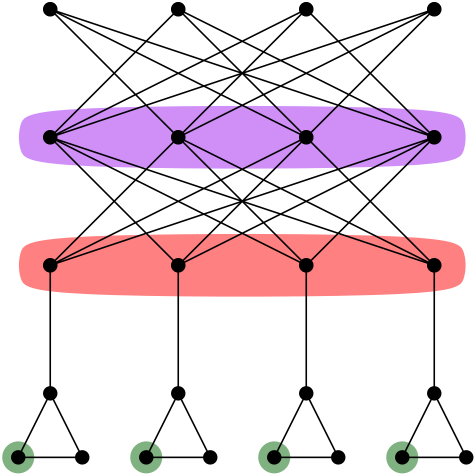

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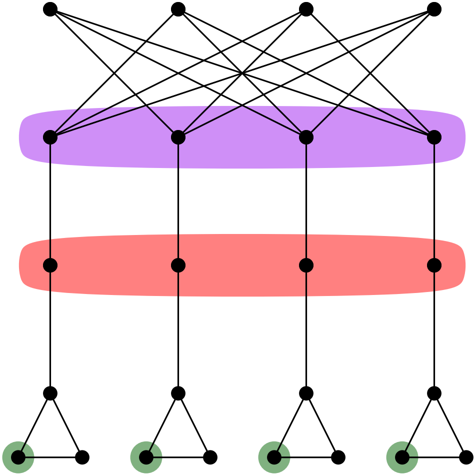

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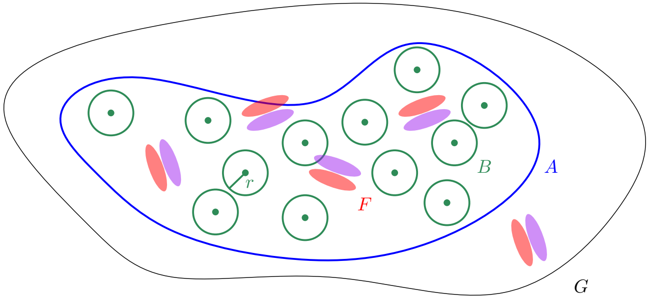

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Flip-Flatness: Towards Model Checking

Qualitative properties of monadic stability:

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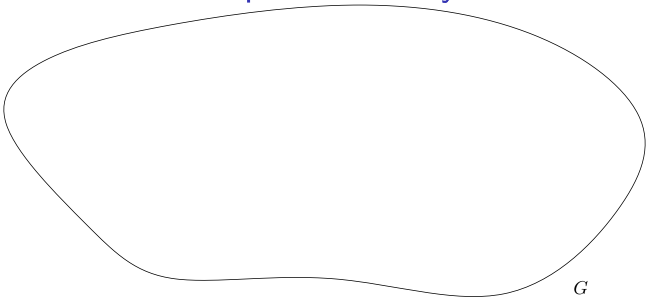
To solve model checking we combine both aspects to build:

- small treelike neighborhood decompositions of bounded depth

Characterizing Monadic NIP: Flip-Breakability



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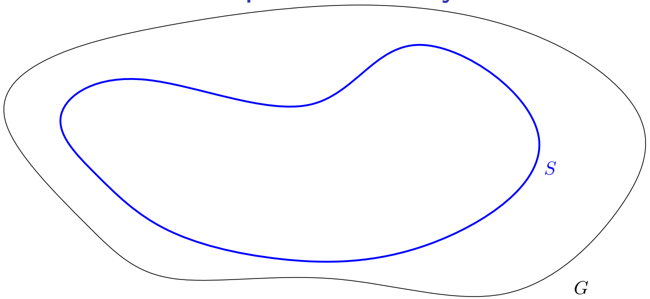

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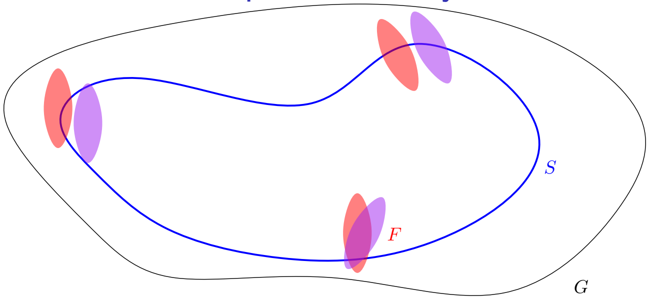

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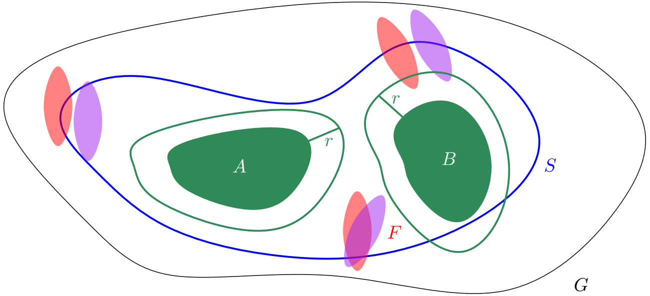

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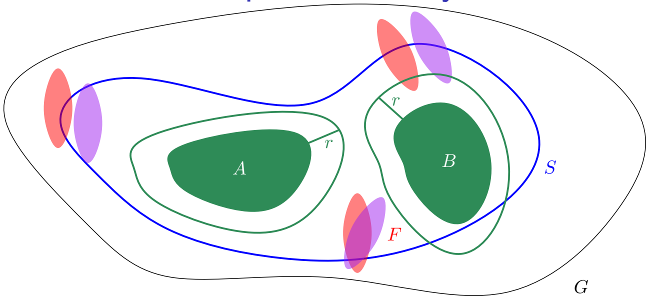

Flip-Breakability (slightly informal)

A class $\mathcal{C}$ is *flip-breakable* if for every radius r, in every large set S we find two large sets A and B that and a flip F of bounded size such that $N^r_{G\oplus F}(A) \cap N^r_{G\oplus F}(B) = \varnothing$.

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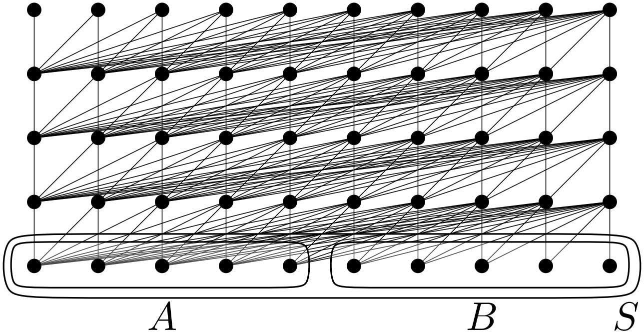

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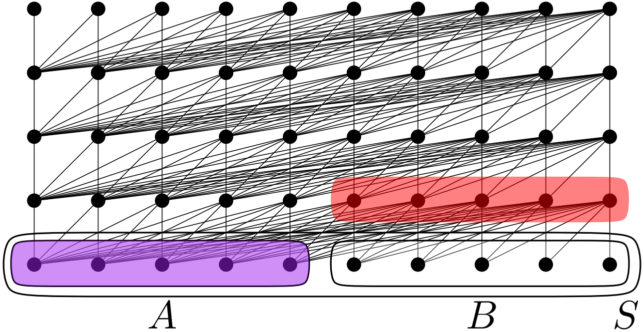

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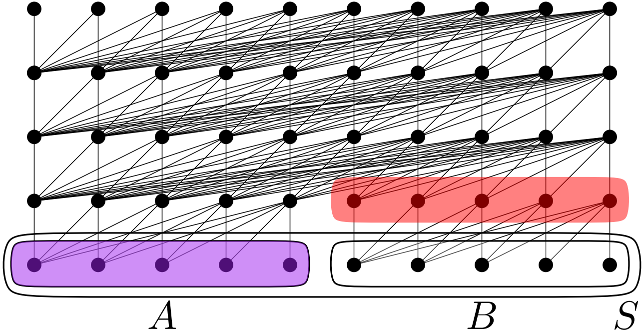

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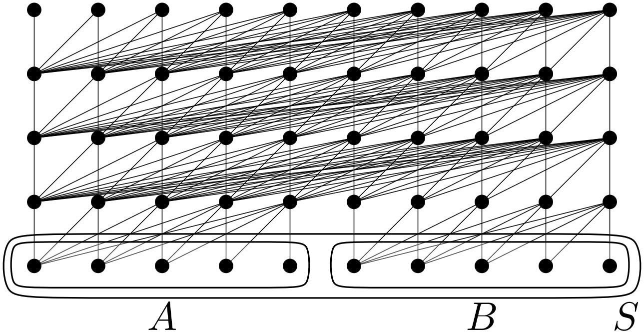

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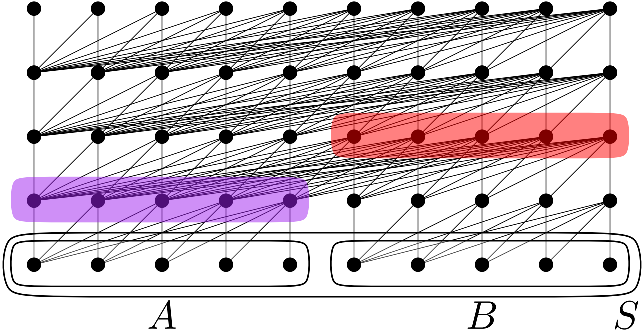

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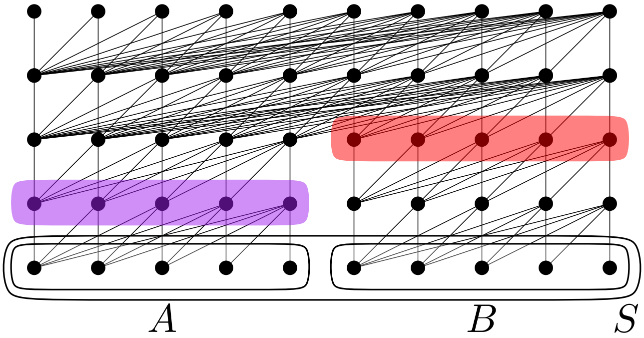

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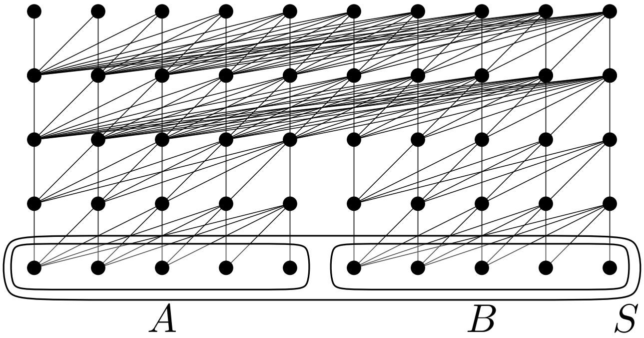

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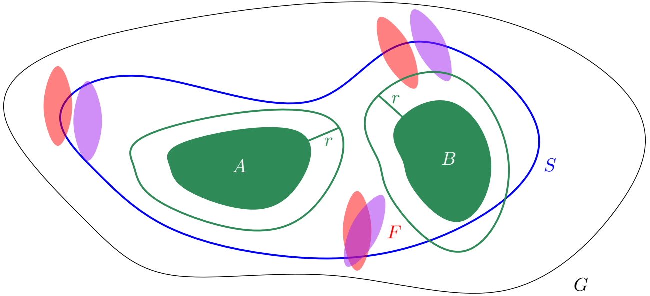

Flip-Breakability (slightly informal)

A class $\mathcal{C}$ is *flip-breakable* if for every radius r, in every large set S we find two large sets A and B that and a flip F of bounded size such that $N^r_{G \oplus F}(A) \cap N^r_{G \oplus F}(B) = \varnothing$.

Theorem [Dreier, Mählmann, Toruńczyk, 2024]

A class $\mathcal{C}$ is flip-breakable if and only if it is monadically NIP.

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.

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	deletion-	**nowhere denseness**	
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dist-∞	flip-	**bd. shrubdepth**	**bd. cliquewidth**
	deletion-	**bd. treedepth**	**bd. treewidth**

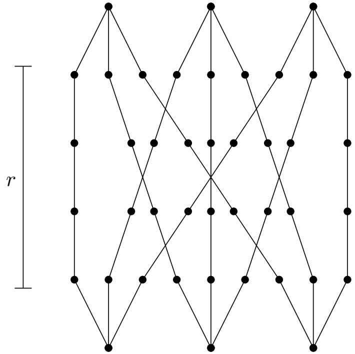
Agenda

Nowhere Dense $\subsetneq$ Monadically Stable $\subsetneq$ Monadically NIP

Goals for today:

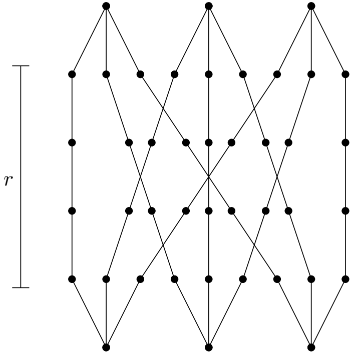
1. Define and motivate mon. stable and mon. NIP classes. ✓
2. Give combinatorial structure characterizations of the two. ✓
   - Build the foundation for fpt model checking. ✓
   - Reveal connections to nowhere denseness and other graph parameters. ✓
3. Give combinatorial non-structure characterizations of the two.
   - Various hardness results are implied.

Characterizing Monadic NIP by Forbidden Induced Subgraphs



star r-crossing
$= r$-subdivided biclique

Characterizing Monadic NIP by Forbidden Induced Subgraphs



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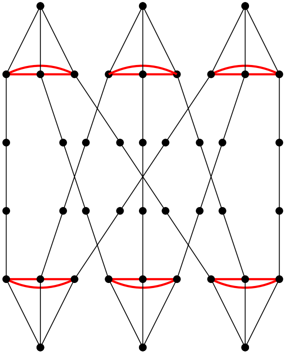
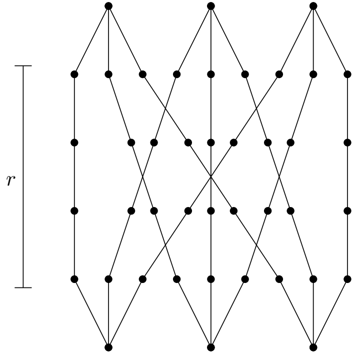


clique r-crossing

Characterizing Monadic NIP by Forbidden Induced Subgraphs



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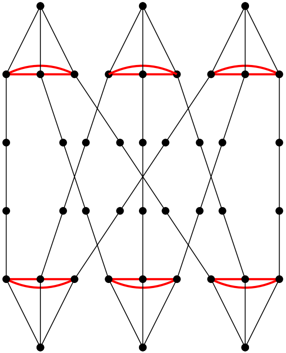
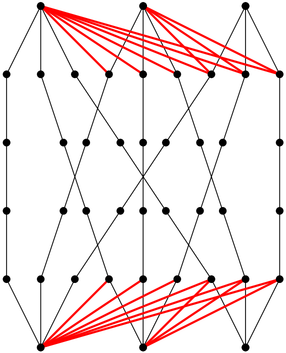
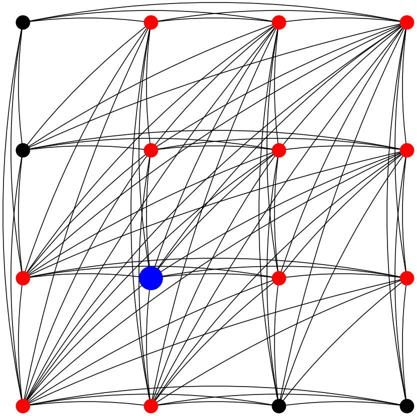


clique r-crossing



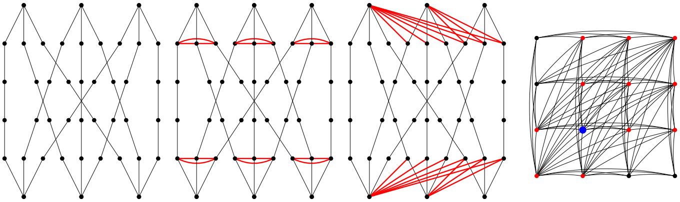
half-graph r-crossing

Characterizing Monadic NIP by Forbidden Induced Subgraphs



comparability grid

Characterizing Monadic NIP by Forbidden Induced Subgraphs



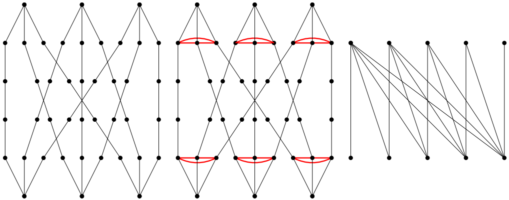
Theorem [Dreier, Mählmann, Toruńczyk, 2024]

Let $\mathcal{C}$ be a graph class. Then $\mathcal{C}$ is monadically NIP if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $\mathcal{C}$ excludes as induced subgraphs

- all layerwise flipped star r-crossings of order k, and
- all layerwise flipped clique r-crossings of order k, and
- all layerwise flipped half-graph r-crossings of order k, and
- the comparability grid of order k.

$\Rightarrow$ Model checking is hard on every hereditary graph class that is not monadically NIP.

Characterizing Monadic Stability by Forbidden Induced Subgraphs

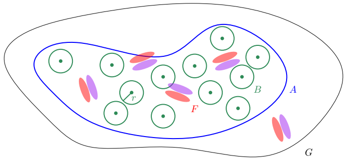
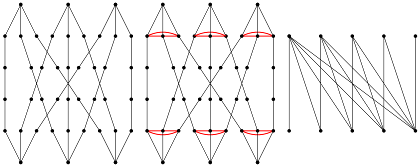
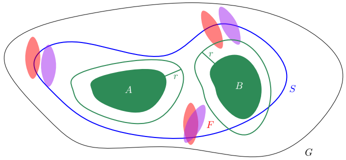
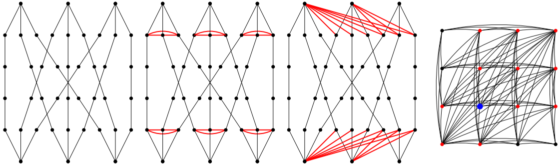


Theorem [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2023]

Let $\mathcal{C}$ be a graph class. Then $\mathcal{C}$ is monadically stable if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $\mathcal{C}$ excludes as induced subgraphs

- all layerwise flipped star r-crossings of order k, and
- all layerwise flipped clique r-crossings of order k, and
- all semi-induced halfgraphs of order k

Summary

<table>
<tr><th></th><th>structure</th><th>non-structure</th></tr>
<tr><td>m. stable</td><td>
</td><td></td></tr>
<tr><td>m. NIP</td><td>
</td><td></td></tr>
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Vielen Dank!